

THE DIAMETER OF AN IMMERSED RIEMANNIAN MANIFOLD WITH BOUNDED MEAN CURVATURE

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Abstract

Let M be an n -dimensional complete Riemannian manifold with Ricci curvature bounded from below. Let \bar{M} be an N -dimensional ($N > n$) complete, simply connected Riemannian manifold with nonpositive sectional curvature. We shall prove in this note that if there exists an isometric immersion φ of M into \bar{M} with the property that the immersed manifold is contained in a ball of radius R and that the mean curvature vector H of the immersion has bounded norm $\|H\| < H_0$ ($H_0 > 0$) then $R > H_0^{-1}$.

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Generalizing a result by Aminov (1973), Hasanis and Koutroufiotis proved that if there exists an isometric immersion $\varphi: M \rightarrow E^N$ such that $\|H\| < H_0$ and $\varphi(M) \subset B_R$, then $R \geq H_0^{-1}$, where H is the mean curvature vector of the immersion and B_R a ball of radius R . In this paper we shall generalize the above result considering isometric immersions of M into \bar{M} . To prove the main theorem we need the following

LEMMA 1. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. If a smooth function f on M has an upper bound, then for any $\varepsilon > 0$ there is a point where the Laplacian $\Delta f < \varepsilon$.*

This is obtained easily from Theorem A of Omori (1967). For a proof see Hasanis and Koutroufiotis (1979).

LEMMA 2. Let M be a complete, simply connected Riemannian manifold with nonpositive sectional curvature. Let $\gamma: [0, 1] \rightarrow M$ be a geodesic parametrized proportional to arc length. If X is a Jacobi vector field along γ such that $X(0) = 0$, then

$$\langle \nabla_T X, X \rangle \geq \langle X, X \rangle,$$

where ∇ is the Riemannian connection and T the tangent vector of γ .

PROOF. We consider the function

$$(1) \quad \psi = \langle t \nabla_T X - X, X \rangle.$$

We have

$$\begin{aligned} \frac{d\psi}{dt} &= t \langle \nabla_T \nabla_T X, X \rangle + \langle t \nabla_T X - X, \nabla_T X \rangle \\ (2) \quad &= t \langle -R(X, T)T, X \rangle + t \langle \nabla_T X, \nabla_T X \rangle - \langle X, \nabla_T X \rangle \\ &\geq t \langle \nabla_T X, \nabla_T X \rangle - \langle X, \nabla_T X \rangle. \end{aligned}$$

Since $\langle X, \nabla_T X \rangle^2 \leq \langle X, X \rangle \langle \nabla_T X, \nabla_T X \rangle$ we have

$$(3) \quad \langle \nabla_T X, \nabla_T X \rangle \geq \langle X, \nabla_T X \rangle^2 \langle X, X \rangle \quad \text{for } t > 0.$$

From (1), (2), (3) we get

$$(4) \quad \frac{d\psi}{dt} \geq \frac{\langle X, \nabla_T X \rangle}{\langle X, X \rangle} \psi.$$

We put $h = \langle X, X \rangle^{1/2}$. It follows that

$$\frac{1}{h} \frac{dh}{dt} = \frac{\langle X, \nabla_T X \rangle}{\langle X, X \rangle}.$$

Hence (4) gives

$$\frac{h \frac{d\psi}{dt} - \psi \frac{dh}{dt}}{h} > 0.$$

Thus

$$(5) \quad \frac{d}{dt} \left(\frac{\psi}{h} \right) > 0 \quad \text{for } t > 0.$$

But $\lim_{t \rightarrow 0} \psi/h = 0$ and so from (5) we have $\psi/h > 0$ and hence $\psi > 0$ which implies the desired inequality.

THEOREM. Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let \bar{M} be a complete, simply connected Riemannian manifold with nonpositive sectional curvature. If there exist an isometric immersion $\varphi: M \rightarrow \bar{M}$ such that $\|H\| \leq H_0 = \text{const} > 0$ and $\varphi(M) \subset B_R$, then $R > H_0^{-1}$.

PROOF. Let Q be the centre of the metric ball B_R in \bar{M} . Consider a vector X tangent to M at P and a geodesic $\beta(u)$ in M such that $\beta(0) = P, \beta'(0) = X$. Let $\sigma(t, u), 0 < t < 1$ be a parametrization of the unique geodesic in \bar{M} from Q to $\beta(u)$ (proportional to arc length). Let T, X be the vector fields along σ defined by

$$T = \sigma_* \frac{\partial}{\partial t}, \quad X = \sigma_* \frac{\partial}{\partial u}.$$

It follows easily from this construction that, for fixed u, X is a Jacobi field along the geodesic $t \rightarrow \sigma(t, u)$ with $X(0) = 0$.

Now, we consider a function F on M defined by

$$F(P) = \frac{1}{2} \{d(Q, P)\}^2,$$

where $d(Q, P)$ is the distance from Q to P in \bar{M} . We will estimate the Hessian form $\nabla^2 F$.

We put $f(u) = d(Q, \beta(u)) = \int_0^1 \|T\| dt$. We obtain easily that

$$f'(u) = \frac{1}{\|T\|} \langle T, X \rangle_{t=1}.$$

Since $\|T\| = f(u)$ we get $f(u) \cdot f'(u) = \langle T, X \rangle_{t=1}$, that is

$$F'(u) = \langle T, X \rangle_{t=1}.$$

If $\bar{\nabla}$ is the Riemannian connection of \bar{M} then

$$F''(u) = \langle \bar{\nabla}_X T, X \rangle_{t=1} + \langle T, \bar{\nabla}_X X \rangle_{t=1} = \langle \bar{\nabla}_T X, X \rangle_{t=1} + \langle T, \bar{\nabla}_X X \rangle_{t=1}$$

since $[X, T] = 0$.

Using the Gauss equation, the fact $F''(u) = \nabla^2 F(X, X)$, and Lemma 2 we obtain

$$\nabla^2 F(X, X) \geq \langle X, X \rangle + \langle T, a(X, X) \rangle,$$

where a is the second fundamental form of immersion. Taking the trace of this, we have

$$\Delta F \geq n + n \langle T, H \rangle.$$

But by assumption $|\langle T, H \rangle| < RH_0$. Hence $\Delta F \geq n(1 - H_0 R)$. If we had $R < H_0^{-1}$ then $1 - H_0 R = c > 0$ and so $\Delta F \geq nc = \text{const} > 0$ but this contradicts Lemma 1, hence $R \geq H_0^{-1}$.

COROLLARY. *If M is a complete, simply connected Riemannian manifold with nonpositive sectional curvature, then a complete minimal submanifold of M , whose Ricci curvature is bound from below, is extrinsically unbounded.*

References

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