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ON EXPONENTIAL SUMS OVER PRIME NUMBERS

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Abstract

In this article we establish an estimate for a sum over primes that is the analogue of an estimate for a sum over consecutive integers which has proved to be very useful in applications of exponential sums to problems in number theory.

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1. Notation

Let $c_0, c_1, \ldots$ denote effectively computable positive absolute constants. For any real number A, we write $\min(A, 1/0) = A$. For any real number x let $[x]$ denote the greatest integer less than or equal to x, let $\{x\} = x - [x]$ denote the fractional part of x and let $\|x\| = \min(\{x\}, 1 - \{x\})$ denote the distance from x to the nearest integer. We write $e^{2\pi i x} = e(x)$. Further, for any positive integer n let $\phi(n)$ denote the number of positive integers less than or equal to n and coprime with n.

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2. Introduction

In number theoretical applications of exponential sums we often use estimates for sums of the form

$$\sum_{n\leq N} \min(y, \|n\alpha\|^{-1}) \tag{1}$$

where y and α are real numbers and N is a positive integer (see, for example, [5, page 24]). The purpose of this paper is to derive similar estimates for sums of the form

$$\sum_{p\leq N} \min(y, \|p\alpha\|^{-1}), \tag{2}$$

where the summation is taken over primes instead of consecutive integers. We expect our estimates will be widely applicable. In fact, a problem in additive number theory (see [3]) first led us to the study of sums of the form (2). By using the result below we are able to simplify the proof of the main theorem of [3].

Theorem. *Let ε be a positive real number. There exists an effectively computable positive absolute constant c_1 and a positive real number N_0 which is effectively computable in terms of ε such that if N is a positive integer with $N > N_0$ and y is a real number with*

$$3 \leq y \leq N^{\frac{1}{4}-\varepsilon}, \tag{3}$$

then

$$\sum_{p\leq N} \min(y, \|p\alpha\|^{-1}) < c_1 \frac{N \log y \log\log y}{\log N},$$

for all real numbers α with $1/N \leq \alpha \leq 1 - 1/N$.

This paper is devoted to a proof of the above theorem. We shall use some ideas from [3]. In particular the treatment of the "major arcs" will be nearly the same as in [3].

3. Preliminary lemmas

Lemma 1. *There exists an effectively computable positive real number c_2 such that*

$$\phi(n) > c_2 \frac{n}{\log\log n}$$

for $n \geq 3$.

Proof. See [2, page 24].

Lemma 2. *There exists an effectively computable positive real number c_3 such that for any integers a and b with $b \geq 2$,*

$$\sum_{\substack{1 \leq n \leq b \\ (n+a,b)=1}} \frac{1}{n} < c_3 \frac{\phi(b)}{b} \log b.$$

Proof. This is [3, Lemma 5].

Lemma 3. *Let h, a and q be integers with $a > 0$, $q > 1$ and $(a, q) = 1$. Let $\rho(n)$ be a real valued function defined for those integers n with $h \leq n < h + q$ and $(n, q) = 1$. Put*

$$\lambda = \max_{\substack{h \leq n < h+q \\ (n,q)=1}} \rho(n) - \min_{\substack{h \leq n < h+q \\ (n,q)=1}} \rho(n)$$

and

$$\psi(n) = \frac{1}{q}(an + \rho(n)).$$

There is an effectively computable positive absolute constant c_4 such that if $\lambda \leq 1$ and if E is a real number satisfying $2 \leq E \leq q$ then

$$\sum_{\substack{h \leq n < h+q \\ (n,q)=1}} \min\left(E, \frac{1}{\|\psi(n)\|}\right) < c_4 \phi(q) \log E.$$

Proof. This is [3, Lemma 6].

Lemma 4. *Let δ be a real number satisfying $0 < \delta \leq 1/2$. Then there exists a periodic function $\psi(x, \delta)$, with period 1, such that*

(i) *$\psi(x, \delta) \geq 1$ in the integral $-\delta \leq x \leq \delta$,*
(ii) *$\psi(x, \delta) \geq 0$ for all x,*
(iii) *$\psi(x, \delta)$ has a Fourier series of the form*

$$\psi(x, \delta) = a_0 + \sum_{0 < j \leq (1/2\delta) - 1} a_j \cos 2\pi jx$$

where $|a_0| \leq \pi^2 \delta$ and $|a_j| < 2\pi^2 \delta$ for $0 < j \leq (1/2\delta) - 1$.

Proof. Put $N = [1/2\delta]$ and

$$\psi(x, \delta) = \frac{\pi^2}{4N^2} \left| \sum_{k=1}^{N} e(kx) \right|^2.$$

Then (ii) holds trivially. Certainly $|1-e(\alpha)| = 2|\sin \pi\alpha|$ and $|\sin\alpha| \le |\alpha|$ for all α, while $|\sin\alpha| \ge 2|\alpha|/\pi$ for $|\alpha| \le \pi/2$. Therefore for $|x| \le \delta \le 1/(2N)$ we have

$$\psi(x,\delta) = \frac{\pi^2}{4N^2}\left|\frac{1-e(Nx)}{1-e(x)}\right|^2 = \frac{\pi^2}{4N^2}\frac{|\sin\pi Nx|^2}{|\sin\pi x|^2} \ge \frac{\pi^2}{4N^2}\frac{\left(\frac{2}{\pi}\cdot\pi Nx\right)^2}{(\pi x)^2} = 1,$$

and so (i) also holds.

Finally, we have

$$\begin{aligned}\psi(x,\delta) &= \frac{\pi^2}{4N^2}\left|\sum_{k=1}^{N} e(kx)\right|^2 = \frac{\pi^2}{4N^2}\sum_{k=1}^{N} e(kx)\sum_{l=1}^{N} e(-lx)\\ &= \frac{\pi^2}{4N^2}\left(N + \sum_{j=1}^{N-1}(N-j)(e(j\alpha)+e(-j\alpha))\right)\\ &= \frac{\pi^2}{4N} + \sum_{j=1}^{N-1}\frac{\pi^2(N-j)}{2N^2}\cos j\alpha = a_0 + \sum_{0<j\le N-1} a_j\cos j\alpha\\ &= a_0 + \sum_{0<j\le(1/2\delta)-1} a_j\cos j\alpha\end{aligned}$$

where

$$a_0 = \frac{\pi^2}{4N} = \frac{\pi^2}{4[1/2\delta]} \le \frac{\pi^2}{2(1/2\delta)} = \pi^2\delta$$

and

$$a_j = \frac{\pi^2(N-j)}{2N^2} < \frac{\pi^2 N}{2N^2} = \frac{\pi^2}{2N} = 2a_0 \le 2\pi^2\delta \quad \text{for } 0 < j \le (1/2\delta) - 1,$$

which completes the proof of Lemma 4.

We shall also require the Brun-Titchmarsh theorem and a refinement, due to Vaughan, of Vinogradov's fundamental lemma.

Let x be a positive real number and let l and k be positive integers. As usual we denote the number of primes less than or equal to x by $\pi(x)$, and the number of primes less than or equal to x and congruent to l modulo k by $\pi(x,k,l)$.

LEMMA 5 (Brun-Titchmarsh theorem). *Let x and y be positive real numbers and let k and l be relatively prime positive integers with $y > k$. Then*

$$\pi(x+y,k,l) - \pi(x,k,l) < \frac{2y}{\phi(k)\log(y/k)}.$$

PROOF. See [1, Theorem 2].

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LEMMA 6. *If α is a real number and a, q, H and N are positive integers with $(a,q)=1$, $q \le N$, $H < N$ and $|\alpha - a/q| \le q^{-2}$ then*

$$\sum_{h=1}^{H}\left|\sum_{p\le N} e(hp\alpha)\right| < c_5(\log N)^6(HNq^{-1/2} + HN^{3/4} + (HNq)^{1/2} + H^{3/5}N^{4/5}\exp(2\log N/\log\log N)),$$

where c_5 is an effectively computable positive absolute constant.

PROOF. This follows from [2, Satz 5.2] and [4, Theorem 1] by partial summation.

4. Further preliminaries

Put $P = y^2(\log N)^{14}$ and $Q = N/P$.

Let T_1 denote the set of those α in the interval $(1/N, 1-1/N)$ for which there exist positive integers a and b with $(a,b)=1$, such that

(4) $$\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^2}$$

and

(5) $$P \le b \le Q = N/P.$$

Put $T' = (1/N, 1-1/N) - T_1$, so that T' consists of the real numbers α in $(1/N, 1-1/N)$ which are not in T_1. Suppose that $\alpha \in T'$. Then by Dirichlet's theorem there exist integers a and b with

(6) $$\left|\alpha - \frac{a}{b}\right| \le \frac{1}{bQ},$$

$0 \le a$, $0 < b < Q$ and $(a,b)=1$. Plainly

(7) $$\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^2},$$

and thus,

(8) $$0 < b < P.$$

To each α in T' we shall associate a pair of coprime integers a and b with $a \ge 0$ and $b > 0$ satisfying (6) and (8) and we shall put $\beta = \alpha - \frac{a}{b}$. Let us define subsets T_2, T_3 and T_4 of T' in the following way:

$$T_2 = \{\alpha \in T' : 1 \le b \le y, |\beta| \le 1/2bN\},$$
$$T_3 = \{\alpha \in T' : 1 \le b \le y, |\beta| > 1/2bN\},$$
$$T_4 = \{\alpha \in T' : y < b\}.$$

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Further put

$$S(\alpha) = \sum_{p \le N} \min(y, \|p\alpha\|^{-1}).$$

Since $(1/N, 1 - 1/N) = T_1 \cup T_2 \cup T_3 \cup T_4$ it suffices to show that

$$\max_{\alpha \in T_i} S(\alpha) < c_6 \frac{N \log y \cdot \log\log y}{\log N} \tag{9}$$

for $i = 1, 2, 3, 4$ when $N > N_0$. For $i = 1$ ("minor arcs"), (9) will be established in Section 5, while cases $i = 2, 3, 4$ ("major arcs") will be dealt with in Section 6.

5. Minor arcs

Assume that $\alpha \in T_1$ and let $N_0, N_1, N_2, \ldots$ denote real numbers which are effectively computable in terms of ε.

For $\beta > 0$, put

$$Z(N, \alpha, \beta) = \sum_{\substack{p \le N \\ \|p\alpha\| < \beta}} 1.$$

Then by the prime number theorem, for $N > N_1$,

$$\begin{aligned}
(10)\quad S(\alpha) &= \sum_{p \le N} \min(y, \|p\alpha\|^{-1}) \\
&= \sum_{\substack{p \le N \\ \|p\alpha\| < 1/y}} \min(y, \|p\alpha\|^{-1}) + \sum_{j=2}^{[y/2]+1} \sum_{\substack{p \le N \\ \frac{j-1}{y} \le \|p\alpha\| < \frac{j}{y}}} \min(y, \|p\alpha\|^{-1}) \\
&\le \sum_{\substack{p \le N \\ \|p\alpha\| < 1/y}} y + \sum_{j=2}^{[y/2]+1} \sum_{\substack{p \le N \\ \frac{j-1}{y} \le \|p\alpha\| < \frac{j}{y}}} \min\left(y, \left(\frac{j-1}{y}\right)^{-1}\right) \\
&= yZ(N, \alpha, 1/y) + \sum_{j=2}^{[y/2]+1} \frac{y}{j-1}(Z(N, \alpha, j/y) - Z(N, \alpha, (j-1)/y)) \\
&= y \sum_{j=2}^{[y/2]} Z(n, \alpha, j/y) \left(\frac{1}{j-1} - \frac{1}{j}\right) + \frac{y}{[y/2]} Z(N, \alpha, ([y/2]+1)/y) \\
&\le y \sum_{j=2}^{[y/2]} \frac{Z(n, \alpha, j/y)}{(j-1)j} + 4 \sum_{p \le N} 1 \\
&< y \sum_{j=2}^{[y/2]} \frac{Z(N, \alpha, j/y)}{(j-1)j} + 5 \frac{N}{\log N}.
\end{aligned}$$

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By Lemma 4 (with j/y in place of δ), for $N > N_2$ and $1 \le j \le y/2$ we have

$$\begin{aligned} Z(N,\alpha,j/y) &= \sum_{\substack{p\le N\\ \|p\alpha\|<j/y}} 1 \le \sum_{p\le N} \psi(p\alpha, j/y) \\ &= \sum_{p\le N}\left(a_0 + \sum_{0<k\le (y/2j)-1} a_k \cos 2\pi k(p\alpha)\right) \\ &= a_0\pi(N) + \sum_{0<k\le (y/2j)-1} a_k \operatorname{Re}\sum_{p\le N} e(kp\alpha) \\ &\le |a_0|\pi(N) + \sum_{0<k\le (y/2j)-1} |a_k| \left|\sum_{p\le N} e(kp\alpha)\right| \\ &\le 2\pi^2\frac{j}{y}\left(\frac{N}{\log N} + \sum_{0<k\le (y/2j)-1}\left|\sum_{p\le N} e(kp\alpha)\right|\right). \end{aligned}$$

Thus, by Lemma 6,

$$\begin{aligned} &Z(N,\alpha,j/y) \\ &\quad < 20\frac{j}{y}\left(\frac{N}{\log N} + c_5(\log N)^6\left(\frac{y}{2j}NP^{-1/2} + \frac{y}{2j}N^{3/4}\left(\frac{y}{2j}\frac{N^2}{P}\right)^{1/2}\right.\right. \\ &\qquad\qquad \left.\left. + \left(\frac{y}{2j}\right)^{3/5} N^{4/5}\exp(2\log N/\log\log N)\right)\right) \\ &\quad < 20\frac{j}{y}\left(\frac{N}{\log N} + c_5(\log N)^6(2yNP^{-1/2} + yN^{3/4} + y^{3/5}N^{4/5}\right. \\ &\qquad\qquad \left. \exp(2\log N/\log\log N))\right). \end{aligned}$$

Since $P = y^2(\log N)^{14}$ it follows from (3) and (11) that for $N > N_3$ and $1 \le j \le y/2$

$$Z(N,\alpha,j/y) < 20\frac{j}{y}\frac{N}{\log N} + c_6\frac{j}{y}\frac{N}{\log N} < c_7\frac{j}{y}\frac{N}{\log N}.$$

Thus from (10),

$$\begin{aligned} (12)\quad S(\alpha) &< y\sum_{j=2}^{[y/2]}\frac{1}{(j-1)j}\cdot c_7\frac{j}{y}\frac{N}{\log N} + 5\frac{N}{\log N} \\ &< 5\frac{N}{\log N} + c_7\frac{N}{\log N}\sum_{j=2}^{[y/2]}\frac{1}{j-1} < c_8\frac{N\log y}{\log N} \quad (\text{for } \alpha\in T_1). \end{aligned}$$

6. Major arcs

Put

$$S(\alpha, b) = \sum_{\substack{p \le N \\ (p,b)=1}} \min(y, \|p\alpha\|^{-1}).$$

In view of (3) for $N > N_5$ we have for any real number α and positive integer $b \le N$, that

$$
\begin{aligned}
(13) \qquad S(\alpha) &= \sum_{p \le N} \min(y, \|p\alpha\|^{-1}) \\
&\le \sum_{p|b} y + \sum_{\substack{p \le N \\ (p,b)=1}} \min(y, \|p\alpha\|^{-1}) \\
&= y \sum_{p|b} 1 + S(\alpha, b) < c_9 y \log b + S(\alpha, b) \\
&\le c_9 y \log N + S(\alpha, b) < \frac{N}{\log N} + S(\alpha, b).
\end{aligned}
$$

Assume first that $\alpha \in T_2$. Notice that we may assume that $b > 1$, since if $b = 1$ then $|\beta| \le 1/2N$ and consequently α is not in $(1/N, 1 - 1/N)$. Further since $b \ne 1$ we may assume that $a \ne 0$.

For $(p, b) = 1$ we have

$$
\begin{aligned}
\|p\alpha\| &= \left\| p\left(\frac{a}{b} + \beta\right)\right\| \ge \left\|\frac{ap}{b}\right\| - p|\beta| \ge \left\|\frac{ap}{b}\right\| - N\frac{1}{2bN} \\
&= \left\|\frac{ap}{b}\right\| - \frac{1}{2b} \ge \frac{1}{2}\left\|\frac{ap}{b}\right\|
\end{aligned}
$$

since $b > 1$ and $(ap, b) = 1$. Thus

$$
\begin{aligned}
S(\alpha, b) &\le \sum_{\substack{p \le N \\ (p,b)=1}} \min(y, 2\|ap/b\|^{-1}) \\
&= \sum_{\substack{0 \le h < b \\ (h,b)=1}} \sum_{\substack{p \le N \\ ap \equiv h \pmod b}} 2\|h/b\|^{-1} \\
&\le 2\left(\max_{\substack{0<l<b \\ (l,b)=1}} \pi(N, b, l)\right) \sum_{\substack{0 \le h < b \\ (h,b)=1}} \|h/b\|^{-1}.
\end{aligned}
$$

By Lemma 5, (3) and $b \le y$, we have

$$S(\alpha, b) \le \frac{4N}{\phi(b)\log(N/b)}\, 2 \sum_{\substack{1 \le h \le b/2 \\ (h,b)=1}} \frac{b}{h} \le \frac{11N}{\phi(b)\log N} \sum_{\substack{1 \le h \le b/2 \\ (h,b)=1}} b/h$$

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and so, by Lemma 2,

$$S(\alpha, b) \le c_{10} \frac{N \log b}{\log N} \le c_{10} \frac{N \log y}{\log N} \qquad (\text{for } \alpha \in T_2) \tag{14}$$

as required.

We shall assume next that $\alpha \in T_3$, whence

$$\frac{1}{2bN} < |\beta| \le \frac{1}{bQ}. \tag{15}$$

Put $L = 1/2b|\beta|$. It follows from (15) that

$$\frac{Q}{2} \le L < N. \tag{16}$$

We have

$$\begin{aligned} S(\alpha, b) &= \sum_{\substack{p \le N \\ (p,b)=1}} \min(y, \|p\alpha\|^{-1}) \\ &\le \sum_{j=1}^{[N/L]+1} \sum_{\substack{(j-1)L<p\le jL \\ (p,b)=1}} \min(y, \|p\alpha\|^{-1}) \\ &= \sum_{j=1}^{[N/L]+1} \sum_{k=1}^{2y} \sum_{\substack{(j-1)L<p\le jL \\ (p,b)=1 \\ \frac{k-1}{2y} \le \{p\alpha\} < \frac{k}{2y}}} \min(y, \|p\alpha\|^{-1}). \end{aligned}$$

Since $(k-1)/(2y) \le \{p\alpha\} < k/(2y)$ implies that

$$\frac{1}{\|p\alpha\|} \le \frac{1}{\left\| \frac{k-1}{2y} \right\|} + \frac{1}{\left\| \frac{k}{2y} \right\|}$$

where, as before, we write $a \le 1/0 + b$ and $1/0 \le 1/0 + a$ for all real numbers a and b, we have

$$\begin{aligned} S(\alpha, b) \le \sum_{j=1}^{[N/L]+1} \sum_{k=1}^{2y} \Bigg(\min\left(y, \left\| \frac{k-1}{2y} \right\|^{-1}\right) \\ + \min\left(y, \left\| \frac{k}{2y} \right\|^{-1}\right) \Bigg) \sum_{\substack{(j-1)L<p\le jL \\ \frac{k-1}{2y} \le \{p\alpha\} < \frac{k}{2y} \\ (p,b)=1}} 1. \end{aligned} \tag{17}$$

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If p and p_0 are primes with $(j-1)L < p \le jL$, $(k-1)/(2y) \le \{p\alpha\} < (k/2y)$ and $(j-1)L < p_0 \le jL$, $(k-1)/(2y) \le \{p_0\alpha\} < k/(2y)$ then

$$\frac{1}{2y} > \|(p-p_0)\alpha\| = \left\|(p-p_0)\left(\frac{a}{b}+\beta\right)\right\| \ge \left\|(p-p_0)\frac{a}{b}\right\| - |p-p_0||\beta|$$
$$> \left\|(p-p_0)\frac{a}{b}\right\| - L|\beta| = \left\|(p-p_0)\frac{a}{b}\right\| - \frac{1}{2b}.$$

Thus $\|(p-p_0)a/b\| < 1/2y + 1/(2b) \le 1/b$, whence $p \equiv p_0 \pmod b$. Therefore

$$\frac{1}{2y} > \|p\alpha - p_0\alpha\| = \left\|(p-p_0)\frac{a}{b} + (p-p_0)\beta\right\| = \|(p-p_0)\beta\|. \tag{18}$$

Since $|(p-p_0)\beta| < L|\beta| = 1/(2b) \le 1/2$, it follows from (18) that $1/(2y) > |p-p_0||\beta|$, and hence

$$|p-p_0| < \frac{1}{2|\beta|y}.$$

Thus, either there are no primes p with $(j-1)L < p \le jL$, $(p,b) = 1$ and $(k-1)/(2y) \le \{p\alpha\} < k/(2y)$, or for some p_0 we have

$$\sum_{\substack{(j-1)L<p\le jL\\ (p,b)=1\\ \frac{k-1}{2y}\le\{p\alpha\}<\frac{k}{2y}}} 1 \le \sum_{\substack{p_0-\frac{1}{2|\beta|y}<p<p_0+\frac{1}{2|\beta|y}\\ p\equiv p_0 \pmod b}} 1 \tag{19}$$
$$\le \pi\left(p_0+\frac{1}{2|\beta|y}, b, p_0\right) - \pi\left(p_0-\frac{1}{2|\beta|y}, b, p_0\right).$$

By (15), $1/(|\beta|y) \ge bQ/y$. Thus, for $N > N_6$, the right-hand side of inequality (19) is, by (3) and Lemma 5, at most

$$\frac{\frac{2}{|\beta|y}}{\phi(b)\log\frac{1}{|\beta|yb}} \le \frac{4bL}{y\phi(b)\log\frac{Q}{y}} \le c_{11}\frac{bL}{y\phi(b)\log N}.$$

In view of (16) it now follows from (17), that

$$S(\alpha,b) \le \sum_{j=1}^{[N/L]+1}\sum_{k=1}^{2y}\left(\min\left(y, \left\|\frac{k-1}{2y}\right\|^{-1}\right) + \min\left(y, \left\|\frac{k}{2y}\right\|^{-1}\right)\right) c_{11}\frac{bL}{y\phi(b)\log N}$$
$$\le \left(\left[\frac{N}{L}\right]+1\right) c_{12}\frac{bL}{y\phi(b)\log N}\sum_{k=0}^{y}\min\left(y, \left\|\frac{k}{2y}\right\|^{-1}\right)$$
$$\le c_{13}\frac{Nb}{y\phi(b)\log N}\left(y + \sum_{k=1}^{y}\frac{2y}{k}\right) \le c_{14}\frac{Nb\log y}{\phi(b)\log N}.$$

Thus, by Lemma 1, $S(\alpha, b) \le c_{15} N \log y \log\log b / \log N$. Since $b \le y$ we have

$$(20) \qquad S(\alpha, b) \le c_{15} \frac{N \log y \log\log y}{\log N} \qquad (\text{for } \alpha \in T_3)$$

provided that $N > N_7$.

Finally we assume that $\alpha \in T_4$. Put $M = \min(N, 1/(|\beta| y))$. Then

$$(21) \qquad \begin{aligned} S(\alpha, b) &= \sum_{\substack{p \le N \\ (p,b)=1}} \min(y, \|p\alpha\|^{-1}) \\ &\le \sum_{j=1}^{[N/M]+1} \sum_{\substack{(j-1)M < p \le jM \\ (p,b)=1}} \min(y, \|p\alpha\|^{-1}). \end{aligned}$$

Now if $\|p\alpha\|^{-1} < y$ with $(j-1)M < p \le jM$ and n is defined by $p \equiv n \pmod b$ with $jM - b < n \le jM$ then

$$\|p\alpha\| = \left\| p\left(\frac{a}{b} + \beta\right) \right\| = \left\| \frac{an}{b} + n\beta + (p-n)\beta \right\| \ge \left\| \frac{1}{b}(an + nb\beta) \right\| - |p-n||\beta|.$$

Since $b > y$ it follows from (6) that $1/(|\beta| y) > Q$, and hence, for $N > N_8$, that $M > Q$. Consequently $b < M$ and so $|p-n| < M$ and $|p-n||\beta| < M|\beta| \le 1/y < \|p\alpha\|$. Thus $2\|p\alpha\| \ge \|(1/b)(an + nb\beta)\|$, and hence

$$\begin{aligned} \min(y, \|p\alpha\|^{-1}) &\le \min\left(y, 2\left\| \frac{1}{b}(an + nb\beta) \right\|^{-1} \right) \\ &\le 2 \min\left(y, \left\| \frac{1}{b}(an + nb\beta) \right\|^{-1} \right). \end{aligned}$$

Therefore, by (21),

(22)

$$S(\alpha, b) \le \sum_{j=1}^{[N/M]+1} \sum_{\substack{jM - b < n \le jM \\ (n,b)=1}} 2 \min\left(y, \left\| \frac{1}{b}(an + nb\beta) \right\|^{-1} \right) \sum_{\substack{(j-1)M < p \le jM \\ p \equiv n \pmod b}} 1.$$

For $N > N_9$ we have, by (3) and (8), that

$$(23) \qquad \frac{M}{b} > \frac{Q}{b} > \frac{N}{P^2} = \frac{N}{y^4 (\log N)^{28}} > N^{2\varepsilon},$$

whence, from Lemma 5,

$$(24) \qquad \sum_{\substack{(j-1)M < p \le jM \\ p \equiv n \pmod b}} 1 < \frac{2M}{\phi(b) \log \frac{M}{b}} \le \frac{M}{\varepsilon \phi(b) \log N}.$$

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Combining (22) and (24), we obtain

$$S(\alpha,b) \le \frac{M}{\varepsilon\phi(b)\log N}\sum_{j=1}^{[N/M]+1}\sum_{\substack{jM-b<n\le jM\\(n,b)=1}}\min\left(y, \left\|\frac{1}{b}(an+nb\beta)\right\|^{-1}\right).$$

We may estimate the inner sum above by means of Lemma 3 with $h = jM - b + 1$, $q = b$ and $\rho(n) = nb\beta$. Then by (6) and (23),

$$\lambda = \max_{\substack{jM-b<n\le jM\\(n,b)=1}} nb\beta - \min_{\substack{jM-b<n\le jM\\(n,b)=1}} nb\beta \le b^2|\beta| < \frac{b}{Q} < N^{-2\varepsilon} < 1$$

for $N > N_{10}$. Thus

$$S(\alpha,b) \le \frac{M}{\varepsilon\phi(b)\log N}\left(\left[\frac{N}{M}\right]+1\right)c_4\phi(b)\log y,$$

and, since $M \le N$,

$$S(\alpha,b) \le c_{16}\frac{N\log y}{\varepsilon\log N} \qquad (\text{for } \alpha\in T_4). \tag{25}$$

If $y < N^{1/10}(\log N)^{-7}$ then we may replace 2ε in (23) by $1/2$ and consequently ε in (25) by 1. On the other hand if $y \ge N^{1/10}(\log N)^{-7}$ then certainly $1/\varepsilon < \log\log y$ for $N > N_{11}$. Thus in either case, we obtain from (25) that

$$S(\alpha,b) \le c_{17}\frac{N\log y\log\log y}{\log N}(\text{for } N > N_{11}, \alpha\in T_4). \tag{26}$$

Thus (9) follows from (12), (13), (14), (20) and (26), and this completes the proof of the theorem.

7. Addendum

We would like to thank the referee for his valuable suggestions and remarks. In particular, the referee drew our attention to reference [4] which allowed us to improve our original exponent of $\frac{1}{5}$ in (3) to $\frac{1}{4}$.

Further, the referee remarked that our estimate for $S(\alpha)$ is essentially best possible for a special choice of y. In fact, by means of a slight generalization of the referee's idea, we shall show that there exist effectively computable positive constants c_{18} and c_{19} such that if $N > c_{18}$ then for all real numbers y with $3 \le y \le N^{1/4}$ we have

$$\max_{\frac{1}{N}\le\alpha\le 1-\frac{1}{N}}\sum_{p\le N}\min(y, \|p\alpha\|^{-1}) > c_{19}\frac{N\log y\log\log y}{\log N}. \tag{26}$$

Therefore our main theorem gives the correct order of magnitude for $S(\alpha)$.

We shall now establish (26). Define the integer x by

$$\prod_{p \leq x} p \leq y^{2/3} < \prod_{p \leq x+1} p$$

and put $b = \prod_{p \leq x} p$. Note that $x \geq 2$ since $y \geq 3$. We have

$$x > c_{20} \log y \tag{27}$$

and

$$\phi(b) = b \prod_{p \leq x} \left(1 - \frac{1}{p}\right) < c_{21} \frac{b}{\log x}. \tag{28}$$

Thus

$$\sum_{\substack{1 \leq a \leq b \\ (a,b)=1}} S\left(\frac{a}{b} - \frac{1}{bN}\right) = \sum_{\substack{1 \leq a \leq b \\ (a,b)=1}} \sum_{p \leq N} \min\left(y, \left\| \frac{ap}{b} - \frac{p}{bN} \right\|^{-1}\right)$$
$$= \sum_{p \leq N} \sum_{\substack{1 \leq a \leq b \\ (a,b)=1}} \min\left(y, \left\| \frac{ap}{b} - \frac{p}{bN} \right\|^{-1}\right).$$

Since $\frac{N}{4} > N^{1/4} > y^{2/3} \geq b$ for $N > 8$,

$$\sum_{\substack{1 \leq a \leq b \\ (a,b)=1}} S\left(\frac{a}{b} - \frac{1}{bN}\right) \geq \sum_{\frac{N}{4} \leq p \leq N} \sum_{\substack{1 \leq h \leq b \\ (h,b)=1}} \min\left(y, \left\| \frac{h}{b} - \frac{p}{bN} \right\|^{-1}\right)$$
$$\geq \sum_{\frac{N}{4} < p \leq N} \min\left(y, \left\| \frac{1}{b} - \frac{p}{bN} \right\|^{-1}\right)$$
$$\geq \sum_{i=1}^{[3y/4b]} \sum_{N - \frac{ibN}{y} < p \leq N - (i-1)\frac{bN}{y}} \min\left(y, \left\| \frac{1}{b} - \frac{1}{b} \frac{(N - ibN/y)}{N} \right\|^{-1}\right)$$
$$\geq \sum_{i=1}^{[3y/4b]} \min\left(y, \left\| \frac{i}{y} \right\|^{-1}\right) (\pi(N - (i-1)bN/y) - \pi(N - ibN/y)).$$

Since $\pi(x+z) - \pi(x) > c_{22} z / \log x$, for $z > x^{3/4}$ and x sufficiently large, we find that for $N > c_{23}$,

$$\pi\left(N - (i-1)\frac{bN}{y}\right) - \pi(N - ibN/y) > c_{24} \frac{bN}{y \log N}.$$

Thus

$$\sum_{\substack{1 \leq a \leq b \\ (a,b)=1}} S\left(\frac{a}{b} - \frac{1}{bN}\right) > \sum_{i=1}^{[3y/4b]} \frac{c_{24} bN}{i \log N} > c_{25} \frac{bN \log y}{\log N}.$$

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Therefore

$$\max_{\substack{1\le a\le b\\(a,b)=1}} S\left(\frac{a}{b}-\frac{1}{bN}\right) > c_{25}\frac{b}{\phi(b)}\frac{N\log y}{\log N}$$

and so, by (27) and (28),

$$\max_{\substack{1\le a\le b\\(a,b)=1}} S\left(\frac{a}{b}-\frac{1}{bN}\right) > c_{26}\frac{N\log y\log\log y}{\log N},$$

which proves (26).

Finally, the L^1 mean of $S(\alpha)$ is asymptotically $2(1+\log(y/2))\pi(N)$ and the referee asked whether $S(\alpha)$ has this size outside of a small set. We remark that by our proof, we have

(29) $$\max_{\substack{\alpha\in[0,1]\\ \alpha\notin T_3\\ \|\alpha\|>1/N}} S(\alpha) < c_{27}\log y\ \pi(N).$$

Further the measure of $T_2\cup T_3$ is, by (6), at most

$$\sum_{b=1}^{[y]}\frac{\phi(b)}{bQ}\le\frac{y}{Q}=\frac{y^3(\log N)^{14}}{N}.$$

Thus (29) holds for all α in $[0,1]$ except for a set of measure at most $(2+y^3(\log N)^{14})/N$. In fact we can be more precise if we make the minor arcs slightly smaller. For example, put $P_1=y^2(\log N)^{20}$ and $Q_1=N/P_1$. It is possible to show that $S(\alpha)$ is $2(1+\log(y/2))\pi(N)(1+o(1))$ for all α in $(1/N,1-1/N)$ for which there exist coprime positive integers a and b with $|\alpha-\frac{a}{b}|<b^{-2}$ and $P_1\le b\le Q_1$. Notice that the complement of this set in $(0,1)$ has measure at most $2/N+P_1/Q_1=(2+y^4(\log N)^{40})/N$. To prove this requires a more careful analysis of $S(\alpha)$ on the minor arcs. In particular we must replace the function $\psi(x,\delta)$ with its finite Fourier series by a function that is a better approximation to the function f where

$$f(x)=\begin{cases}1 & \text{for } \|x\|\le\delta,\\ 0 & \text{for } \delta<\|x\|\le 1/2.\end{cases}$$

Such a function can be found by an appropriate truncation of the Fourier series expansion of f.

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