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A note on Omori-Lie groups

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The theory of differentiation in locally convex spaces constructed by the author in *Memoirs Amer. Math. Soc.* 17 (1979) is used to give a new form of the definition of Omori-Lie groups.

An *Omori-Lie group* (a "strong ILB-Lie group" in Omori's terminology) is defined in [*6*] as follows. Let

$$\{E, E^k : k \geq 0\}$$

be a Sobolev chain, that is,

(1) all E^k are Banach spaces;

(2) E^{k+1} is linearly and densely imbedded in E^k ;

(3) E is the intersection of all E^k and has the inverse limit topology defined by $\{E^k\}$.

Then, a topological group G is called an *Omori-Lie group* if the following seven conditions are satisfied.

(OL.1) There is an open neighborhood U of zero in E^0 and a homeomorphism

$$\xi : U \cap E \to \tilde{U}$$

such that $\xi(0) = e$ (the unit of G), where $U \cap E$ is given the relative topology from E and $\tilde{U}$ is an open neighborhood of e in G .

(OL.2) There is an open neighborhood V of zero in E^0 such that

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$$\xi(V \cap E) = \xi(V \cap E)^{-1} \text{ and } \xi(V \cap E)^2 \subset \xi(V \cap E) .$$

(OL.3) Put $\eta(u, v) = \xi^{-1}[\xi(u)\xi(v)]$; then, for all $k \geq 0$ and $r \geq 0$, η can be extended to a C^r-map of $(V \cap E^{k+r}) \times (V \cap E^k)$ into $U \cap E^k$.

(OL.4) Put $\eta_v(u) = \eta(u, v)$; then, for each $v \in V \cap E^k$ and $k \geq 0$, η_v can be extended to a C^∞-map of $V \cap E^k$ into itself.

(OL.5) Put $\theta(w, u, v) = (d\eta_v)_u(w)$; then, for all $k \geq 0$ and $r \geq 0$, θ can be extended to a C^r-map of $E^{k+r} \times (V \cap E^{k+r}) \times (V \cap E^k)$ into E^k .

(OL.6) Put $i(u) = \xi^{-1}[\xi(u)^{-1}]$; then, for all $k \geq 0$ and $r \geq 0$, i can be extended to a C^r-map of $V \cap E^{k+r}$ into $V \cap E^k$.

(OL.7) For any $g \in G$ there is an open neighborhood W of zero in E^0 such that $g^{-1}\xi(W \cap E)g \subset \xi(V \cap E)$ and the map

$$A_g : u \mapsto \xi^{-1}[g^{-1}\xi(u)g]$$

can be extended to a C^∞-map of $W \cap E^k$ into itself for every $k \geq 0$.

Examples of the Omori-Lie group include the group $D(M)$ of all C^∞-diffeomorphisms of a compact manifold M and its various subgroups. In fact, the notion of Omori-Lie groups has been introduced in order to develop a general theory which covers these groups of diffeomorphisms. It is the only general theory in existence today which has gained some success in such an attempt.

In [7], I have introduced a notion of differentiability for maps in locally convex spaces, which was called the Γ-*differentiability*, and it was used to define the Γ-manifolds. An outline of this study was also published in [5]. In this note, we shall use this method to define the Γ-*Lie groups* and then show a way to obtain another form of the definition of Omori-Lie groups. This new method opens a way to the study of the group $D(M)$ with noncompact M .

The basic concepts in [7], such as "calibrations", "Γ-families", "Γ-continuous maps", and "Γ-differentiable maps", will be used without explanation.

A notion of differentiability similar to ours has been proposed by Fischer [1], which contains various topics on the manifolds modelled on locally convex spaces and the groups of smooth diffeomorphisms on compact manifolds.

1. Gradings of calibrations

Let F be a Γ-family. Hence, F is a family of locally convex spaces and Γ is a family of maps on F such that the value p_E of $p \in \Gamma$ at $E \in F$ is a continuous semi-norm on E, and the set

$$\Gamma_E = \{p_E : p \in \Gamma\} ,$$

which is called the *E-component* of Γ, induces the topology of E.

A *grading* of Γ is a sequence

$$\sigma = (\sigma_k)_{k=0,1,2,\ldots}$$

of maps

$$\sigma_k : \Gamma \to \Gamma$$

such that

$$\sigma_{k+1}(p) \geq \sigma_k(p) \quad \text{and} \quad \sigma_0(p) = p .$$

Obviously, each $\sigma_k(\Gamma)$ is a calibration for F. We shall put

$$\Gamma_k = \sigma_k(\Gamma) , \quad k \geq 0 .$$

Since Γ_k also is a calibration for F, it has its E-component for each $E \in F$. The space E equipped with this calibration is denoted by $E_{(k)}$.

Furthermore, we put

$$F_{(k)} = \{E_{(k)} : E \in F\}$$

and

$$F_\sigma = \bigcup\{F_{(k)} : k \geq 0\} .$$

For each $p \in \Gamma$, we define a semi-norm map $\sigma(p)$ on F_σ by

$$\sigma(p)_{E_{(k)}} = \sigma_k(p)_E ,$$

and put

$$\Gamma_\sigma = \{\sigma(p) : p \in \Gamma\} .$$

In other words, the E_k-component of Γ_σ is defined to be the E-component of Γ_k ; that is,

$$(\Gamma_\sigma)_{E_{(k)}} = (\Gamma_k)_E .$$

(1.1). *Γ_σ is a calibration for F_σ which is an extension of the calibration Γ for F .*

Proof. For $E \in F$,

$$\sigma(p)_E = \sigma(p)_{E_0} = \sigma_0(p)_E = p_E .$$

(1.2). *For each $E \in F$, $E_{(k)} = E$ as topological linear spaces.*

Proof. Since $(\Gamma_k)_E \subset \Gamma_E$, the topology of $E_{(k)}$ is weaker that that of E . The converse follows from

$$\sigma_k(p) \geq \sigma_0(p) = p .$$

(1.3). *For $E, F \in F$, if $n \leq k$ and $j \leq m$, then*

$$L_{\Gamma_\sigma}\left(E_{(j)}, F_{(k)}\right) \subset L_{\Gamma_\sigma}\left(E_{(m)}, F_{(n)}\right) ,$$

and the inclusion is $B\Gamma_\sigma$-continuous.

Proof. For $u \in L_{\Gamma_\sigma}\left(E_{(j)}, F_{(k)}\right)$, we have

$$\begin{aligned}\sigma(p)_{\left(E_{(m)},F_{(n)}\right)}(u) &= \sup\{\sigma_n(p)[u(x)] : \sigma_m(p)(x) \leq 1\} \\ &\leq \sup\{\sigma_k(p)[u(x)] : \sigma_j(p)(x) \leq 1\} \\ &= \sigma(p)_{\left(E_{(j)},F_{(k)}\right)}(u) .\end{aligned}$$

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In particular, if $j \le m$,

$$L_{\Gamma_\sigma}\left(E_{(j)}, F\right) \subset L_{\Gamma_\sigma}\left(E_{(m)}, F\right) ,$$

and the inclusion map is $B\Gamma_\sigma$-continuous.

For each $p \in \Gamma$, let us denote by $E[p]$ the space E that is regarded as a semi-normed space with respect to the semi-norm p . Then,

$$(1.4).\quad L_{\Gamma_\sigma}\left(E_{(k)}, F_{(l)}\right) = \bigcap_{p\in\Gamma} L\left(E\left[\sigma_k(p)\right], F\left[\sigma_l(p)\right]\right) .$$

Proof. $u \in L_{\Gamma_\sigma}\left(E_{(k)}, F_{(l)}\right)$ if and only if, for each $p \in \Gamma$, there exists $\gamma = \gamma(p, k, l) > 0$ such that

$$\sigma_l(p)[u(x)] \le \gamma\sigma_k(p)(x) \quad \text{for all} \quad x \in E ,$$

which is equivalent to $u \in L\left(E\left[\sigma_k(p)\right], F\left[\sigma_l(p)\right]\right)$ for all $p \in \Gamma$.

2. Gelfand families and their gradings

A *Gelfand space* is a locally convex space which has a calibration consisting of an increasing sequence of norms:

$$\|\cdot\|_n , \quad n = 0, 1, 2, \ldots ,$$

which are *pairwise coordinated*: if a sequence of element is a Cauchy sequence with respect to the nth norm and converges to zero with respect to the $(n-1)$th norm, then it converges to zero with respect to the nth norm. For detailed description of properties of Gelfand spaces, we refer to [2], [3], and [4]. We owe the name "Gelfand space" to [2].

The most basic property of the Gelfand space is the following fact: *a complete locally convex space* E *is a Gelfand space if and only if there is a sequence* $\{E_n\}$ *of Banach spaces such that* E_{n+1} *is linearly and densely imbedded in* E_n *for each* n *and* E *is the intersection of all* E_n *with the inverse limit topology.*

When E is a Gelfand space, the Banach spaces E_n can be chosen as the completions of E with respect to the nth norms.

Now let F be a family of Gelfand spaces. Then each space E in F

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has a calibration consisting of

$$\|\cdot\|_{E,n}\ ,\quad n = 0, 1, 2, \ldots .$$

Therefore, we can equip F with a calibration Γ which consists of countable (semi-)norm maps:

$$p_n\ ,\quad n = 0, 1, 2, \ldots ,$$

such that

$$(p_n)_E = \|\cdot\|_{E,n} .$$

A family of Gelfand spaces equipped with this calibration will be called a *Gelfand family*. The calibration will be called the *natural calibration* for this family.

Assume that F is a Gelfand family, and let Γ be the natural calibration. Then we can define a grading of Γ by

$$\sigma_k(p_n) = p_{n+k}\ ,\quad k, n = 0, 1, 2, \ldots .$$

This grading will be called the *natural grading* of Γ. In this case, we have

$$(\Gamma_k)_E = \{\|\cdot\|_{E,k}, \|\cdot\|_{E,k+1}, \ldots\}$$

and

$$\sigma(p_n)_{E_{(k)}} = \sigma_k(p_n)_E = \|\cdot\|_{E,n+k} \quad \text{for each } p_n \in \Gamma .$$

In the sequel, we shall denote the E-component of p_n by $\|\cdot\|_n$, without specifying the space E when there is no possibility of confusion Further, the normed space $E[p_n]$ will sometimes be denoted by $E[n]$.

3. σ-smoothness

Let F be a Γ-family. We recall two facts from [7].

First, let $E \in F$; then a subset U of E is said to be *Γ-open* i it is p-open for every $p \in \Gamma$, that is, for each $p \in \Gamma$ and $x \in U$, there exists a positive number δ such that

$$x + y \in U \quad \text{if} \quad p_E(y) < \delta .$$

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Some properties of Γ-open subsets have been given in [7, Chapter I, §4]. When U is a Γ-open subset of E, it is obvious that U is an open subset of the semi-normed space $E[p]$. The set U regarded as an open subset of $E[p]$ will be denoted by $U[p]$.

Secondly, let U be a Γ-open subset of E. Then we have proved in [7, Chapter II, §2] the following fact:

Let $F \in \mathrm{F}$ be sequentially complete. Then a map $f : U \to F$ is of class C^r_Γ if and only if f is of class C^r as a map of $U[p]$ into $F[p]$ for every $p \in \Gamma$.

Now we assume that this calibration Γ has a grading $\sigma = (\sigma_k)$.

When U is a Γ-open subset of E in F, it is a Γ_k-open subset of $E_{(k)}$. The set U regarded as a Γ_k-open subset of $E_{(k)}$ is denoted by $U_{(k)}$.

Let $F \in \mathrm{F}$; then a map $f : U \to F$ is said to be *σ-smooth* if, for every $k \geq 0$, it is a $C^k_{\Gamma_\sigma}$-map of $U_{(k)}$ into F. Then the following fact follows immediately from the second remark given above.

(3.1). *Let Γ be a graded calibration for F, $E, F \in \mathrm{F}$, and F be sequentially complete. Let U be a Γ-open subset of E. Then a map $f : U \to F$ is σ-smooth if and only if, for every $p \in \Gamma$ and $k \geq 0$, f is a C^k-map of $U[\sigma_k(p)]$ into $F[p]$.*

When F is a Gelfand family with the natural calibration Γ, the map f is σ-smooth if and only if, for every $k \geq 0$ and $n \geq 0$, it is of class C^k as a map of $U[n+k]$ into $F[n]$.

Further, let E, F, and G be members of a Γ-family with a grading σ. Let U and V be Γ-open subsets of E and F respectively. Then a map

$$f : U \times V \to G$$

is said to be *(σ, Γ)-smooth* if $E \times F$ is a Γ-product and, for every $k \geq 0$, f is a C^k-map of $U_{(k)} \times V$ into G.

The following fact can be proved in the same way as in the case of (3.1).

(3.2). *Let* E, F, G, U, V, *and* f *be as above. Then* f *is* (σ, Γ)*-smooth if and only if, for every* $p \in \Gamma$ *and* $k \geq 0$, f *is a* C^k*-map of* $U[\sigma_k(p)] \times V[p]$ *into* $G[p]$.

When F is a Gelfand family with the natural calibration Γ and its natural grading σ, the map is (σ, Γ)-smooth if and only if it is a C^k-map of $U[n+k] \times V[n]$ into $G[n]$ for every $n \geq 0$ and $k \geq 0$.

4. Γ-Lie groups

A *Γ-Lie group* is a topological group G such that there is a Γ-family with a grading σ, and the following conditions are satisfied:

(ΓL.1) G is a Γ-manifold of class C^∞;

(ΓL.2) the product operation

$$(g, h) \mapsto gh : G \times G \to G$$

is (σ, Γ)-smooth;

(ΓL.3) the inverse operation

$$g \mapsto g^{-1} : G \to G$$

is σ-smooth.

In particular, when Γ is the natural calibration for a Gelfand family and σ is the natural grading of Γ, the Γ-Lie group will be called a *Gelfand-Lie group*. The Omori-Lie groups are Gelfand-Lie groups; the conditions (OL.3) and (OL.6) imply (ΓL.2) and (ΓL.3), respectively. In order to have the inverse implications, we need a new notion of "completional continuity", which will be discussed in the next section.

5. Completional continuity

Let F be a Γ-family and $E, F \in \mathsf{F}$.

Let U be a p-open subset of E for $p \in \Gamma$. Then a map $f : U \to F$ is said to be *completionally* p*-continuous* if, for arbitrary p-Cauchy sequences $\{x_i\}$ and $\{y_i\}$ contained in U such that

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$$\lim_{i\to\infty} p_E(x_i - y_i) = 0 ,$$

we have

$$\lim_{i\to\infty} p_F(f(x_i) - f(y_i)) = 0 .$$

This definition includes the case when all y_i are equal to an element. Hence, the following statement is obvious.

(5.1). *Completionally p-continuous maps are p-continuous.*

A p-continuous map does not always transform a p-Cauchy sequence into a p-Cauchy sequence. However:

(5.2). *If f is a completionally p-continuous map on U and $\{x_i\}$ is a p-Cauchy sequence contained in U, then $\{f(x_i)\}$ is also a p-Cauchy sequence.*

Proof. If the sequence $\{f(x_i)\}$ is not p-Cauchy, there are $\delta > 0$ and subsequences $\{x_{i_n}\}$ and $\{x_{j_n}\}$ such that $i_n \to \infty$, $j_n \to \infty$, and

$$p_F[f(x_{i_n}) - f(x_{j_n})] \geq \delta .$$

However, since $\{x_i\}$ is a p-Cauchy sequence, its subsequences $\{x_{i_n}\}$ and $\{x_{j_n}\}$ are also p-Cauchy sequences and

$$\lim_{n\to\infty} p_E(x_{i_n} - x_{j_n}) = 0 ,$$

which is a contradiction.

The following statement is also obvious.

(5.3). *All p-Lipschitz maps are completionally p-continuous.*

In particular, *every p-continuous linear map is completionally p-continuous.* Furthermore, since every C_p^1-map is locally lipschitzian, we have the following.

(5.4). *Let $f : U \to F$ be a C_p^1-map. Then, for each $a \in U$, there*

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is an open p-ball $B(a, \gamma)$ *around* a *with radius* $\gamma > 0$ *such that* $B(a, \gamma) \subset U$ *and* f *is completionally p-continuous on* $B(a, \gamma)$.

We denote the completion of E with respect to p_E by $\hat{E}[p]$, and the extension of p_E over $\hat{E}[p]$ by $\hat{p}_E$. Therefore, each element $\hat{x}$ of $\hat{E}[p]$ is an equivalence class of p-Cauchy sequences $\{x_i\}$, and

$$\hat{p}_E(\hat{x}) = \lim_{i\to\infty} p_E(x_i) \ .$$

It is easy to see that $\hat{p}_E$ defines a norm on $\hat{E}[p]$ and $\hat{E}[p]$ is a Banach space with this norm. This space will be called the *p-completion* of E .

A subset U of E is called a *completionally p-open subset* if there is an open subset $\hat{U}$ in $\hat{E}[p]$ such that $U = E \cap \hat{U}$. Obviously, completionally p-open subsets are p-open.

(5.5). *Let* U *be a completionally p-open subset of* E . *Then, for each* $\hat{a} \in \hat{U}$, *there is a p-Cauchy sequence* $\{a_i\}$ *in* U *such that*

$$\lim_{i\to\infty} \hat{p}_E(a_i-\hat{a}) = 0 \ .$$

Conversely, if $\{a_i\}$ *is a p-Cauchy sequence in* U *and* $\hat{a}$ *is the class containing* $\{a_i\}$, *then* $\hat{a}$ *belongs to the p-closure of* $\hat{U}$ *in* $\hat{E}[p]$.

Proof. Let $\{a_i\}$ be a p-Cauchy sequence contained in $\hat{a}$. Then, since

$$\lim_{i\to\infty} \hat{p}_E(a_i-\hat{a}) = 0 \ ,$$

we have

$$a_i \in \hat{U} \cap E \quad \text{for large } i \ .$$

Conversely, if $\{a_i\}$ is a p-Cauchy sequence contained in U and $\hat{a}$ is the class containing $\{a_i\}$, we have

$$a_i \in \hat{U} \quad \text{and} \quad \lim_{i\to\infty} \hat{p}_E(a_i-\hat{a}) = 0 \ ,$$

which imply that $\hat{a}$ belongs to the closure of $\hat{U}$.

Now we can give a characterization of the completional p-continuity.

(5.6). *Let* U *be a completionally* p*-open subset of* E *. Then a map* $f : U \to F$ *is completionally* p*-continuous on* U *if and only if* f *has a* $\hat{p}$*-continuous extension* $\hat{f} : \overline{\hat{U}} \to \hat{F}[p]$ *.*

Proof. Assume that f is completionally p-continuous on U . We define $\hat{f}$ as follows: for $\hat{a} \in \hat{U}$ we put

$$\hat{f}(\hat{a}) = \lim_{i\to\infty} f(a_i) \text{ in } \hat{F}[p] ,$$

or $\hat{f}(\hat{a})$ is the class containing $\{f(a_i)\}$ for a p-Cauchy sequence $\{a_i\}$ in $\hat{a}$. This is possible because $\{f(a_i)\}$ is also a p-Cauchy sequence by (5.2).

Therefore, in order to show that f can be extended to $\overline{\hat{U}}$, we only need to show that such $\{a_i\}$ can be chosen for every $\hat{a} \in \overline{\hat{U}}$.

If $\hat{a} \in \hat{U}$, such $\{a_i\}$ exists by (5.5).

If $\hat{a} \in \overline{\hat{U}}\backslash\hat{U}$, there is a sequence $\hat{a}_n \in \hat{U}$ such that

$$\lim_{n\to\infty} \hat{a}_n = \hat{a} \text{ in } \hat{E}[p] .$$

Then there are p-Cauchy sequences $\{a_{n,i}\}$ in U such that

$$\lim_{i\to\infty} a_{n,i} = \hat{a}_n \text{ in } \hat{E}[p] ,$$

and also there is a p-Cauchy sequence $\{a_i\}$ in E which is contained in $\hat{a}$. Then

$$\lim_{n\to\infty} \lim_{i\to\infty} p_E(a_{n,i}-a_i) = 0 .$$

Hence there are $\{n_k\}$ and $\{i_k\}$ such that

$$p_E(a_{n_k,i_k}-a_{i_k}) < 1/k .$$

Since $\{a_{i_k}\}$ is p-Cauchy and $\{a_{n_k,i_k}-a_{i_k}\}$ is p-null, $\{a_{n_k,i_k}\}$ is p-Cauchy, and it is contained in U . Furthermore,

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$$\lim_{k\to\infty} a_{n_k,i_k} = \lim_{k\to\infty} a_{i_k} + \lim_{k\to\infty} \left(a_{n_k,i_k} - a_{i_k}\right) = \hat{a} \quad \text{in} \quad \hat{E}[p] .$$

Thus, for any $\hat{a} \in \overline{\hat{U}}$, we can find a p-Cauchy sequence in U which converges to $\hat{a}$ in $\hat{E}[p]$.

Next, to prove the $\hat{p}$-continuity of $\hat{f}$ thus defined, we assume that

$$\lim_{n\to\infty} \hat{p}_E\left(\hat{a}_n - \hat{a}\right) = 0 , \quad \hat{a}_n, \hat{a} \in \overline{\hat{U}} ,$$

and

$$\hat{p}_F\left[\hat{f}\left(\hat{a}_n\right) - \hat{f}(\hat{a})\right] < \delta \quad \text{for all} \quad n ,$$

for some positive number δ. We take p-Cauchy sequences $\{a_{n,i}\}$ and $\{a_i\}$ in U such that

$$\lim_{i\to\infty} a_{n,i} = \hat{a}_n \quad \text{and} \quad \lim_{i\to\infty} a_i = \hat{a} \quad \text{in} \quad \hat{E}[p] .$$

Then these assumptions are equivalent to the following:

$$\lim_{n\to\infty} \lim_{i\to\infty} p_E\left(a_{n,k} - a_i\right) = 0$$

and

$$\lim_{i\to\infty} p_F\left[f\left(a_{n,i}\right) - f\left(a_i\right)\right] > \delta .$$

From the first equality, we can find $\{n_k\}$ and $\left\{i_k^{(1)}\right\}$ such that

$$p_E\left(a_{n_k,i} - a_i\right) < 1/k \quad \text{if} \quad i \geq i_k^{(1)} .$$

Since

$$\lim_{i\to\infty} p_F\left[f\left(a_{n_k,i}\right) - f\left(a_i\right)\right] > \delta$$

from the second inequality, we have $\left\{i_k^{(2)}\right\}$ such that

$$p_F\left[f\left(a_{n_k,i}\right) - f\left(a_i\right)\right] > \delta \quad \text{if} \quad i \geq i_k^{(2)} .$$

Therefore, for $i_k \geq \max\left\{i_k^{(1)}, i_k^{(2)}\right\}$, we have

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$$p_E(a_{n_k, i_k} - a_{i_k}) < 1/k \quad \text{and} \quad p_F[f(a_{n_k, i_k}) - f(a_{i_k})] > \delta \ .$$

This is a contradiction, because $\{a_{n_k, i_k}\}$ is also a p-Cauchy sequence contained in U .

Conversely, suppose that f has a $\hat{p}$-continuous extension, and suppose that $\{x_i\}$ and $\{y_i\}$ are p-Cauchy sequences in U such that

$$\lim_{i \to \infty} p_E(x_i - y_i) = 0 \ .$$

Then there exists $\hat{a} \in \overline{\hat{U}}$ such that

$$\lim_{i \to \infty} x_i = \lim_{i \to \infty} y_i = \hat{a} \ .$$

Hence

$$\lim_{i \to \infty} f(x_i) = \lim_{i \to \infty} f(y_i) = \hat{f}(\hat{a}) \quad \text{in} \quad \hat{F}[p] \ ,$$

which implies

$$\lim_{i \to \infty} p_F[f(x_i) - f(y_i)] = 0 \ .$$

A subset U of E is said to be *completionally* Γ-*open* if it is completionally p-open for every $p \in \Gamma$. Obviously, completionally Γ-open subsets are Γ-open.

Let U be a completionally Γ-open subset of E . Then a map $f : U \to F$ is said to be *completionally* Γ-*continuous* on U if it is completionally p-continuous for every $p \in \Gamma$. Hence f is completionally Γ-continuous if and only if, for each $p \in \Gamma$, it has a $\hat{p}$-continuous extension from $\overline{\hat{U}}$ into $\hat{F}[p]$.

Again, let U be a completionally Γ-open subset of E . A map $f : U \to F$ is said to be k-*times completionally continuously* Γ-*differentiable* or *of class* CC_Γ^k on U if it is of class C_Γ^k and the derivatives

$$f^{(i)} : U \to L_\Gamma^i(E, F) \quad (0 \le i \le k)$$

are completionally Γ-continuous.

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(5.7). *Let* $E, F \in \mathfrak{F}$ *and* F *be sequentially complete. Let* U *be a completionally* Γ*-open subset of* E . *Let* $f : U \to F$ *be* k*-times Gateaux differentiable on* U . *Then* f *is of class* CC_Γ^k *on* U *if and only if, for each* $p \in \Gamma$, f *has a* C^k*-extension*

$$\hat{f} : \overline{\hat{U}} \to \hat{F}[p] .$$

Proof. Since f is k-times Gâteaux differentiable on U , we have

$$f^{(i)} : U \to L^i(E, F) \quad (0 \leq i \leq k) .$$

Assume that f is of class CC_Γ^k ; then each $f^{(i)}$ is completionally Γ-continuous. Hence, for each $p \in \Gamma$, we have a continuous extension

$$\widehat{f^{(i)}} : \overline{\hat{U}} \to L^i(\hat{E}[p], \hat{F}[p]) .$$

In particular, we have a continuous extension

$$\hat{f} : \overline{\hat{U}} \to \hat{F}[p] ,$$

and we shall show that $\widehat{f^{(i)}}$ is the ith derivative map of $\hat{f}$; that is,

$$\hat{f}^{(i)} = \widehat{f^{(i)}} .$$

Now assume that $\widehat{f'}(\hat{a})$ is not the derivative of $\hat{f}$ at $\hat{a}$. Then there is a null sequence $\{\hat{x}_n\}$ in $\hat{E}[p]$ such that $\hat{a} + \hat{x}_n \in \overline{\hat{U}}$ and

$$\hat{p}_E(\hat{x}_n)^{-1}\hat{p}_F[\hat{f}(\hat{a}+\hat{x}_n)-\hat{f}(\hat{a})-\widehat{f'}(\hat{a})(\hat{x}_n)] > \delta \quad \text{for all } n ,$$

for some positive number δ . If $\{x_{n,i}\}$ are p-Cauchy sequences contained in $\hat{x}_n$, this assumption is equivalent to

$$\lim_{n\to\infty} \lim_{i\to\infty} p_E(x_{n,i}) = 0$$

and

$$\lim_{i\to\infty} p_E(x_{n,i})^{-1} p_F[f(a_i+x_{n,i})-f(a_i)-f'(a_i)(x_{n,i})] > \delta ,$$

where $\{a_i\}$ is a p-Cauchy sequence contained in $\hat{a}$.

In exactly the same way as in (5.6), we choose $\{n_k\}$ and $\{i_k\}$ such that

$$\lim_{k\to\infty} p_E(x_{n_k,i_k}) = 0$$

and

$$p_E(x_{n_k,i_k})^{-1} p_F[f(a_{i_k}+x_{n_k,i_k})-f(a_{i_k})-f'(a_{i_k})(x_{n_k,i_k})] > \delta .$$

From the second inequality, together with the mean value theorem, we have

$$p_{(E,F)}[f'(a_{i_k}+\theta_k x_{n_k,i_k})-f'(a_{i_k})] > \delta ,$$

which contradicts the completional continuity of f' .

We can prove the cases of higher derivatives in exactly the same way.

Conversely, assume that there is a C^k-extension

$$\hat{f} : \bar{\hat{U}} \to \hat{F}[p]$$

for every $p \in \Gamma$. Since f is assumed to be k-times Gâteaux differentiable on U , we have a map

$$f' : U \to L(E, F) ,$$

and, for $a \in U$ and $x \in E$,

$$\begin{aligned} f'(a)(x) &= \lim_{\varepsilon\to\infty} \varepsilon^{-1}[f(a+\varepsilon x)-f(a)] \\ &= \lim_{\varepsilon\to\infty} \varepsilon^{-1}[\hat{f}(a+\varepsilon x)-\hat{f}(a)] = \hat{f}'(a)(x) . \end{aligned}$$

In other words,

$$\hat{f}' : U \to L(\hat{E}[p], \hat{F}[p])$$

is a continuous extension of f' . Therefore, f' is completionally p-continuous on U , and this holds for every $p \in \Gamma$.

We can prove the cases of higher derivatives similarly.

We shall call a Γ-manifold of class C^k a *completional* Γ-*manifold of class* C^k or Γ-*manifold of class* CC^k if there is an atlas whose transition maps are all of class CC^k_Γ .

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6. Completional Γ-Lie groups

A Γ-Lie group is said to be *completional* if all the smoothnesses involved in its definition are of class CC_Γ^∞ . In other words, a *completional* Γ-*Lie group* is a topological group G which satisfies the following conditions:

(CΓL.1) G is a completional Γ-manifold of class C^∞ ;

(CΓL.2) the product operation is completionally (σ, Γ)-smooth;

(CΓL.3) the inverse operation is completionally σ-smooth.

Obviously, Omori-Lie groups are completional Gelfand-Lie groups. Conversely, a completional Gelfand-Lie group is a Omori-Lie group if it satisfies additional smoothness conditions corresponding to (0.4), (0.5), and (0.7).

Thus, when M is a compact C^∞-manifold without boundary, the group $D(M)$ of all C^∞-diffeomorphisms on M and various subgroups of $D(M)$ are completional Gelfand-Lie groups. We leave it as a conjecture that $D(M)$ for noncompact M will also be a completional Γ-Lie group for a suitably chosen Γ .

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