A THEORY OF INTEGRATION

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1. Introduction. The usual development of the Lebesgue integral starts with a measure that may have been derived from some simpler set function, its associated class of measurable sets, the corresponding set of measurable functions, and operations which ultimately define the integral of any given function of this class, except for certain ones which are unbounded above and below. Here we propose to define a process of integration with respect to a set function more general than a measure. This process allows us to integrate virtually all functions real-valued on our space. The integrals thus obtained are all completely additive on a certain completely additive class of sets. Under rather mild hypotheses, we are able to delineate this class of sets explicitly.

Certain questions of additivity with respect to the integrands are taken up in §5. This study is continued in §6. The end result is a demonstration that, under our general hypotheses, our integrals possess all the additivity and convergence properties of Lebesgue integrals, and in addition Lebesgue integrals turn out to represent a special case of our considerations.

2. Some basic preliminaries. We begin by stating a number of definitions and conventions which will be used throughout the paper.

If $\mathbf{F}$ is any family of sets, we agree to let $\cup \mathbf{F}$ denote the *union* of $\mathbf{F}$; that is, the set of points belonging to at least one member of $\mathbf{F}$. We say that a family of sets $\mathbf{F}$ *covers* a set A if and only if $A \subset \cup \mathbf{F}$. We agree to let $\cap \mathbf{F}$ denote the *intersection* of a family of sets $\mathbf{F}$; that is, the set of points belonging to each member of $\mathbf{F}$. For finite or infinite sequences of sets we sometimes use the notations $\cup_n A_n$ and $\cap_n A_n$ to denote, respectively, the union and intersection. We understand tacitly that the index n is to run over a finite set of positive integers or all of them according to whether the sequence is finite or infinite. We sometimes follow a similar convention in summing numerical sequences. If A and B are sets, then by $A - B$ we mean the set of those points that are in A but not in B. We agree to denote by $\langle x \rangle$ the set whose only member is x. We further agree to let 0 denote the null set as well as zero.

We allow real-valued functions to take on the values $+\infty$ or $-\infty$. We agree that $0\cdot(+\infty) = 0\cdot(-\infty) = 0$, $c\cdot(+\infty) = +\infty$, $c\cdot(-\infty) = -\infty$ if $c > 0$; the signs are reversed if $c < 0$. Also $c + \infty = +\infty$, if c is a real number or $c = +\infty$, $c - \infty = -\infty$ if c is a real number or $c = -\infty$.

If A is a set of real numbers, we denote by $\sup A$ and $\inf A$ the supremum

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and infimum of A, respectively. We keep in mind that $\sup 0 = -\infty$, $\inf 0 = +\infty$. If F is a real-valued function and A is a subset of its domain, we use the notations $\sup_{x\in A} F(x)$ and $\inf_{x\in A} F(x)$ to denote, respectively, the supremum and infimum of the values assumed by $F(x)$ for all x in A. We note that empty sums are zero.

We say that ϕ is an *outer measure on* S if and only if the domain of ϕ is the set of all subsets of S and

$$0 \leqslant \phi(A) \leqslant \sum_{\beta\in\mathbf{F}} \phi(\beta)$$

whenever $\mathbf{F}$ is a finite or countably infinite family and $A \subset \bigcup \mathbf{F} \subset S$. This is equivalent to the usual definition, namely $\phi(0) = 0$; $\phi(A) \leqslant \phi(B)$ whenever $A \subset B \subset S$; and

$$\phi(\bigcup \mathbf{F}) \leqslant \sum_{\beta\in\mathbf{F}} \phi(\beta)$$

whenever $\mathbf{F}$ is a finite or countably infinite class of subsets of S.

We say that the set $A \subset S$ is ϕ*-measurable* if and only if ϕ is an outer measure on S and $\phi(E) = \phi(A \cap E) + \phi(E - A)$ whenever $E \subset S$.

A family of sets $\mathbf{F}$ is said to be a *ring* if and only if $E \cup F \in \mathbf{F}$ and $(E - F) \in \mathbf{F}$ whenever E and F belong to $\mathbf{F}$. It follows readily that if $\mathbf{F}$ is a ring, then $E \cap F \in \mathbf{F}$ whenever E and F belong to $\mathbf{F}$; also, $\mathbf{F}$ is closed with respect to finite unions of its members. In case a ring is closed with respect to countably infinite unions of its members, it is called a σ*-ring*. If $\mathbf{F}$ is a σ*-ring* of subsets of S, and $S \in \mathbf{F}$, then $\mathbf{F}$ is called a σ*-algebra* of sets. Complements in S of members of such a σ-algebra are again members of the σ-algebra. It is well known that if ϕ is any outer measure on a set S, then the class of all ϕ-measureable subsets of S is a σ-algebra which contains all sets E for which $\phi(E) = 0$ (in particular the null set itself) and their complements (in particular S) (cf. **1**, pp. 89–90).

If $\mathbf{F}_0$ is any non-empty family of subsets of a given set S and g is any non-negative function whose domain is $\mathbf{F}_0$, then for any set $A \subset S$, we let $\mathfrak{M}(A)$ denote the family of all finite or countably infinite subfamilies $\mathbf{G}$ of $\mathbf{F}_0$ that cover A ($\mathfrak{M}(A)$ may happen to be empty). We so define the function $\bar{g}$ that

$$\bar{g}(A) = \inf_{\mathbf{G}\in\mathfrak{M}(A)} \sum_{\beta\in\mathbf{G}} g(\beta).$$

Owing to our assumptions on g and our convention on empty sums, it follows that $\bar{g}$ is a non-negative real-valued function on the class of all subsets of S. It is well known that $\bar{g}$ is an outer measure on S (cf. **1**, pp. 90–91). Without further assumptions about g and $\mathbf{F}_0$, it is not possible to describe the associated class of $\bar{g}$-measurable sets in greater detail than above. However, significant facts emerge under the following relatively mild hypotheses.

2.1. THEOREM. *Let $\mathbf{F}_0$ denote such a non-empty family of subsets of a set S that $\alpha \cap \beta \in \mathbf{F}_0$ and $(\alpha - \beta) \in \mathbf{F}_0$ whenever α and β belong to $\mathbf{F}_0$. (The condition*

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$(\alpha - \beta) \in \mathbf{F}_0$ alone implies $\alpha \cap \beta \in \mathbf{F}_0$.) Also, let g denote such a non-negative function on $\mathbf{F}_0$ that

$$g(\alpha) \geqslant g(\alpha \cap \beta) + g(\alpha - \beta)$$

whenever α and β belong to $\mathbf{F}_0$. Then $\mathbf{F}_0$ is contained in the class of $\bar{g}$-measurable sets.

Proof. We consider an arbitrary set $\beta \in \mathbf{F}_0$, and an arbitrary subset A of S. if $\mathbf{H}$ is an arbitrary member of $\mathfrak{M}(A)$, we define the families

$$\mathbf{G} = \{\gamma : \gamma = \alpha \cap \beta \text{ for some } \alpha \in \mathbf{H}\},$$
$$\mathbf{G}' = \{\gamma' : \gamma' = \alpha - \beta \text{ for some } \alpha \in \mathbf{H}\}.$$

Evidently $\mathbf{G}$ and $\mathbf{G}'$ are finite or countably infinite subfamilies of $\mathbf{F}_0$ by our hypotheses above. We see that $\mathbf{G}$ covers $A \cap \beta$ and $\mathbf{G}'$ covers $A - \beta$; therefore

$$\sum_{\alpha \in \mathbf{H}} g(\alpha) \geqslant \sum_{\alpha \in \mathbf{H}} g(\alpha \cap \beta) + \sum_{\alpha \in \mathbf{H}} g(\alpha - \beta)$$
$$\geqslant \sum_{\gamma \in \mathbf{G}} g(\gamma) + \sum_{\gamma' \in \mathbf{G}'} g(\gamma') \geqslant \bar{g}(A \cap \beta) + \bar{g}(A - \beta).$$

From the arbitrary nature of $\mathbf{H} \in \mathfrak{M}(A)$ in this last relation, we infer that $\bar{g}(A) \geqslant \bar{g}(A \cap \beta) + \bar{g}(A - \beta)$. Since the reverse inequality holds because $\bar{g}$ is an outer measure on S, it follows that β is $\bar{g}$-measurable.

2.2. LEMMA. *If g and $\mathbf{F}_0$ satisfy the conditions of Theorem* 2.1, *and $\mathbf{H}$ is any finite or countably infinite subfamily of $\mathbf{F}_0$, then there exists a finite or countably infinite disjoint family $\mathbf{H}' \subset \mathbf{F}_0$, each of whose members is contained in some member of $\mathbf{H}$, such that $\cup \mathbf{H}' = \cup \mathbf{H}$ and*

$$\sum_{\alpha \in \mathbf{H}} g(\alpha) \geqslant \sum_{\beta \in \mathbf{H}'} g(\beta).$$

Furthermore, if $\mathbf{H}$ is finite, so is $\mathbf{H}'$.

Proof. There is nothing to prove if $\mathbf{H} = 0$ or if $\mathbf{H}$ consists of a single element, so we may assume it has at least two elements. We now arrange $\mathbf{H}$ in the form of a non-repetitive finite or infinite sequence α. We define a corresponding sequence β such that

$$\beta_1 = \alpha_1, \beta_n = \alpha_n - (\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_{n-1})$$

for each positive integer $m \geqslant 2$, not exceeding the number of members in $\mathbf{H}$. We see that each such set $\beta_n \in \mathbf{F}_0$ owing to our hypotheses. We take $\mathbf{H}'$ to be the range of the sequence β. From our hypotheses it follows that $\beta_n \subset \alpha_n$, $g(\beta_n) \leqslant g(\alpha_n)$ for any n in the domain of the sequence α; hence

$$\sum_{\beta \in \mathbf{H}'} g(\beta) \leqslant \sum_{\alpha \in \mathbf{H}} g(\alpha) \; ; \qquad \cup \mathbf{H}' \subset \cup \mathbf{H}$$

It is apparent that $\mathbf{H}'$ is disjoint and that each point $x \in \cup \mathbf{H}$ belongs to exactly one member of $\mathbf{H}'$, whence $\cup \mathbf{H} \subset \cup \mathbf{H}'$, and so finally $\cup \mathbf{H}' = \cup \mathbf{H}$. The fact that $\mathbf{H}'$ is finite if $\mathbf{H}$ is finite is apparent from the above.

2.3. LEMMA. *If g and $\mathbf{F}_0$ satisfy the conditions of Theorem* 2.1, $\gamma \in \mathbf{F}_0$, $\mathbf{H}$ *is a finite or countably infinite disjoint subfamily of* $\mathbf{F}_0$, *and* $\mathbf{G}$ *is any finite or countably infinite subfamily of* $\mathbf{F}_0$, *then*

$$g(\gamma) \geqslant \sum_{\alpha \in \mathbf{H}} g(\gamma \cap \alpha) ;$$

$$\sum_{\beta \in \mathbf{G}} g(\beta) \geqslant \sum_{\beta \in \mathbf{G}} \Big(\sum_{\alpha \in \mathbf{H}} g(\beta \cap \alpha) \Big).$$

Proof. We find it convenient to represent $\mathbf{H}$ in the form of a non-repetitive sequence α, finite or infinite. We may assume $\mathbf{H}$ has at least two elements or there is nothing to prove. Then

$$g(\gamma) \geqslant g(\gamma \cap \alpha_1) + g(\gamma - \alpha_1) \geqslant g(\gamma \cap \alpha_1) + g((\gamma - \alpha_1) \cap \alpha_2) + g((\gamma - \alpha_1) - \alpha_2).$$

Since $\alpha_1 \cap \alpha_2 = 0$ by hypothesis, then $(\gamma - \alpha_1) \cap \alpha_2 = \gamma \cap \alpha_2$ and the above may be written $g(\gamma) \geqslant g(\gamma \cap \alpha_1) + g(\gamma \cap \alpha_2) + g((\gamma - (\alpha_1 \cup \alpha_2))$. This relation can obviously be extended inductively so that for any positive integer n not exceeding the number of members of $\mathbf{H}$,

$$g(\gamma) \geqslant \sum_{i=1}^{n} g(\gamma \cap \alpha_i) + g(\gamma - (\alpha_1 \cup \ldots \cup \alpha_n)),$$

whence follows the first statement of the lemma. The second is an obvious consequence of the first.

2.4. THEOREM. *If g_1 and g_2 are functions satisfying the conditions of Theorem* 2.1, $A \subset S$, *and* $h = g_1 + g_2$, *then* $\bar{h}(A) = \bar{g}_1(A) + \bar{g}_2(A)$.

Proof. We assume that there exists some family $\mathbf{H} \in \mathfrak{M}(A)$, otherwise $\bar{g}_1(A)$, $\bar{g}_2(A)$, and $\bar{h}(A)$ would all be infinite, and the statement of the theorem would be obviously true. Given $\epsilon > 0$, then, there exist finite or countably infinite subfamilies $\mathbf{H}_1$ and $\mathbf{H}_2$ of $\mathbf{F}_0$ such that

$$\sum_{\alpha \in \mathbf{H}_1} g_1(\alpha) \leqslant \bar{g}_1(A) + \epsilon/2, \qquad \sum_{\beta \in \mathbf{H}_2} g_2(\beta) \leqslant \bar{g}_2(A) + \epsilon/2. \tag{1}$$

By virtue of Lemma 2.2, we may also assume without loss of generality that $\mathbf{H}_1$ and $\mathbf{H}_2$ are disjoint. Now we construct the family

$$\mathbf{H} = \{\gamma : \gamma = \alpha \cap \beta \text{ for some } \alpha \in \mathbf{H}_1 \text{ and some } \beta \in \mathbf{H}_2\}.$$

Using Lemma 2.3, we see that

$$\begin{aligned} \sum_{\beta \in \mathbf{H}_2} g_2(\beta) &\geqslant \sum_{\beta \in \mathbf{H}_2} \Big(\sum_{\alpha \in \mathbf{H}_1} g_2(\alpha \cap \beta) \Big) \geqslant \sum_{\gamma \in \mathbf{H}} g_2(\gamma), \\ \sum_{\alpha \in \mathbf{H}_1} g_1(\alpha) &\geqslant \sum_{\alpha \in \mathbf{H}_1} \Big(\sum_{\beta \in \mathbf{H}_2} g_1(\alpha \cap \beta) \Big) \geqslant \sum_{\gamma \in \mathbf{H}} g_1(\gamma). \end{aligned} \tag{2}$$

Since $\mathbf{H}$ is clearly a finite or countably infinite disjointed subfamily of $\mathbf{F}_0$ covering A, we obtain from (1) and (2) the relations

$$(3) \qquad \bar{h}(A) \leqslant \sum_{\gamma \in \mathbf{H}} h(\gamma) = \sum_{\gamma \in \mathbf{H}} (g_1(\gamma) + g_2(\gamma)) \leqslant \bar{g}_1(A) + \bar{g}_2(A) + \epsilon;$$

thus $\bar{h}(A) \leqslant \bar{g}_1(A) + \bar{g}_2(A)$.

On the other hand, if $\mathbf{G}$ is any finite or countably infinite subfamily of $\mathbf{F}_0$ covering A, then

$$\bar{g}_1(A) + \bar{g}_2(A) \leqslant \sum_{\gamma \in \mathbf{G}} g_1(\gamma) + \sum_{\gamma \in \mathbf{G}} \bar{g}_2(\gamma) = \sum_{\gamma \in \mathbf{G}} h(\gamma) \ ;$$

thus $\bar{g}_1(A) + \bar{g}_2(A) \leqslant \bar{h}(A)$ and the proof is complete.

2.5. COROLLARY. *If, in Theorem* 2.1, *h is defined on $\mathbf{F}_0$ in such a way that $h(\beta) \leqslant g_1(\beta) + g_2(\beta)$ for each $\beta \in \mathbf{F}_0$, then $\bar{h}(A) \leqslant \bar{g}_1(A) + \bar{g}_2(A)$ for each set $A \subset S$.*

Proof. The steps leading to (3) in Theorem 2.4 are still valid under the above hypotheses; and this yields the desired conclusion.

3. The theory of integration. For the remainder of the paper, S will denote a fixed set, $\mathbf{F}_0$ a fixed non-empty family of subsets of S. We will assume that $\mathfrak{M}(S) \neq 0$. Also, μ will denote a non-negative finite-valued function whose domain is $\mathbf{F}_0$. In later sections we will impose additional conditions on $\mathbf{F}_0$ and μ. We shall denote by $\mathbf{S}(\mathbf{F}_0)$ the smallest σ-algebra of sets containing $\mathbf{F}_0$. Also, for any set $A \subset S$, we shall denote by $\tilde{A}$ the complement of A in S; that is, $\tilde{A} = S - A$.

Next we consider an arbitrary function f non-negative and real-valued on S. We so define the associated non-negative function f^* on $\mathbf{F}_0$ that

$$f^*(\beta) = \mu(\beta) \cdot \sup_{x \in \beta} f(x) \text{ if } 0 \neq \beta \in \mathbf{F}_0; f^*(0) = 0 \text{ if } 0 \in \mathbf{F}_0.$$

Applying the procedure of §2 to f^* yields an outer measure $\bar{f}^*$.

In case f is bounded on S, we agree to give to $\bar{f}^*$ the special name $\int f$, and we denote the value $\bar{f}^*$ (A) by $\int_A f$ for any set $A \subset S$. Thus for each bounded non-negative function f on S, $\int f$ is an outer measure on S. If f is non-negative on S but not bounded above, we so define $f^{(n)}$ for each positive integer n that $f^{(n)}(x) = f(x)$ if $f(x) < n$, $f^{(n)}(x) = n$ if $f(x) \geqslant n$; and we agree to define $\int_A f = \lim_n \int_A f^{(n)}$ for each set $A \subset S$. Since $f^{(n)} \leqslant f^{(n+1)}$ holds for each positive integer n throughout S, it follows that the limit in question exists. Furthermore, it is easily seen that $\int f$ is again an outer measure on S.

If f is any function real-valued on S, we so define f_+ and f_- that

$$f_+(x) = f(x) \text{ if } f(x) > 0; \qquad f_+(x) = 0 \text{ if } f(x) \leqslant 0;$$
$$f_-(x) = -f(x) \text{ if } f(x) < 0; \qquad f_-(x) = 0 \text{ if } f(x) \geqslant 0.$$

Clearly f_+ and f_- are non-negative on S and $f = f_+ - f_-$; also, $(-f)_+ = f_-$ and $(-f)_- = f_+$. For such a function f, we agree to define $\int_A f = \int_A f_+ - \int_A f_-$ for each set $A \subset S$. Since $f = f_+$ and $f_- = 0$ whenever f is a non-negative function on S, and since our assumption that $\mathfrak{M}(S) \neq 0$ guarantees that the integral of the identically zero function is zero over any subset of S, we see that the definition just given reduces to the previous one in case f is non-negative throughout S, and so is compatible with it. We shall say that f is *integrable* if at least one of the integrals $\int_S f_+$ and $\int_S f_-$ is finite; it is *summable* if both are finite.

Since $\int f$ is an outer measure on S whenever f is a function non-negative on S, each such integral determines its own class of $\int f$-measurable subsets of S. We let $\mathbf{S}$ denote the intersection of all such classes associated with all such functions. Clearly $\mathbf{S}$ is a σ-algebra of subsets of S on which every integral of the type just mentioned is completely additive; moreover, since the integral of any function real-valued on S is representable as the difference between two integrals of the type just considered, we see that the integral of any integrable function is completely additive on $\mathbf{S}$. Of course, without further information about μ, $\mathbf{F}_0$, and S, all we can say about $\mathbf{S}$ is that 0 and S are members.

We denote by K_A the characteristic function of any given set $A \subset S$; that is, $K_A(x) = 1$ if $x \in A$; $K_A(x) = 0$ if $x \in \tilde{A}$. The particular function K_S plays a special role among the functions integrable on S, and we give the integral $\int K_S$ the special name $\bar{\mu}$, in keeping with the notation of §2.

3.1. THEOREM. (i) *If f is integrable on S and c is a real number, then $\int_A cf = c\int_A f$ for each set $A \subset S$.*

(ii) *If f and g are integrable functions such that $f(x) \leqslant g(x)$ holds for each $x \in S$, then $\int_A f \leqslant \int_A g$ whenever $A \subset S$.*

Proof. (i). This follows directly from our definition of the integral in case $0 \leqslant c$. If $c < 0$, then $0 < -c$, $-cf_+ = (cf)_-$, $-cf_- = (cf)_+$. We can now use the result just mentioned to obtain $\int_A (cf)_- = \int_A -cf_+ = -c\int f_+$. Putting these together in accordance with the definition of the integral gives the desired result.

(ii) It is easily checked that

$$0 \leqslant f_+(x) \leqslant g_+(x), \qquad 0 \leqslant f_+^{(n)}(x) \leqslant g_+^{(n)}(x),$$
$$0 \leqslant g_-(x) \leqslant f_-(x), \qquad 0 \leqslant g_-^{(n)}(x) \leqslant f_-^{(n)}(x) \tag{1}$$

hold for each $x \in S$ and each positive integer n. In case both f_+ and g_+ are bounded on S, we use the first of these relations to infer that

$$\sum_{\beta \in \mathbf{F}} f_+^*(\beta) \leqslant \sum_{\beta \in \mathbf{F}} g_+^*(\beta) \tag{2}$$

whenever $\mathbf{F} \in \mathfrak{M}(A)$, whence $\int_A f_+ \leqslant \int_A g_+$. Similarly, if both f and g are unbounded on S, we infer that $\int_A f_+^{(n)} \leqslant \int_A g_+^{(n)}$ for each positive integer n, and so again $\int_A f_+ \leqslant \int_A g_+$. If f_+ is bounded on S and g_+ is not, then $f_+ = f_+^{(n)}$ for suitably large values of n, whence $f_+ \leqslant g_+^{(n)}$ and $\int_A f_+ \leqslant \int_A g_+^{(n)}$ hold for large n; consequently $\int_A f_+ \leqslant \int_A g_+$. Applying the same arguments to the second pair of inequalities in (1), we derive $\int_A g_- \leqslant \int_A f_-$. Putting these results together gives the desired conclusion.

We note that an important special case occurs if a and b are real numbers and $a \leqslant f(x) \leqslant b$ holds for each $x \in S$. In this case, we have $a\bar{\mu}(A) = \int_A aK_S \leqslant \int_A f \leqslant \int_A bK_S = b\bar{\mu}(A)$. This leads to the following observation.

3.2. COROLLARY. *If f is real-valued on S, $A \subset S$, and $\bar{\mu}(A) = 0$, then $\int_A f = 0$.*

Proof. If f is bounded on S, this is an immediate consequence of the above remark; otherwise, it follows from our definition of integrals of unbounded functions as limits of integrals of bounded functions.

4. An additional restriction on $\mathbf{F}_0$ and $\boldsymbol{\mu}$. In this section we assume that $\alpha \cap \beta \in \mathbf{F}_0$, $\alpha \cap \tilde{\beta} \in \mathbf{F}_0$, and

$$\mu(\alpha) \geqslant \mu(\alpha \cap \beta) + \mu(\alpha \cap \tilde{\beta})$$

whenever α and β belong to $\mathbf{F}_0$. Since $\mathbf{F}_0$ is assumed non-empty, there is some $\alpha \in \mathbf{F}_0$; taking $\beta = \alpha$ we infer that $0 \in \mathbf{F}_0$. Also taking $\alpha = \beta = 0$ we see that $\mu(0) \geqslant \mu(0) + \mu(0)$, whence $\mu(0) = 0$.

4.1 LEMMA. $\mathbf{S}$ *contains* $\mathbf{S}(\mathbf{F}_0)$ (recall the definitions at the opening of §3).

Proof. It is sufficient to show that if f is any function non-negative on S, then the class of all $\int f$-measurable sets includes $\mathbf{F}_0$, for then it necessarily includes $\mathbf{S}(\mathbf{F}_0)$, and so then does $\mathbf{S}$.

Accordingly we take such a function f and consider the function f^* associated with f as in §3. We also assume that f is bounded on S; this restriction will be removed later. We take arbitrary sets α and β belonging to $\mathbf{F}_0$. If both $\alpha \cap \beta$ and $\alpha \cap \tilde{\beta}$ are non-empty, then by our sectional hypotheses on μ we have

$$(1) \quad f^*(\alpha) = \mu(\alpha) \cdot \sup_{x \in \alpha} f(x) \geqslant \mu(\alpha \cap \beta) \cdot \sup_{x \in \alpha \cap \beta} f(x) + \mu(\alpha \cap \tilde{\beta}) \cdot \sup_{x \in \alpha \cap \tilde{\beta}} f(x) = f^*(\alpha \cap \beta) + f^*(\alpha \cap \tilde{\beta}).$$

In case $\alpha \cap \beta = 0$ or $\alpha \cap \tilde{\beta} = 0$, we must have, respectively, $\alpha \cap \tilde{\beta} = \alpha$ or $\alpha \cap \beta = \alpha$, leading to either

$$f^*(\alpha) = f^*(\alpha \cap \tilde{\beta}) = f^*(\alpha \cap \tilde{\beta}) + f^*(\alpha \cap \beta)$$

or

$$f^*(\alpha) = f^*(\alpha \cap \beta) = f^*(\alpha \cap \beta) + f^*(\alpha \cap \tilde{\beta}),$$

respectively. Thus (1) holds in any case and f^* satisfies the hypotheses of Theorem 2.1. Hence α is f^*-measurable if $\alpha \in \mathbf{F}_0$. Consequently if $E \subset S$ and $\alpha \in \mathbf{F}_0$ then we must have

$$(2) \qquad \int_E f = f^*(E) \geqslant f^*(E \cap \alpha) + f^*(E \cap \tilde{\alpha}) = \int_{E \cap \alpha} f + \int_{E \cap \tilde{\alpha}} f.$$

Now suppose f is non-negative on S, but unbounded. Then we infer from (2) that

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$$\int_E f^{(n)} \geqslant \int_{E \cup \alpha} f^{(n)} + \int_{E \cap \tilde{\alpha}} f^{(n)}$$

whenever n is a positive integer, $E \subset S$ and $\alpha \in \mathbf{F}_0$. Thus, taking limits of the three expressions with respect to n, we obtain $\int_E f \geqslant \int_{E\cap\alpha} f + \int_{E\cap\tilde{\alpha}} f$. But this relation is the criterion that α be an $\int f$-measurable set, and the proof is complete.

4.2. THEOREM. $\mathbf{S}$ *consists of all sets of the form* $D - N$, *where* $D \in \mathbf{S}(\mathbf{F}_0)$ *and* $\bar{\mu}(N) = 0$.

Proof. If f is any function non-negative on S, then $\int_N f = 0$ whenever $N \subset S$ and $\bar{\mu}(N) = 0$ by Corollary 3.2. Since $\int f$ is an outer measure on S, then N is $\int f$-measurable and so $N \in \mathbf{S}$. But, by Lemma 4.1, if $D \in \mathbf{S}(\mathbf{F}_0)$ then $D \in \mathbf{S}$. Thus $(D - N) \in \mathbf{S}$.

On the other hand, consider any set $A \in \mathbf{S}$. Since $\mathfrak{M}(S) \neq 0$, there exists a finite or countably infinite subfamily of $\mathbf{F}_0$, say $\mathbf{F}$, such that $\bigcup \mathbf{F} = S$. By our sectional hypotheses and Lemma 2.2, we may take $\mathbf{F}$ to be disjoint. We choose an arbitrary set $B \in \mathbf{F}$. By well-known methods (cf. **1**, Theorem 12.3, p. 97), it is easily shown that there exists a set D_B which is the intersection of a finite or infinite sequence of sets, each of which is in turn the union of a finite or countably infinite subfamily of $\mathbf{F}_0$, $A \cap B \subset D_B \subset B$, and $\bar{\mu}(A \cap B) = \bar{\mu}(D_B) < \infty$. Both $A \cap B$ and D_B belong to $\mathbf{S}$, and so both sets are $\bar{\mu}$-measurable. Consequently, $D_B - A \cap B = N_B$, where $\bar{\mu}(N_B) = 0$. It follows that

$$A = \bigcup_{B \in \mathbf{F}} (D_B - N_B) = \bigcup_{B \in \mathbf{F}} D_B - \bigcup_{B \in \mathbf{F}} N_B,$$

where the first set on the right of this equation belongs to $\mathbf{S}(\mathbf{F}_0)$ and the second is a set of $\bar{\mu}$-measure zero.

5. Some results on additivity of the integral with respect to the integrand. We will operate in this section under the general hypotheses of §4. As we just saw, our definition of the integral assured its countably infinite additivity with respect to a large class of sets. However, additivity with respect to the integrand is more restrictive. We will begin our study of this question now. We start with a lemma extending somewhat one result achieved in Theorem 3.1.

5.1. LEMMA. *If* $B \in \mathbf{F}_0$, f *and* g *are integrable functions such that* $f(x) \leqslant g(x)$ *holds for each* $x \in B$, *then* $\int_A f \leqslant \int_A g$ *holds whenever* $A \subset B$. *In particular, if* a *and* b *are real numbers and* $a \leqslant f(x) \leqslant b$ *holds for each* $x \in B$, *then* $a\bar{\mu}(A) \leqslant \int_A f \leqslant b\bar{\mu}(A)$ *whenever* $A \subset B$.

Proof. We note that the statements (1) and (2) occurring in the proof of Theorem 3.1 (ii) are still valid provided we restrict x and $\mathbf{F}$, respectively, so that $x \in B$ and the members of $\mathbf{F}$ are subsets of B. However, if $A \subset B$

and if we choose any family $\mathbf{G} \in \mathfrak{M}(A)$, we may intersect its members with B to obtain a family $\mathbf{F}$ whose members are subsets of B. Owing to our sectional hypotheses, it follows that $\mathbf{F} \in \mathfrak{M}(A)$; moreover, the sum on the right side of (2) is not greater than the corresponding sum over $\mathbf{G}$. With this observation, we are free to take over the remainder of the proof of Theorem 3.1 verbatim to obtain the desired conclusion, including the note at the end.

5.2. DEFINITION. *For any set $\beta \subset S$ and any function f real-valued on β we define*

$$\Omega(f, \beta) = \sup_{x \in \beta} f(x) - \inf_{x \in \beta} f(x)$$

if $0 \neq \beta$; $\Omega(f, 0) = 0$.

5.3. DEFINITION. *If $B \subset S$, then we agree to denote by $\mathbf{C}(B)$ the family of all functions f real-valued and bounded on B, such that for each $\epsilon > 0$ there exists a family $\mathbf{F} \in \mathfrak{M}(B)$ with a finite subfamily $\mathbf{F}' \subset \mathbf{F}$ for which*

(i) $\Omega(f, \beta \cap B) < \epsilon$ whenever $\beta \in \mathbf{F}'$,

(ii) $\sum_{\beta \in (\mathbf{F}-\mathbf{F}')} \mu(\beta) < \epsilon$.

The family $\mathbf{F}$ occurring in the above definition may be assumed to be disjoint without loss of generality. For, following the lines of proof of Lemma 2.2, there exists a disjoint family $\mathbf{G} \in \mathfrak{M}(B)$, each member α of which is a subset of exactly one corresponding set β in $\mathbf{F}$. Thus $\mathbf{G}$ contains a finite subfamily $\mathbf{G}'$ corresponding to $\mathbf{F}'$ in $\mathbf{F}$, and for each $\alpha \in \mathbf{G}'$ with its corresponding β in $\mathbf{F}'$ we have, by (i),

$$\Omega(f, \alpha \cap B) \leqslant \Omega(f, \beta \cap B) < \epsilon.$$

Likewise it follows from (ii) that

$$\sum_{\alpha \in (\mathbf{G}-\mathbf{G}')} \mu(\alpha) \leqslant \sum_{\beta \in (\mathbf{F}-\mathbf{F}')} \mu(\beta) < \epsilon.$$

5.4. THEOREM. *If $B \subset S$, $A \subset B$, c is a real number, $f \in \mathbf{C}(B)$ and $g \in \mathbf{C}(B)$ then*

(i) $f \in \mathbf{C}(A)$

(ii) $f_+ \in \mathbf{C}(B)$, $f_- \in \mathbf{C}(B)$

(iii) $cf \in \mathbf{C}(B)$

(iv) $(f + g) \in \mathbf{C}(B)$; $|f| \in \mathbf{C}(B)$

(v) $f \cdot g \in \mathbf{C}(B)$.

Proof. (i) This follows at once from Definitions 5.2 and 5.3 combined with the fact that $\mathfrak{M}(B) \subset \mathfrak{M}(A)$.

(ii) It is easily checked that $\Omega(f_+, \beta) \leqslant \Omega(f, \beta)$ and $\Omega(f_-, \beta) \leqslant \Omega(f, \beta)$ whenever $\beta \subset S$. From this the desired conclusion follows at once.

(iii) Clearly $\Omega(cf, \beta) = |c|\Omega(f, \beta)$ whenever $\beta \subset S$; thus $cf \in \mathbf{C}(B)$.

(iv) It is easy to see that $\Omega(f+g), \beta) \leqslant \Omega(f, \beta) + \Omega(g, \beta)$ whenever $\beta \subset B$. Now, given $\epsilon > 0$, there exist families $\mathbf{F} \in \mathfrak{M}(B)$, $\mathbf{G} \in \mathfrak{M}(B)$ with finite subfamilies $\mathbf{F}' \subset \mathbf{F}$, $\mathbf{G}' \subset \mathbf{G}$ such that

$$\sum_{\alpha \epsilon (\mathbf{F}-\mathbf{F}')} \mu(\alpha) < \epsilon/2, \qquad \sum_{\beta \epsilon (\mathbf{G}-\mathbf{G}')} \mu(\beta) < \epsilon/2,$$

$\Omega(f, \alpha \cap B) < \epsilon/2$ and $\Omega(f, \beta \cap B) < \epsilon/2$ whenever $\alpha \in \mathbf{F}'$, $\beta \in \mathbf{G}'$.

We so define $\mathbf{H}'$ and $\mathbf{H}$ that

$$\mathbf{H}' = \{\gamma : \gamma = \alpha \cap \beta \text{ for some } \alpha \in \mathbf{F}' \text{ and some } \beta \in \mathbf{G}'\}$$
$$\mathbf{H} = \mathbf{H}' \cup (\mathbf{F} - \mathbf{F}') \cup (\mathbf{G} - \mathbf{G}').$$

Clearly $\mathbf{H} \in \mathfrak{M}(B)$ and $\mathbf{H}'$ is a finite subfamily of $\mathbf{H}$. Also, if $\gamma = \alpha \cap \beta$, where $\alpha \in \mathbf{F}'$ and $\beta \in \mathbf{G}'$, then evidently $\Omega(f, \gamma \cap B) \leqslant \Omega(f, \alpha \cap B) < \epsilon/2$ and $\Omega(g, \gamma \cap B) \leqslant \Omega(g, \beta \cap B) < \epsilon/2$. Thus, by our opening observation, it follows that $\Omega(f+g, \gamma \cap B) < \epsilon$ whenever $\gamma \epsilon \mathbf{H}'$. Also, since $(\mathbf{H} - \mathbf{H}') \subset (\mathbf{F} - \mathbf{F}') \cup (\mathbf{G} - \mathbf{G}')$, then

$$\sum_{\gamma \epsilon (\mathbf{H}-\mathbf{H}')} \mu(\gamma) \leqslant \sum_{\alpha \epsilon (\mathbf{F}-\mathbf{F}')} \mu(\alpha) + \sum_{\beta \epsilon (\mathbf{G}-\mathbf{G}')} \mu(\beta) < \epsilon.$$

Finally, since $f + g$ is bounded on B, we conclude that $(f+g) \in \mathbf{C}(B)$. In particular, from (ii) and the fact that $f = f_+ + f_-$, we derive $|f| \in \mathbf{C}(B)$.

(v) Since f and g are bounded on B, we may select a positive number M serving as an upper bound for both f and g on B.

Given $\epsilon > 0$, we select families $\mathbf{F} \in \mathfrak{M}(B)$ and $\mathbf{G} \in \mathfrak{M}(B)$ with finite subfamilies $\mathbf{F}' \subset \mathbf{F}$ and $\mathbf{G}' \subset \mathbf{G}$ such that

$$\sum_{\alpha \epsilon (\mathbf{F}-\mathbf{F}')} \mu(\alpha) < \epsilon/(2+3M) \text{ and } \Omega(f, \alpha \cap B) < \epsilon/(2+3M)$$

whenever $\alpha \in \mathbf{F}'$;

$$\sum_{\beta \epsilon (\mathbf{G}-\mathbf{G}')} \mu(\beta) < \epsilon/(2+3M) \text{ and } \Omega(g, \beta \cap B) < \epsilon/(2+3M)$$

whenever $\beta \in \mathbf{G}'$.

We define sets $\mathbf{H}'$ and $\mathbf{H}$ from $\mathbf{F}$ and $\mathbf{G}$ as in (iv) above. Thus if $\gamma = \alpha \cap \beta$, where $\alpha \in \mathbf{F}'$, $\beta \in \mathbf{G}'$, and $\gamma \cap B \neq 0$, then $\inf_{x \epsilon \gamma \cap B} f(x) = m$ is a finite number and we see that

$$\begin{aligned}
\sup_{x \epsilon \gamma \cap B} f(x) \cdot g(x) &\leqslant \sup_{x \epsilon \gamma \cap B} g(x) \cdot (f(x) - m) + \sup_{x \epsilon \gamma \cap B} mg(x) \\
&\leqslant M\Omega(f, \gamma \cap B) + \sup_{x \epsilon \gamma \cap B} mg(x); \\
\inf_{x \epsilon \gamma \cap B} f(x) \cdot g(x) &\geqslant \inf_{x \epsilon \gamma \cap B} g(x) \cdot (f(x) - m) + \inf_{x \epsilon \gamma \cap B} mg(x) \\
&\geqslant - M\Omega(f, \gamma \cap B) + \inf_{x \epsilon \gamma \cup B} mg(x).
\end{aligned}$$

By subtraction, we obtain from these relations

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$$\Omega(f\cdot g, \gamma \cap B) \leqslant 2M\Omega(f, \gamma \cap B) + \Omega(mg, \gamma \cap B)$$
$$\leqslant 2M\Omega(f, \alpha \cap B) + M\Omega(g, \beta \cap B) < 2\epsilon/3 + \epsilon/3 = \epsilon.$$

This last relation clearly holds if $\gamma \cap B = 0$, hence it holds for each $\gamma \in \mathbf{H}'$. Also, $(\mathbf{H} - \mathbf{H}') \subset (\mathbf{F} - \mathbf{F}') \cup (\mathbf{G} - \mathbf{G}')$, thus

$$\sum_{\gamma\epsilon(\mathbf{H}-\mathbf{H}')} \mu(\gamma) \leqslant \sum_{\alpha\epsilon(\mathbf{F}-\mathbf{F}')} \mu(\alpha) + \sum_{\beta\epsilon(\mathbf{G}-\mathbf{G}')} \mu(\beta).$$

Finally, since $f\cdot g$ is bounded on B, we conclude that $f\cdot g \in \mathbf{C}(B)$.

There is an obvious analogy between the class $\mathbf{C}(B)$ and the class of functions bounded and continuous almost everywhere on a closed interval. It is strengthened in the next theorem.

5.5. LEMMA. *If* $B \in \mathbf{F}_0$, $f \in \mathbf{C}(B)$, *and* $g \in \mathbf{C}(B)$, *then* $\int_B(f+g) = \int_B f + \int_B g$.

Proof. We let $h = f + g$. By Theorem 5.4, $h \in \mathbf{C}(B)$. We select a positive number M, serving as a bound for $|f|$, $|g|$, and $|h|$ on B, and not less than $\bar{\mu}(B)$.

We suppose $\epsilon > 0$ and select families $\mathbf{F} \in \mathfrak{M}(B)$, $\mathbf{G} \in \mathfrak{M}(B)$ with finite subfamilies $\mathbf{F}' \subset \mathbf{F}$, $\mathbf{G}' \subset \mathbf{G}$ such that $\Omega(f, \alpha \cap B) < \epsilon/8M$, $\Omega(g, \beta \cap B) < \epsilon/8M$ whenever $\alpha \in \mathbf{F}'$ and $\beta \in \mathbf{G}'$,

$$\sum_{\alpha\epsilon(\mathbf{F}-\mathbf{F}')} \mu(\alpha) < \epsilon/8M, \qquad \sum_{\beta\epsilon(\mathbf{G}-\mathbf{G}')} \mu(\beta) < \epsilon/8M.$$

Owing to the remarks following Definition 5.3, we may and do assume that $\mathbf{F}$ and $\mathbf{G}$ are disjoint. Also, we may further assume without loss of generality that the members of $\mathbf{F}$ and $\mathbf{G}$ are subsets of B, since otherwise we could intersect their elements with B without altering the above inequalities. Thus we have

(1) $\Omega(f, \alpha) < \epsilon/8M$ and $\Omega(g, \beta) < \epsilon/8M$ whenever $\alpha \in \mathbf{F}'$ and $\beta \in \mathbf{G}'$;

and

$$\sum_{\alpha\epsilon(\mathbf{F}-\mathbf{F}')} \mu(\alpha) < \epsilon/8M, \qquad \sum_{\beta\epsilon(\mathbf{G}-\mathbf{G}')} \mu(\beta) < \epsilon/8M. \tag{2}$$

We now define $\mathbf{K}' = \{\gamma : \gamma = \alpha \cap \beta$ for some $\alpha \in \mathbf{F}'$ and some $\beta \in \mathbf{G}'\}$. Clearly $\mathbf{K}'$ is a finite subfamily of $\mathbf{F}_0$, thus $(B - \cup\mathbf{K}') \in \mathbf{F}_0$. It is easy to see that $\mathbf{K}' \cup (\mathbf{F} - \mathbf{F}') \cup (\mathbf{G} - \mathbf{G}') \in \mathfrak{M}(B)$, and consequently $(\mathbf{F} - \mathbf{F}') \cup (\mathbf{G} - \mathbf{G}') \in \mathfrak{M}(B - \cup\mathbf{K}')$. Thus from (2) we obtain

$$\bar{\mu}(B - \cup\mathbf{K}') < \epsilon/4M \tag{3}$$

For each $x \in (B - \cup\mathbf{K}')$ we have $-M \leqslant f(x) \leqslant M$, $-M \leqslant g(x) \leqslant M$, $-M \leqslant h(x) \leqslant M$; so, by Lemma 5.1 and (3), we have

$$\left| \int_{B-\cup\mathbf{K}'} h - \int_{B-\cup\mathbf{K}'} f - \int_{B-\cup\mathbf{K}'} g \right| \leqslant 3M\bar{\mu}(B - \cup\mathbf{K}') < 3\epsilon/4 \tag{4}$$

If $0 \neq \alpha \cap \beta = \gamma$, where $\alpha \in \mathbf{F}'$ and $\beta \in \mathbf{G}'$, we let $m(\gamma) = \inf_{x \epsilon \gamma} f(x)$ and $m'(\gamma) = \inf_{x \epsilon \gamma} g(x)$. From (1) we see that for each $x \in \gamma$,

(5) $\quad m(\gamma) \leqslant f(x) \leqslant m(\gamma) + \epsilon/8M, m'(\gamma) \leqslant g(x) \leqslant m'(\gamma) + \epsilon/8M,$

whence $m(\gamma) + m'(\gamma) \leqslant h(x) \leqslant m(\gamma) + m'(\gamma) + \epsilon/4M$. Applying Lemma 5.1 to (5) we obtain

$$m(\gamma)\bar{\mu}(\gamma) \leqslant \int_{\gamma} f \leqslant m(\gamma)\bar{\mu}(\gamma) + \epsilon\bar{\mu}(\gamma)/8M,$$

$$m'(\gamma)\bar{\mu}(\gamma) \leqslant \int_{\gamma} g \leqslant m'(\gamma)\bar{\mu}(\gamma) + \epsilon\bar{\mu}(\gamma)/8M,$$

$$[m(\gamma) + m'(\gamma)]\bar{\mu}(\gamma) \leqslant \int_{\gamma} h \leqslant [m(\gamma) + m'(\gamma)]\bar{\mu}(\gamma) + \epsilon\bar{\mu}(\gamma)/4M.$$

From these relations we obtain

$$\left| \int_{\gamma} h - \left(\int_{\gamma} f + \int_{\gamma} g \right) \right| \leqslant \epsilon\bar{\mu}(\gamma)/4M. \tag{6}$$

This last relation holds if $\gamma = 0$, hence it holds for each set $\gamma \in \mathbf{K}'$. From the complete additivity of our integrals on $\mathbf{S}(\mathbf{F}_0)$, the disjointedness of $\mathbf{K}'$, and (6) we infer

$$\left| \int_{\cup \mathbf{K}'} h - \left(\int_{\cup \mathbf{K}'} f + \int_{\cup \mathbf{K}'} g \right) \right| \leqslant \epsilon\bar{\mu}(\cup \mathbf{K}')/4M \leqslant \epsilon\bar{\mu}(B)/4M \leqslant \epsilon/4.$$

Putting this together with (4) completes the proof.

5.6. Theorem. *If $B \in \mathbf{F}_0$, $f \in \mathbf{C}(B)$, $g \in \mathbf{C}(B)$, $A \in \mathbf{S}$, and $A \subset B$, then* $\int_A (f + g) = \int_A f + \int_A g$.

Proof. The statement is true if $A \in \mathbf{F}_0$ by virtue of Theorem 5.4 (i) and Lemma 5.5. If A is the union of a finite or countably infinite subfamily $\mathbf{F}$ of $\mathbf{F}_0$, we may, by virtue of Lemma 2.2, take $\mathbf{F}$ as disjoint, whence by what was just said we obtain

$$\int_A (f + g) = \sum_{\beta \epsilon \mathbf{F}} \int_{\beta} (f + g) = \sum_{\beta \epsilon \mathbf{F}} \left(\int_{\beta} f + \int_{\beta} g \right) = \int_A f + \int_A g.$$

Next we take up the case where A is the intersection of a finite or infinite sequence D of sets, each term in the sequence being itself the union of a finite or countably infinite subfamily of $\mathbf{F}_0$. By repetition of the terms of D if necessary, we may assume that D is an infinite sequence. For an arbitrary positive integer n, we let $\mathbf{F}_n$ denote a finite or countably infinite subfamily of $\mathbf{F}_0$ for which $D_n = \cup \mathbf{F}_n$. By intersecting the members of $\mathbf{F}_{n+1}$ with those of $\mathbf{F}_n$, if necessary, we may assume that $D_{n+1} = \cup \mathbf{F}_{n+1} \subset \cup \mathbf{F}_n = D_n$ for each n. By well-known properties of completely additive set functions, we infer that

$$\int_A (f + g) = \lim_n \int_{D_n} (f + g) = \lim_n \left(\int_{D_n} f + \int_{D_n} g \right) = \int_A f + \int_A g.$$

Finally we take up the case where $A = D - N$, where D is a set of the type considered in the preceding paragraph, and $\bar{\mu}(N) = 0$. Clearly $\bar{\mu}(D \cap N) = 0$, and $D = A \cup (D \cap N)$, whence by what we just proved and Corollary 3.2 we obtain

$$\int_A (f+g) = \int_A (f+g) + \int_{D\cap N} (f+g) = \int_D (f+g) = \int_D f + \int_D g$$
$$= \left(\int_A f + \int_{D\cap N} f\right) + \left(\int_A g + \int_{D\cap N} g\right) = \int_A f + \int_A g.$$

This completes the proof, since each set $A \in \mathbf{S}$, $A \subset B \in \mathbf{F}_0$ is of the type just considered, as may be seen from an inspection of the proof of Theorem 4.2.

We note in passing that if $B \in \mathbf{F}_0$, $c_1, c_2, \ldots, c_n$ are real numbers, and $C_1, C_2, \ldots, C_n$ are mutually disjoint members of $\mathbf{F}_0$, then $f = \sum_{i=1}^n c_i K_{C_i}$ belongs to $\mathbf{C}(B)$. We will show that the integrals of such functions have the values one might expect for them.

5.7. Lemma. *If n is a positive integer, $C_1, C_2, \ldots, C_n$ are mutually disjoint members of $\mathbf{S}$, $A \subset S$, and $0 \leqslant c_1, c_2, \ldots, c_n < \infty$, then*

$$\sum_{i=1}^n c_i \bar{\mu}(A \cap C_i) \leqslant \int_A \sum_{i=1}^n c_i K_{C_i}.$$

Proof. We let $f = \sum_{i=1}^n c_i K_{C_i}$. For any positive integer j, $1 \leqslant j \leqslant n$, we consider $\int_{A\cap C_j} f$. In case any one of these integrals is infinite, the desired conclusion holds. In case they are finite for each such j, we take an arbitrary $\epsilon > 0$. For each such j we select a family $\mathbf{F}_j \in \mathfrak{M}(A \cap C_j)$ such that, in the terminology of §3,

$$\sum_{\beta\in\mathbf{F}_j} f^*(\beta) \leqslant \int_{A\cap C_j} f + \epsilon/n. \tag{1}$$

We may assume without loss of generality that for each $\beta \in \mathbf{F}_j$, $\beta \cap A \cap C_j \neq 0$, since otherwise we could discard from $\mathbf{F}_j$ those members failing to intersect $A \cap C_j$ to obtain a subfamily still satisfying (1) and belonging to $\mathfrak{M}(A \cap C_j)$.

For any $\beta \in \mathbf{F}_j$, then, we see that $c_j \leqslant \sup_{x\in\beta} f(x)$, whence $c_j\mu(\beta) \leqslant f^*(\beta)$. from (1) and the definition of $\bar{\mu}$ we see that

$$c_j\bar{\mu}(A \cap C_j) \leqslant c_j \sum_{\beta\in\mathbf{F}_j} \mu(\beta) \leqslant \sum_{\beta\in\mathbf{F}_j} f^*(\beta) \leqslant \int_{A\cap C_j} f + \epsilon/n. \tag{2}$$

Then, using (2), the disjointedness of the sets C_j, the additivity of the integral, and the fact that $A \cap (C_1 \cup C_2 \cup \ldots \cup C_n) \subset A$ we obtain

$$\sum_{i=1}^n c_i\bar{\mu}(A \cap C_i) \leqslant \int_A f + \epsilon.$$

Since ϵ is arbitrary, the proof is complete.

5.8. Lemma. *If $A \in \mathbf{S}$ and $B \in \mathbf{F}_0$ then $\int_A K_B \leqslant \bar{\mu}(A \cap B)$.*

Proof. We take an arbitrary family $\mathbf{H} \in \mathfrak{M}(A)$. By Lemma 2.2, we may assume $\mathbf{H}$ is disjoint. Given $\epsilon > 0$, we select such a family $\mathbf{K} \in \mathfrak{M}(A \cap B)$ that

$$(1) \qquad \sum_{\beta \epsilon \mathbf{K}} \mu(\beta) \leqslant \bar{\mu}(A \cap B) + \epsilon.$$

We now define

$$\mathbf{D} = \{\gamma : \gamma = \alpha \cap \beta \cap B \text{ for some } \alpha \in \mathbf{H} \text{ and some } \beta \in \mathbf{K}\}$$
$$\mathbf{D}' = \{\gamma : \gamma = \alpha \cap \tilde{B} \text{ for some } \alpha \in \mathbf{H}\}.$$

We see that $\mathbf{D} \in \mathfrak{M}(A \cap B)$, $(\mathbf{D} \cup \mathbf{D}') \in \mathfrak{M}(A)$. Since $\gamma \cap B = 0$ for each $\gamma \in \mathbf{D}'$, then $K^*{}_B(\gamma) = 0$ for each such γ and so

$$(2) \qquad \int_A K_B \leqslant \sum_{\gamma \epsilon (\mathbf{D} \cup \mathbf{D}')} K_B^*(\gamma) = \sum_{\gamma \epsilon \mathbf{D}} K_B^*(\gamma) \leqslant \sum_{\gamma \epsilon \mathbf{D}} \mu(\gamma)$$
$$\leqslant \sum_{\beta \epsilon \mathbf{K}} \Big(\sum_{\alpha \epsilon \mathbf{H}} \mu(\alpha \cap \beta \cap B)\Big) \leqslant \sum_{\beta \epsilon \mathbf{K}} \Big(\sum_{\alpha \epsilon \mathbf{H}} \mu(\alpha \cap \beta)\Big).$$

Recalling Lemma 2.3, we derive from (1) and (2),

$$\int_A K_B \leqslant \sum_{\beta \epsilon \mathbf{K}} \mu(\beta) \leqslant \bar{\mu}(A \cap B) + \epsilon.$$

The arbitrary nature of ϵ gives the desired result.

5.9. Corollary. *If $A \in \mathbf{S}$ and $B \in \mathbf{F}_0$ then $\int_A K_B = \bar{\mu}(A \cap B)$.*

Proof. This comes from Lemmas 5.7 and 5.8.

5.10. Corollary. *If $A \in \mathbf{S}$, $C_1, C_2, \ldots C_n$ are mutually disjoint members of $\mathbf{F}_0$, $c_1, c_2, \ldots, c_n$ are real numbers, then*

$$\int_A \sum_{i=1}^{n} c_i K_{C_i} = \sum_{i=1}^{n} c_i \bar{\mu}(A \cap C_i).$$

Proof. Since $\mathfrak{M}(S) \neq 0$ by hypothesis, there exists a finite or countably infinite subfamily $\mathbf{F}$ of $\mathbf{F}_0$ with $\cup \mathbf{F} = S$; we may assume $\mathbf{F}$ to be disjoint. For each $\beta \in \mathbf{F}$ it is easily seen that $K_{C_i} \in \mathbf{C}(\beta)$, thus

$$\int_{A \cap \beta} \sum_{i=1}^{n} c_i K_{C_i} = \sum_{i=1}^{n} c_i \int_{A \cap \beta} K_{C_i}$$

by Theorems 5.6 and 3.1. The conclusion follows from Corollary 5.9 and summing the integrals over the family $\mathbf{F}$.

5.11. Theorem. *If $f_1, f_2, \ldots, f_m$ are functions non-negative on S, then*

$$\int_A \Big(\sum_{i=1}^{m} f_i\Big) \leqslant \sum_{i=1}^{m} \int_A f_i$$

whenever $A \subset S$.

Proof. We take up the case where $m = 2$ and let $g = f_1 + f_2$. We assume

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first that both f_1 and f_2 are bounded on S. Using the terminology of §3, we see that if $0 \neq \beta \in \mathbf{F}_0$, then

$$(1) \quad g^*(\beta) = \mu(\beta) \sup_{x\epsilon\beta} g(x) \leqslant \mu(\beta)(\sup_{x\epsilon\beta} f_1(x) + \sup_{x\epsilon\beta} f_2(x)) = f_1^*(\beta) + f_2^*(\beta).$$

In case $\beta = 0$, we have $g^*(\beta) = f^*_1(\beta) = f^*_2(\beta) = 0$, whence (1) holds for each $\beta \in \mathbf{F}_0$. By virtue of Corollary 2.5, we see that if $A \subset S$, then $\int_A g = \bar{g}^*(A) \leqslant \bar{f}^*_1(A) + \bar{f}^*_2(A) = \int_A f_1 + \int_A f_2$.

If both f_1 and f_2 are unbounded on S, then we see that $g^{(n)} \leqslant f_1^{(n)} + f_2^{(n)}$ holds throughout S for each positive integer n, whence by an argument similar to that just given we conclude that $\int_A g^{(n)} \leqslant \int_A f_1^{(n)} + \int f_2^{(n)}$ holds for each such n. Taking limits with respect to n completes the work. In case just one of the two functions f_1 and f_2 is unbounded on S, a minor obvious modification of the above procedure leads to the desired result. Ordinary induction takes care of the general case.

5.12. Corollary. *If $A \subset S$ and the functions $f_1, f_2, \ldots, f_n$ each have a finite integral over A, then so does $\sum_{i=1}^n f_i$.*

Proof. We let $g = \sum_{i=1}^n f_i$. It is easy to see that

$$g_+(x) \leqslant \sum_{i=1}^{n} f_{i+}(x), \qquad g_-(x) \leqslant \sum_{i=1}^{n} f_{i-}(x)$$

for each $x \in S$, whence by Theorem 5.11 and 3.1, $\int_A g$ is finite.

We conclude this section with a result concerning uniform convergence.

5.13. Lemma. *If $A \subset S$, f and g are functions real-valued on S with finite integrals over A, then $|\int_A f - \int_A g| \leqslant 2\int_A |f - g|$.*

Proof. It is easily verified that $f_+(x) - g_+(x) \leqslant |f(x) - g(x)|$ and $g_-(x) - f_-(x) \leqslant |f(x) - g(x)|$ whenever $x \in S$. Thus $f_+(x) \leqslant g_+(x) + |f(x) - g(x)|$ and $g_-(x) \leqslant f_-(x) + |f(x) - g(x)|$ hold whenever $x \in S$. Applying Theorem 3.1 and 5.11 we see that

$$\int_A f_+ \leqslant \int_A g_+ + \int_A |f - g|, \qquad \int_A g_- \leqslant \int_A f_- + \int_A |f - g|.$$

Adding and re-grouping the terms in these relations, we obtain

$$\left(\int_A f_+ - \int_A f_-\right) - \left(\int_A g_+ - \int_A g_-\right) \leqslant 2\int_A |f - g|,$$

hence $\int_A f - \int_A g \leqslant 2\int_A |f - g|$. Reversing the roles of f and g completes the proof.

5.14. Corollary. *If $A \subset S$, $\bar{\mu}(A) < \infty$, f is an infinite sequence of functions real-valued on S, each with a finite integral over A, g is real-valued on S, and f converges uniformly to g on S, then $\lim_n \int_A f_n = \int_A g$ and $\lim_n \int_A |f_n - g| = 0$. If $B \in \mathbf{F}_0$ and the uniform convergence holds only on B, then the conclusion is still valid provided $A \subset B$.*

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Proof. Given $\epsilon > 0$ we select such a positive integer N that $|f_n(x) - g(x)| < \epsilon$ holds for each $x \in S$ and each $n > N$. Thus $-\epsilon + f_{N+1}(x) < g(x) < f_{N+1}(x) + \epsilon$, for each $x \in S$. From the remarks following Theorem 3.1 and Corollary 5.12 we conclude that $\int_A g$ is finite. Thus we may apply Lemma 5.13 and the remarks following Theorem 3.1 to infer that $|\int_A f_n - \int_A g| \leqslant 2\int_A |f_n - g| < 2\epsilon\bar{\mu}(A)$. The conclusion follows from the arbitrariness of ϵ. Under the alternative hypotheses, the inequalities developed in the proof just given are still valid provided $x \in A$ and $A \subset B$, due to Lemma 5.1 instead of Theorem 3.1, and so the stated conclusion is valid.

6. Further additivity and convergence properties with respect to the integrand. To obtain the usual Lebesgue theorems we need to extend Corollary 5.10 to cover the case where B is an arbitrary member of $\mathbf{S}$, and to extend Theorem 5.11 to cover the case of infinite sequences. It is possible to construct examples showing that this cannot be done under the general hypotheses of §§4 and 5. Accordingly, in this section we will add to those hypotheses the following: if $B \in \mathbf{F}_0$, $\epsilon > 0$, $A \subset B$, and A is the union of a finite or countably infinite subfamily of $\mathbf{F}_0$, then there exists a family $\mathbf{F} \in \mathfrak{M}(B - A)$ such that the subfamily $\mathbf{F}'$ consisting of those members of $\mathbf{F}$ that intersect A satisfies the relation

$$\sum_{\beta \in \mathbf{F}'} \mu(\beta) < \epsilon.$$

This new condition would be satisfied if $\mathbf{F}_0$ were endowed with the property that $(B - A) \in \mathbf{F}_0$ whenever A is the union of a finite or countably infinite subfamily of $\mathbf{F}_0$. The above hypotheses would be satisfied still more particularly if $\mathbf{F}_0$ were a completely additive family of subsets of $\mathbf{F}_0$ and μ were a measure on $\mathbf{F}_0$. It will be apparent that our integral will agree with the Lebesgue integral on $\mathbf{F}_0$ when applied to μ-measurable functions in this special case.

In what follows, we will be dealing with the class of $\bar{\mu}$-measurable functions. We shall assume that the properties of such functions are known (cf. **2**, pp. 12–15).

6.1. LEMMA. *If $A \in \mathbf{S}$, b is a non-negative real number, f is a function non-negative on S for which $0 \leqslant f(x) \leqslant b$ whenever $x \in A$, then $0 \leqslant \int_A f \leqslant b\bar{\mu}(A)$.*

Proof. We show this first under the assumption that f is bounded throughout S. We choose a positive number $M \geqslant b$ serving as an upper bound for f on S. We select a family $\mathbf{F} \in \mathfrak{M}(S)$ and use Lemma 2.2 to justify the assumption that it is disjoint. We take an arbitrary set $B \in \mathbf{F}$ and an arbitrary $\epsilon > 0$. By Lemma 5.1 we see that

$$(1) \qquad 0 \leqslant \int_Q f \leqslant M\bar{\mu}(Q) < \epsilon$$

whenever $Q \subset B$ and $\bar{\mu}(Q) < \epsilon/M$.

We select such a family $\mathbf{G} \in \mathfrak{M}(B - A)$ that

$$\bar{\mu}(\bigcup \mathbf{G}) \leqslant \sum_{\beta \epsilon \mathbf{G}} \mu(\beta) < \bar{\mu}(B - A) + \epsilon/2M,$$

whence, setting $C = \bigcup \mathbf{G}$, we readily check that

$$\bar{\mu}(A \cap B \cap C) < \epsilon/2M. \tag{2}$$

By intersecting the members of $\mathbf{G}$ with B if necessary, there is no loss of generality in assuming $\bigcup \mathbf{G} = C \subset B$.

We now use our sectional hypotheses to find a family $\mathbf{H} \in \mathfrak{M}(B - C)$ such that the subfamily $\mathbf{H}'$ consisting of those members of $\mathbf{H}$ that intersect C satisfies the relation

$$\bar{\mu}(\bigcup \mathbf{H}') \leqslant \sum_{\alpha \epsilon \mathbf{H}'} \mu(\alpha) < \epsilon/2M. \tag{3}$$

Following the lines of Lemma 2.2, we are justified in assuming that $\mathbf{H}$ is disjoint and its members are subsets of B.

We let $D = \bigcup (\mathbf{H} - \mathbf{H}')$, $E = \bigcup \mathbf{H}'$. For any set $\alpha \in (\mathbf{H} - \mathbf{H}')$, we have $\alpha \cap C = 0$, and so $\alpha \subset A \cap B$. Thus, by Lemma 5.1, we have $0 \leqslant \int_\alpha f \leqslant b\bar{\mu}(\alpha)$ for each such α, and by additivity of the integral we obtain

$$0 \leqslant \int_D f \leqslant b\bar{\mu}(D) \ ; \qquad D \subset A \cap B. \tag{4}$$

Since $(B - C) \subset \bigcup \mathbf{H} = D \cup E$ then $[(A \cap B - D) - C] \subset E$. Also $(A \cap B - D) \cap C \subset A \cap B \cap C$. Putting together these relations we find that $(A \cap B - D) \subset E \cup (A \cap B \cap C)$, and so from (2) and (3) we conclude that

$$\bar{\mu}(A \cap B - D) < \epsilon/M. \tag{5}$$

Since $A \cap B = (A \cap B - D) \cup D$, we now use (1), (4), (5) and the additivity of our integral to see that

$$0 \leqslant \int_{A \cap B} f \leqslant b\bar{\mu}(D) + \epsilon \leqslant b\bar{\mu}(A \cap B) + \epsilon.$$

Since ϵ is arbitrary, we conclude that $0 \leqslant \int_{A \cap B} f \leqslant b\bar{\mu}(A \cap B)$. Finally, from the arbitrary nature of $B \in \mathbf{F}$, the disjointedness of $\mathbf{F}$ and the fact that $A \subset \bigcup \mathbf{F}$, we obtain the desired result.

In case f is unbounded on S, we use the result just obtained to infer that $0 \leqslant \int_A f^{(n)} \leqslant b\bar{\mu}(A)$ for each positive integer n, whence the desired conclusion is derived by taking the limit with respect to n.

Our definition of the integral of a given function over a set A may involve values assumed by the function outside of A. This explains the need for the following result.

6.2. Corollary. *If $A \in \mathbf{S}$, f and g are functions real-valued on S, such that $0 \leqslant f(x) \leqslant g(x)$ for each $x \in S$, and $f(x) = g(x)$ for each $x \in A$, then $\int_A f = \int_A g$.*

Proof. We so define h on S that $h = g - f$; then $g = f + h$. All three functions are non-negative on S and by Theorem 5.11, $\int_A g \leqslant \int_A f + \int_A h$. Since $h(x) = 0$

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for each $x \in A$, we may apply Lemma 6.1 with $b = 0$ to infer that $\int_A h = 0$, when $\int_A g \leqslant \int_A f$. The reverse inequality follows from Theorem 3.1.

6.3. Corollary. *If $A \in \mathbf{S}$, $C_1, C_2, \ldots, C_n$ are mutually disjoint members of $\mathbf{S}$, $0 \leqslant c_1, c_2, \ldots, c_n < \infty$, then*

$$\int_A \sum_{i=1}^{n} c_i K_{C_i} = \sum_{i=1}^{n} c_i \bar{\mu}(A \cap C_i).$$

Proof. We let

$$f = \sum_{i=1}^{n} c_i K_{C_i}, \qquad D = \bigcup_{i=1}^{n} C_i.$$

Since $f(x) = 0$ whenever $x \in (A - D)$ and $f(x) = c_i$ whenever $x \in C_i$, we infer from Lemma 6.1 that $\int_{A-D} f = 0$ and $\int_{A \cup C_i} f \leqslant c_i \bar{\mu}(A \cap C_i)$ for each positive integer i, $1 \leqslant i \leqslant n$. Then, using Lemma 5.7, we see that

$$\sum_{i=1}^{n} c_i \bar{\mu}(A \cap C_i) \leqslant \int_A f = \int_{A \cap D} f + \int_{A-D} f = \sum_{i=1}^{n} \int_{A \cap C_i} f \leqslant \sum_{i=1}^{n} c_i \bar{\mu}(A \cap C_i).$$

Functions of the type considered in Corollary 6.3, without, however, requiring that $c_1, c_2, \ldots, c_n$ be non-negative, are often called *simple*. It is easily checked that the sum of two simple functions is again simple.

6.4. Corollary. *If f and g are non-negative simple functions and $A \in \mathbf{S}$, then $\int_A (f + g) = \int_A f + \int_A g$.*

Proof. The method of proof is well known and is therefore not given here (cf. **2**, p. 21).

6.5. Lemma. *If $A \in \mathbf{S}$, f is a non-decreasing sequence of non-negative functions on S, each of which is $\bar{\mu}$-measurable, and $g = \lim_n f_n$, then $\int_A g = \lim_n \int_A f_n$.*

Proof. We show this first on the assumption that f is bounded on S. We select a family $\mathbf{F} \in \mathfrak{M}(S)$, which we may take to be disjoint. We consider an arbitrary $B \in \mathbf{F}$ and take $\epsilon > 0$. For each positive integer n, we let

$$Q_n = \{x : x \in B \cap A \text{ and } 0 \leqslant g(x) - f_n(x) < \epsilon\}.$$

Evidently $Q_n \subset Q_{n+1}$ for each positive integer n, and $B \cap A = \lim_n Q_n$, whence $\bar{\mu}(B \cap A) = \lim_n \bar{\mu}(Q_n)$.

From the boundedness of g on S it follows by Lemma 6.1 that $\int_B g$ is finite, and moreover there exists $\delta > 0$ such that

$$(1) \qquad 0 \leqslant \int_D g < \epsilon$$

whenever $D \subset B$, $D \in \mathbf{S}$, and $\bar{\mu}(D) < \delta$. We select N so that

$$0 \leqslant \bar{\mu}(B \cap A) - \bar{\mu}(Q_N) = \bar{\mu}(B \cap A - Q_N) < \delta$$

and let $D = B \cap A - Q_N$.

Using Theorem 3.1 and the fact that $f_n \leqslant f_{n+1}$ holds throughout S for each positive integer n, we have

$$\int_{A\cap B} f_N \leqslant \lim_n \int_{A\cap B} f_n. \tag{2}$$

Also by Lemma 6.1,

$$\int_{Q_N} (g - f_N) \leqslant \epsilon\bar{\mu}(B), \tag{3}$$

and by Theorem 3.1 and (1),

$$0 \leqslant \int_D (g - f_N) \leqslant \int_D g < \epsilon. \tag{4}$$

Now, recalling Theorem 5.11 and using (2), (3), and (4) we get

$$\int_{A\cap B} g \leqslant \int_{A\cap B} f_N + \int_{A\cap B} (g - f_N) \leqslant \lim_n \int_{A\cap B} f_n + \epsilon(1 + \bar{\mu}(B)),$$

whence $\int_{A\cap B} g \leqslant \lim_n \int_{A\cap B} f_n$. The reverse inequality comes from the fact that $0 \leqslant f_n \leqslant g$ holds throughout S and Theorem 3.1. Hence $\int_{A\cap B} g = \lim_n \int_{A\cap B} f_n$. The task is completed by summing over the family $\mathbf{F}$ with a little care.

In case f is not bounded on S, then for any positive integers n and k, the inequalities $f_n^{(k)} \leqslant f_n \leqslant g, f_n^{(k)} \leqslant f_{n+1}^{(k)} \leqslant g^{(k)}, \lim_n f^{(k)} = g^{(k)}$ hold throughout S. Using the result just established on the sequence $f^{(k)}$ with limit $g^{(k)}$, and taking appropriate limits with respect to k, the final result is easily achieved.

6.6. COROLLARY. *If f and g are functions non-negative on S and $\bar{\mu}$-measurable, and $A \in \mathbf{S}$, then $\int_A (f + g) = \int_A f + \int_A g$.*

Proof. As is well known (cf. **2**, p. 14), there exist non-decreasing sequences s and t of non-negative simple functions such that $\lim_n s_n = f$ and $\lim_n t_n = g$ hold throughout S. Evidently $s + t$ is a non-decreasing sequence of non-negative simple functions converging to $f + g$ throughout S. The derived result is an immediate consequence of Corollary 6.4 and Lemma 6.5.

6.7. COROLLARY. *If f and g are real-valued on S and $\bar{\mu}$-measurable, $A \in \mathbf{S}$, and both functions are non-negative on A, then $\int_A (f + g) = \int_A f + \int_A g$.*

Proof. We let $h = f + g$. We see that $0 \leqslant h_+ \leqslant f_+ + g_+$ and $0 \leqslant h_- \leqslant f_- + g_-$ hold throughout S; also $h_+ = f_+ + g_+$ and $0 = h_- = f_- + g_-$ hold on A. Thus, applying Corollary 6.2, Corollary 6.6, and Lemma 6.1 we obtain

$$\int_A h_+ = \int_A (f_+ + g_+) = \int_A f_+ + \int_A g_+ ;$$

$$\int_A h_- = \int (f_- + g_-) = \int_A f_- + \int_A g_- = 0 ;$$

putting these results together completes the proof.

With this in hand, it is possible to prove the following theorem easily (cf. **2**, p. 24). We state it without proof.

6.7. THEOREM. *If $A \in \mathbf{S}$, f and g are real-valued on S and $\bar{\mu}$-measurable, both functions possess a finite integral over A, then $\int_A (f+g) = \int_A f + \int_A g$.*

As we mentioned earlier, our definition of the integral of a function over a set A involves function values outside of A. However, we are finally in a position to show that these values do not affect the values of the integrals of $\bar{\mu}$-measurable functions after all.

6.8. COROLLARY. *If $A \in \mathbf{S}$, f and g are real-valued on S and $\bar{\mu}$-measurable, $\int_A g$ is finite and $f = g$ on A then $\int_A f = \int_A g$.*

Proof. We so define h on S that $h = f - g$. Clearly $h_+ = h_- = 0$ on A, so by Lemma 6.1, $\int_A h_+ = \int_A h_- = 0$, whence $\int_A h = 0$. Since $f = g + h$ on S, we now infer from Theorem 6.7 that $\int_A f = \int_A g + \int_A h = \int_A g$, as required.

Owing to Corollary 3.2 this last result is true if $f = g$ $\bar{\mu}$-almost everywhere on A.

From this point on, no special techniques are needed to prove such theorems as the Fatou lemma and the general Lebesgue convergence theorem, so we omit their proofs (cf. **2**, pp. 29, 30).

The writer proposes to investigate the Fubini theorem and transformation theory from the present point of view in succeeding papers.

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