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The Poincaré–Deligne Polynomial of Milnor Fibers of Triple Point Line Arrangements is Combinatorially Determined

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Abstract. Using a recent result by S. Papadima and A. Suciu, we show that the equivariant Poincaré–Deligne polynomial of the Milnor fiber of a projective line arrangement having only double and triple points is combinatorially determined.

1 Introduction

Let $\mathcal{A}$ be an arrangement of d hyperplanes in $\mathbb{P}^n$, with $d \geq 2$, given by a reduced equation $Q(x) = 0$. Consider the corresponding complement M defined by $Q(x) \neq 0$ in $\mathbb{P}^n$, and the global Milnor fiber F defined by $Q(x) - 1 = 0$ in $\mathbb{C}^{n+1}$ with monodromy action $h: F \to F$, $h(x) = \exp(2\pi i/d) \cdot x$. We refer the reader to [17] for the general theory of hyperplane arrangements.

The following are basic open questions in this area, a positive answer for any question in this list implying the same for the previous ones.

(a) Are the Betti numbers $b_j(F)$ combinatorially determined, *i.e.,* determined by the intersection lattice $L(\mathcal{A})$?
(b) Are the monodromy operators $h^j: H^j(F) \to H^j(F)$ combinatorially determined?
(c) Is the equivariant Poincaré–Deligne polynomial $PD^{\mu_d}(F)$ of F coming from the monodromy action combinatorially determined? Here μ_d is the multiplicative group of d-th roots of unity, and the definition of $PD^{\mu_d}(F)$ is recalled in the next section.

On the positive side, it was shown by N. Budur and M. Saito in [2] that the spectrum $Sp(\mathcal{A})$ of $\mathcal{A}$, whose definition is also recalled in the next section, is combinatorially determined.

We assume in the sequel that $n = 2$ and that the line arrangement $\mathcal{A}$ has only double and triple points. Then a recent result of S. Papadima and A. Suciu [15] shows that the answer to question (b) is positive. More precisely, they have introduced a combinatorial invariant $\beta_3(\mathcal{A}) \in \{0, 1, 2\}$ of the line arrangement $\mathcal{A}$ such that the multiplicity of a cubic root of unity $\lambda \neq 1$ as an eigenvalue for h^1 is exactly $\beta_3(\mathcal{A})$.

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The main result of this note, answering a question raised by Suciu, is the following.

Theorem 1.1 *Let $\mathcal{A}$ be an arrangement of d lines in $\mathbb{P}^2$ such that $\mathcal{A}$ has only double and triple points. Then the equivariant Poincaré–Deligne polynomial $PD^{\mu_d}(F;u,v,t)$ of F coming from the monodromy action is determined by the number d of lines in $\mathcal{A}$, the number $n_3(\mathcal{A})$ of triple points in $\mathcal{A}$ and the Papadima–Suciu invariant $\beta_3(\mathcal{A})$.*

In particular, question (c) has a positive answer in this case. This is rather surprising, given the complexity of the mixed Hodge structure on the cohomology of the Milnor fiber F, as seen from Propositions 3.1 and 3.3. The very explicit formulas given in these two propositions apply to certain monodromy eigenvalues for arbitrary line arrangements; see Remarks 3.2 and 3.4.

For a possible application to the study of some (singular) projective surfaces, see Remark 3.7. Relations to the superabundance or the defect of some linear systems passing through the triple points of $\mathcal{A}$ are described in Remark 3.8.

Note also that there are very few examples of nonisolated (quasi-homogeneous) hypersurface singularities $(X,0)$ for which both the monodromy and the MHS on the corresponding Milnor fibers are well understood, even though the isolated quasi-homogeneous case was settled by J. Steenbrink [18] a long time ago.

The case of a hyperplane arrangement in $\mathbb{P}^{3k-1}$, which is obtained by taking a product $\mathcal{A}_1\times\mathcal{A}_2\times\cdots\times\mathcal{A}_k$ of k line arrangements $\mathcal{A}_j$ having only double and triple points, can now be treated using the results in this note and [5, Theorem 1.4].

In the last section we prove the following related result.

Proposition 1.2 *Let $C : Q = 0$ be a degree d reduced curve in the projective plane $\mathbb{P}^2$, and let $F : Q-1=0$ be the associated Milnor fiber in $\mathbb{C}^3$. Then the equivariant Poincaré–Deligne polynomial $PD^{\mu_d}(F;u,v,t)$ of F coming from the monodromy action is determined by its specialization, the Hodge–Deligne polynomial*

$$HD^{\mu_d}(F;u,v) = PD^{\mu_d}(F;u,v,-1).$$

Since the Hodge–Deligne polynomial (or rather a compactly supported version of it, is additive; see, for instance, [7]), this result might be used in some situations to compute these polynomials. It is an open question whether such a result holds in higher dimensions, even for the hyperplane arrangements.

For similar non-cancellation properties in the case of braid arrangements A_3 and A_4, see [8, Section 6].

2 Equivariant Hodge–Deligne and Poincaré–Deligne Polynomials and Spectra

Recall that if X is a quasi-projective variety over $\mathbb{C}$, one can consider the Deligne mixed Hodge structure (MHS) on the rational cohomology groups $H^*(X,\mathbb{Q})$ of X. We refer to the reader [16] for general notions and results concerning the MHS.

Since this MHS is functorial with respect to algebraic mappings, if a finite group Γ acts algebraically on X, each of the graded pieces

$$H^{p,q}(H^j(X,\mathbb{C})) := Gr_F^p Gr_{p+q}^W H^j(X,\mathbb{C}) \tag{2.1}$$

becomes a Γ-module in the usual functorial way, and these modules are the building blocks of the Hodge-Deligne polynomial $HD^\Gamma(X;u,v) \in R(\Gamma)[u,v]$, defined by

$$HD^\Gamma(X;u,v) = \sum_{p,q} E^{\Gamma;p,q}(X)u^p v^q,$$

where $E^{\Gamma;p,q}(X) = \sum_j (-1)^j H^{p,q}(H^j(X,\mathbb{C})) \in R(\Gamma)$. One can consider an even finer (and hence harder to determine) invariant, namely the equivariant Poincaré–Deligne polynomial

$$PD^\Gamma(X;u,v,t) = \sum_{p,q,j} H^{p,q}\big(H^j(X,\mathbb{C})\big)u^p v^q t^j \in R_+(\Gamma)[u,v,t].$$

Clearly, one has $PD^\Gamma(X;u,v,-1) = HD^\Gamma(X;u,v)$.

The case of interest to us is when $\Gamma = \mu_d$ and the action on F is determined by

$$\exp(2\pi i/d)\cdot x = h^{-1}(x).$$

The reason to use h^{-1} instead of h is either functorial (*i.e.,* to really have a group action when Γ is not commutative, see [8]) or geometrical, as explained in [10], in order to get results compatible with those in [2], which we survey below. Recall that the spectrum of a hyperplane arrangement $\mathcal{A} \subset \mathbb{P}^n$ is the polynomial

$$Sp(\mathcal{A}) = \sum_{\alpha\in\mathbb{Q}} n_\alpha \quad t^\alpha,$$

with coefficients given by

$$n_\alpha = \sum_j (-1)^{j-n} \dim\, Gr_F^p \widetilde{H}^j(F,\mathbb{C})_\lambda,$$

where $p = \lfloor n+1-\alpha \rfloor$, $\lambda = \exp(-2i\pi\alpha)$, with $\widetilde{H}^j(F,\mathbb{C})_\lambda = H^j(F,\mathbb{C})_\lambda$ (eigenspaces with respect to the action of $(h^j)^{-1}$ as explained above) for $j \neq 0$, $\widetilde{H}^0(F,\mathbb{C})_\lambda = 0$ and $\lfloor y \rfloor := \max\{k \in \mathbb{Z} \mid k \le y\}$. It is easy to see that $n_\alpha = 0$ for $\alpha \notin (0, n+1)$.

Theorem 3 in [2] implies the following result.

Theorem 2.1 *Let $\mathcal{A}$ be an arrangement of d lines in $\mathbb{P}^2$ such that $\mathcal{A}$ has only double and triple points. Let $n_3(\mathcal{A})$ denote the number of triple points in $\mathcal{A}$. Then $n_\alpha = 0$ if $\alpha d \notin \mathbb{Z}$, and for $\alpha = \frac{j}{d} \in]0,1]$ with $j \in [1,d] \cap \mathbb{Z}$, the following hold:*

$$n_\alpha = \binom{j-1}{2} - n_3(\mathcal{A})\binom{\lceil 3j/d \rceil - 1}{2},$$

$$n_{\alpha+1} = (j-1)(d-j-1) - n_3(\mathcal{A})(\lceil 3j/d \rceil - 1)(3 - \lceil 3j/d \rceil),$$

$$n_{\alpha+2} = \binom{d-j-1}{2} - n_3(\mathcal{A})\binom{3-\lceil 3j/d \rceil}{2} - \delta_{j,d},$$

where $\lceil y \rceil := \min\{k \in \mathbb{Z} \mid k \ge y\}$, and $\delta_{j,d} = 1$ if $j = d$ and 0 otherwise.

In particular, the spectrum $Sp(\mathcal{A})$ is determined by the number d of lines in $\mathcal{A}$ and the number $n_3(\mathcal{A})$ of triple points.

3 The Proof of Theorem 1.1

Let us consider the cohomology groups $H^j(F,\mathbb{Q})$ one by one and discuss the corresponding MHS and monodromy action. The group $H^0(F,\mathbb{C})$ is clearly one dimensional, of type $(0,0)$, and the monodromy action is trivial, *i.e.*, $H^0(F,\mathbb{C})=H^0(F,\mathbb{C})_1$.

The next group $H^1(F,\mathbb{Q})$ is already more interesting. It has a direct sum decomposition

$$H^1(F,\mathbb{Q})=H^1(F,\mathbb{Q})_1\oplus H^1(F,\mathbb{Q})_{\neq 1}$$

in the category of MHS. The fixed part under the monodromy $H^1(F,\mathbb{Q})_1$ is isomorphic to the cohomology group of the projective complement, namely $H^1(M,\mathbb{Q})$, and hence it has dimension $d-1$ and is a pure Hodge–Tate structure of type $(1,1)$.

The other summand $H^1(F,\mathbb{Q})_{\neq 1}$ is always a pure Hodge structure of weight 1; see [3,9] for two distinct proofs. Moreover, in the case when only double and triple points occur in $\mathcal{A}$, the second summand corresponds to cubic roots of unity and it can be non zero only when d is divisible by 3; see, for instance, Remark 3.2. By combining Papadima–Suciu results in [15] with (the proof) of [6, Theorem 1] (see also [3, Theorem 2] for a more general result and Remark 3.8 for additional information), one gets

$$\begin{aligned} h^{1,0}(H^1(F))_{\gamma'} &= h^{0,1}(H^1(F))_{\gamma}=\beta_3(\mathcal{A}),\\ h^{1,0}(H^1(F))_{\gamma} &= h^{0,1}(H^1(F))_{\gamma'}=0, \end{aligned} \tag{3.1}$$

where $\beta=1/3$, $\gamma=\exp(-2\pi i\beta)$, $\beta'=2/3$, $\gamma'=\exp(-2\pi i\beta')=\overline{\gamma}$. Here and in the sequel we use the notation $h^{p,q}(H^j(F))_\lambda$ to denote the multiplicity of the 1-dimensional μ_d-representation r_λ sending $\exp(2\pi i/d)$ to $\lambda\in\mu_d\subset\mathbb{C}^*=\mathrm{Aut}(\mathbb{C})$ in the μ_d-module $H^{p,q}(H^j(F,\mathbb{C}))$ defined in (2.1). Note that one has

$$\dim Gr_F^pH^j(F,\mathbb{C})_\lambda=\sum_{q\geq j-p}h^{p,q}\big(H^j(F)\big)_\lambda,$$

by the general properties of MHS, F being smooth.

Determination of the equivariant Poincaré–Deligne polynomial $PD^{\mu_d}(F)$ of F is clearly equivalent to determination of all the equivariant mixed Hodge numbers $h^{p,q}(H^j(F))_\lambda$. Until now, we have done this for $j=0$ and $j=1$.

It remains to treat the case $j=2$, which is the most difficult. Again, we have a direct sum decomposition

$$H^2(F,\mathbb{Q})=H^2(F,\mathbb{Q})_1\oplus H^2(F,\mathbb{Q})_{\neq 1}$$

in the category of MHS. The fixed part under the monodromy $H^2(F,\mathbb{Q})_1$ is isomorphic to the cohomology group of the projective complement, namely $H^2(M,\mathbb{Q})$ and hence has dimension $b_2(M)$ and pure Hodge–Tate type $(2,2)$. Since the Euler characteristic $\chi(M)=b_0(M)-b_1(M)+b_2(M)$ can be computed from the combinatorics, it follows that

$$b_2(M)=\binom{d-1}{2}-n_3(\mathcal{A}).$$

We can also write $H^2(F,\mathbb{Q})_{\neq 1}$ as a direct sum of two MHS, namely $H^2(F,\mathbb{Q})_{\neq 1} = H \oplus H'$, where H corresponds to the eigenvalues of h^2 that are cubic roots of unity different from 1, and H' corresponds to all the other eigenvalues.

Proposition 4.1 in [5] implies that H' is a pure Hodge structure of weight 2, *i.e.,* $h^{p,q}(H^2(F))_\lambda = 0$ for $p+q \neq 2$ and λ not a cubic root of unity. On the other hand, [7, Theorem 1.3] implies that the only weights possible for H are 2 and 3, hence $h^{p,q}(H^2(F))_\lambda = 0$ for $p+q \notin \{2,3\}$ and λ a cubic root of unity.

Now we explicitly determine the equivariant mixed Hodge numbers $h^{p,q}(H^2(F))_\lambda$ for $\lambda \neq 1$, the case $\lambda = 1$ already being clear by the above discussion. The above discussion implies also the following result.

Proposition 3.1 *Let $\mathcal{A}$ be an arrangement of d lines in $\mathbb{P}^2$ such that $\mathcal{A}$ has only double and triple points. Let $n_3(\mathcal{A})$ denote the number of triple points in $\mathcal{A}$. Assume that $\lambda = \exp(-2\pi\alpha)$, with $0 < \alpha = j/d < 1$, is not a cubic root of unity. Then one has $h^{2,0}(H^2(F))_\lambda = n_\alpha$, $h^{1,1}(H^2(F))_\lambda = n_{\alpha+1}$ and $h^{0,2}(H^2(F))_\lambda = n_{\alpha+2}$, where the integers $n_\alpha, n_{\alpha+1}, n_{\alpha+2}$ are given by the formulas in Theorem 2.1.*

Remark 3.2 Let $\mathcal{A}$ be any essential arrangement of d lines in $\mathbb{P}^2$; *i.e.,* $\mathcal{A}$ is not a pencil of lines through a point. Then the formulas given in Proposition 3.1 hold for any $\lambda \in \mu_d$ such that there is a line $L \in \mathcal{A}$ with $\lambda^k \neq 1$ whenever there is a point of multiplicity k in $\mathcal{A}$ situated on L. Indeed, this last condition is known to imply that $H^1(F)_\lambda = 0$; see [13]. In such a case, the integers n_α are not given by the formulas in Theorem 2.1, but they are described in [2, Theorem 3].

Now we consider the case of the cubic roots of unity $\gamma = \exp(-2\pi i\beta)$ and $\gamma' = \exp(-2\pi i\beta')$ introduced above. They can be eigenvalues of h^2 only when d is divisible by 3.

Proposition 3.3 *Let $\mathcal{A}$ be an arrangement of d lines in $\mathbb{P}^2$ such that $\mathcal{A}$ has only double and triple points. Let $n_3(\mathcal{A})$ denote the number of triple points in $\mathcal{A}$ and suppose that d is divisible by 3. Then one has the following:*

(i) $h^{2,0}(H^2(F))_\gamma = h^{0,2}(H^2(F))_{\gamma'} = n_{\beta'+2}$;
(ii) $h^{1,1}(H^2(F))_\gamma = h^{1,1}(H^2(F))_{\gamma'} = n_{\beta'+2} + n_{\beta'+1} - n_\beta + \beta_3(\mathcal{A})$;
(iii) $h^{0,2}(H^2(F))_\gamma = h^{2,0}(H^2(F))_{\gamma'} = n_{\beta'+2} + n_{\beta'+1} + n_{\beta'} - n_\beta - n_{\beta+1} + \beta_3(\mathcal{A})$;
(iv) $h^{2,1}(H^2(F))_\gamma = h^{1,2}(H^2(F))_{\gamma'} = n_\beta - n_{\beta'+2}$;
(v) $h^{1,2}(H^2(F))_\gamma = h^{2,1}(H^2(F))_{\gamma'} = n_{\beta+1} + n_\beta - n_{\beta'+1} - n_{\beta'+2} - \beta_3(\mathcal{A})$.

Here, $\beta = 1/3$, $\beta' = 2/3$ and the various integers n_η are given by the formulas in Theorem 2.1.

Proof In the case $\alpha = \beta$, the definition of the spectrum and the above discussion on the mixed Hodge properties of the cohomology group of the Milnor fiber F yield the following relations:

(a) $n_\beta = h^{2,0}(H^2(F))_\gamma + h^{2,1}(H^2(F))_\gamma$;
(b) $n_{\beta+1} = h^{1,1}(H^2(F))_\gamma + h^{1,2}(H^2(F))_\gamma$ (use (3.1));
(c) $n_{\beta+2} = h^{0,2}(H^2(F))_\gamma - h^{0,1}(H^1(F))_\gamma = h^{0,2}(H^2(F))_\gamma - \beta_3(\mathcal{A})$ (use (3.1) again).

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Similarly, for $\alpha = \beta'$, we get the following.

(a) $n_{\beta'} = h^{2,0}(H^2(F))_{\gamma'} + h^{2,1}(H^2(F))_{\gamma'}$;
(b) $n_{\beta'+1} = h^{1,1}(H^2(F))_{\gamma'} + h^{1,2}(H^2(F))_{\gamma'} - \beta_3(\mathcal{A})$ (use (3.1));
(c) $n_{\beta'+2} = h^{0,2}(H^2(F))_{\gamma'}$ (use (3.1) again).

The proof is completed by using the obvious equality

$$h^{p,q}(H^2(F))_\lambda = h^{q,p}(H^2(F))_{\overline{\lambda}},$$

obtained by taking the complex conjugation. ∎

Remark 3.4 Let $\mathcal{A}$ be any essential arrangement of d lines in $\mathbb{P}^2$; *i.e.*, $\mathcal{A}$ is not a pencil of lines through a point. Then the formulas given in Proposition 3.3 where we take $\beta_3(\mathcal{A}) = 0$ clearly hold for any $\lambda \in \mu_d$ such that $H^1(F)_\lambda = 0$, with the integers n_α given by [2, Theorem 3].

Moreover, it is clear that Propositions 3.1 and 3.3 imply Theorem 1.1. They also yield the following corollary.

Corollary 3.5 *Let $\mathcal{A}$ be an arrangement of d lines in $\mathbb{P}^2$ such that $\mathcal{A}$ has only double and triple points. Then the dimensions* $\dim Gr_2^W H^2(F,\mathbb{Q})$ *and* $\dim Gr_3^W H^2(F,\mathbb{Q})$ *of the graded quotients with respect to the weight filtration W depend both on the Papadima–Suciu invariant $\beta_3(\mathcal{A})$.*

Example 3.6 Let $\mathcal{A}$ be the Ceva (or Fermat) arrangement of $d = 9$ lines in $\mathbb{P}^2$ given by the equation

$$Q(x,y,z) = (x^3-y^3)(x^3-z^3)(y^3-z^3).$$

Then the Papadima–Suciu invariant $\beta_3(\mathcal{A})$ is equal to 2; there are $n_3(\mathcal{A}) = 12$ triple points, and a direct computation gives the following formula for the spectrum

$$\begin{aligned} Sp(\mathcal{A}) &= t^{1/3} + 3t^{4/9} + 6t^{5/9} + 10t^{2/3} + 3t^{7/9} + 9t^{8/9} + 16t + 6t^{11/9} + 10t^{4/3} \\ &\quad - 2t^{5/3} + 6t^{16/9} - 8t^2 + 9t^{19/9} + 3t^{20/9} - 2t^{7/3} + 6t^{22/9} + 3t^{23/9} + t^{8/3} - t^3. \end{aligned}$$

Proposition 3.3 implies

$$h^{2,1}(H^2(F))_\gamma = h^{1,2}(H^2(F))_{\gamma'} = n_{1/3} - n_{8/3} = 1 - 1 = 0$$

and

$$\begin{aligned} h^{1,2}(H^2(F))_\gamma = h^{2,1}(H^2(F))_{\gamma'} &= n_{4/3} + n_{1/3} - n_{5/3} - n_{8/3} - \beta_3(\mathcal{A}) \\ &= 10 + 1 + 2 - 1 - 2 = 10. \end{aligned}$$

These values correct a misprint in [7, p. 244] and confirm the computations done by P. Bailet in [1]. This example also shows that the integers n_η may be strictly negative.

Remark 3.7 Let $\mathcal{A}$ be an arrangement of d lines in $\mathbb{P}^2$ such that $\mathcal{A}$ has only double and triple points. Then, in view of [7, Theorem 1.1], the results in Propositions 3.1 and 3.3 can be used to describe the μ_d-action on the cohomology of the associated surface

$$X_Q : Q(x,y,z) - t^d = 0$$

in $\mathbb{P}^3$, which is a singular compactification of the Milnor fiber F.

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Remark 3.8 Let $\mathcal{A}$ be an arrangement of d lines in $\mathbb{P}^2$ such that $\mathcal{A}$ has only double and triple points and $d = 3m$ for some integer m. Let $T \subset \mathbb{P}^2$ be the set of triple points in $\mathcal{A}$. If $S = \mathbb{C}[x, y, z]$ denotes the graded ring of polynomials in x, y, z, consider the evaluation map $\rho: S_{2m-3} \to \mathbb{C}^T$ obtained by picking up a representative s_t in $\mathbb{C}^3$ for each point $t \in T$ and sending a homogeneous polynomial $h \in S_{2m-3}$ to the family $(h(s_t))_{t \in T}$.

Then [3, Theorem 2] and the discussion following it imply the key formula (3.1). This can be reformulated as $\beta_3(\mathcal{A}) = \dim(\operatorname{Coker}\rho)$, and the last integer is by definition the *superabundance* or the *defect* $S_{2m-3}(T)$ of the finite set of points T with respect to the polynomials in S_{2m-3}. Since by the work of Papadima and Suciu we know that $\beta_3(\mathcal{A}) \in \{0, 1, 2\}$, this gives a very strong restriction on the position of the triple points in such a line arrangement. For other relations to algebraic geometry of a similar flavor, we refer the reader to [11, 12, 14].

4 The Proof of Proposition 1.2

We follow the notation from the previous section; in particular, M denotes the complement of C in $\mathbb{P}^2$ given by $Q \neq 0$. To prove Proposition 1.2, we have to check whether for each character r_λ of μ_d, its coefficient in $PD^{\mu_d}(F; u, v, t)$ (which is a polynomial $c_\lambda(u, v, t)$) can be recovered from the polynomial $c_\lambda(u, v, -1)$. In other words, the monomials in u, v coming from the various cohomology groups $H^j(F)$ should not undergo any simplification, and their degree should tell from which cohomology group they come.

Consider first the trivial character r_1. Then $H^0(F)$ contributes to the coefficient $c_1(u, v, t)$ by 1 and $H^1(F)$ contributes by a multiple of the monomial uvt, since $H^1(F)_1 = H^1(M)$ is still of pure type $(1, 1)$ in this more general setting. To see this, one can use the Gysin sequence

$$0 = H^1(\mathbb{P}^2 \smallsetminus \Sigma) \longrightarrow H^1(M) \longrightarrow H^0(C \smallsetminus \Sigma)(-1) \longrightarrow \cdots$$

with Σ denoting the set of singular points of the curve C. The group $H^2(F)_1 = H^2(M)$ has weights at least 2, since M is smooth. On the other hand, the elements of weight 2 are those in the image of the morphism

$$i^*: H^2(\mathbb{P}^2) \longrightarrow H^2(M)$$

induced by the inclusion $i: M \to \mathbb{P}^2$, since $\mathbb{P}^2$ is a compactification of M. But this morphism is trivial, since $H^2(\mathbb{P}^2, \mathbb{Q})$ is spanned by the first Chern class of the line bundle $\mathcal{O}(d)$ and the restriction $\mathcal{O}(d)|M$ is trivial. Indeed, it admits Q as a section without zeroes. It follows that all the classes in $H^2(M)$ have in fact weights 3 and 4, and hence we can recover $c_1(u, v, t)$ from $c_1(u, v, -1)$.

Now consider a nontrivial character r_λ, *i.e.*, $\lambda \neq 1$. Then $H^0(F)$ contributes to the coefficient $c_\lambda(u, v, t)$ by 0 and $H^1(F)$ contributes by a linear form in ut, vt, since $H^1(F)_{\neq 1}$ is still of pure of weight 1 in this more general setting; see [3, Theorem 1.5] or [9, Theorem 4.1]. The contribution of $H^2(F)$ to $c_\lambda(u, v, t)$ is by monomials of the form $u^a v^b t^2$ with $a + b \geq 2$, since F is a smooth variety. This implies again that we can recover $c_\lambda(u, v, t)$ from $c_\lambda(u, v, -1)$, which ends the proof of Proposition 1.2. ∎

Remark 4.1 Note that the information contained in the polynomial $Sp(\mathcal{A})$ is equivalent to the information contained in the specialization $HD^{\mu_d}(F;u,1)$; see [8]. However, even if $Sp(\mathcal{A})$ is known to be combinatorially determined by [2], it is an open question if the same holds for the Hodge–Deligne polynomial $HD^{\mu_d}(F;u,v)$ of the Milnor fiber of a hyperplane arrangement. Moreover, the specialization $HD^{\mu_d}(F;u,1)$ does not determine the Hodge-Deligne polynomial $HD^{\mu_d}(F;u,v)$, even in the case of a line arrangement $\mathcal{A}$ having only double and triple points, since $Sp(\mathcal{A})$ does not determine the Papadima–Suciu invariant $\beta_3(\mathcal{A})$ (which cancels out when we set $v=1$ in $HD^{\mu_d}(F;u,v)$). For an explicit example, we refer the reader to [4, Examples 5.4 and 5.5], where the realizations of the configurations $(9_3)_1$ and $(9_3)_2$ are shown to have distinct $b_1(F)$'s. They have the same spectra by Theorem 2.1, having the same number of lines and triple points.

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