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On BP<1>*(K(Z, 3); Z/p)

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Introduction. In this paper we study the inverse limit cohomology $h^*(K(Z, 3))$ of an Eilenberg-MacLane object $K(Z, 3)$ for certain cohomology theories h. Our main result gives a complete description of all non-trivial differentials of the Atiyah-Hirzebruch spectral sequence (AHSS) $H^*(X;h^*(pt)) \Rightarrow h^*(X)$ for $X = K(Z, 3)$ and h either of the complex K-theories $K^*(\ ;Z/p)$ and $K^*(\ ;Z_{(p)})$. This is achieved inductively using the finite symmetric product spaces SP^kS^3, $k = p^r$. Identification of cycles and boundaries of each non-trivial differential leads to an explicit description of $BP<1>^*(K(\mathbf{Z}, 3); \mathbf{Z}/p)$ and some information about $BP<1>^*(K(\mathbf{Z}, 3))$.

1. The ring $H^*(SP^{p^r}S^3; \mathbf{Z}_{(p)})$. Here we indicate how to obtain the ring $H^*(SP^{p^r}S^3;\mathbf{Z}_{(p)})$ in terms of generators and relations from the well known results of Serre, Cartan and Nakaoka. First, there are the mod p cohomology rings of the infinite symmetric product $SP^\infty S^3$ (which is a $K(\mathbf{Z}, 3)$ by [**6**]).

1.1 ([**11**]) $H^*(SP^\infty S^3; \mathbf{Z}/2) \simeq \mathbf{Z}/2[u_i]_{i \geqq 0}$.

1.2 ([**5**]) For p an odd prime

$$H^*(SP^\infty S^3; \mathbf{Z}/p) \simeq E(u_i)_{i \geqq 0} \otimes \mathbf{Z}/p[v_j]_{j \geqq 1}.$$

In 1.1 and 1.2 u_0 is the fundamental class, u_i is $Sq^{2^i}u_{i-1}$ or $P^{p^{i-1}}u_{i-1}$, $i \geqq 1$, according as $p = 2$ or $p > 2$, $v_j = \beta_p u_j$, and Sq^i, P^i and β_p are the usual Steenrod and Bockstein operations. Then by [**10**] (p any prime)

1.3 $$H^*(SP^{p^r}S^3; \mathbf{Z}/p) \simeq H^*(SP^\infty S^3; \mathbf{Z}/p)/\ker i_r^*$$

where $i_r: SP^{p^r}S^3 \rightarrow SP^\infty S^3$ is the standard axial inclusion. Nakaoka further describes the ideal $\ker i_r^*$ in terms of the Serre-Cartan generators as follows: assign the generators u_0, u_i, v_i p-rank 1, p^i, p^i respectively, and any monomial $x_1x_2 \dots . x_k$ in these generators p-rank the sum of the p-ranks of its factors. Then $\ker i_r^*$ is the ideal $\mathfrak{a}_{p^r}$ generated by all monomials of

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p-rank $> p^r$. This fact together with 1.1, 1.2 describe the ring $H^*(SP^{p^r}S^3; Z/p)$ in terms of generators and relations.

From [**5**, n°11, Theorem 1] $H^*(K(\mathbf{Z}, 3); \mathbf{Z})$, and hence also $H^*(K(\mathbf{Z}, 3); \mathbf{Z}_{(p)})$, has no element of order p^2 for any prime p. Thus the Bockstein exact triangle defined by the coefficient sequence

$$\mathbf{Z}_{(p)} \xrightarrow{p} \mathbf{Z}_{(p)} \xrightarrow{r_p} \mathbf{Z}/p$$

implies that the reduction mod p homomorphism

$$H^*(SP^{p^r}S^3; \mathbf{Z}_{(p)}) \xrightarrow{r_p} H^*(SP^{p^r}S^3; \mathbf{Z}/p)$$

is a monomorphism in dimensions > 3. This makes it possible (and elementary) to extract the following generators-and-relations description of the ring $H^*(SP^{p^r}S^3;\mathbf{Z}_{(p)})$ from the preceding discussion.

If we set $u_I = u_{i_0} u_{i_1} \dots u_{i_s}$ and $v_J{}^N = v_{j_1}^{n_1} v_{j_2}^{n_2} \dots v_{j_t}^{n_t}$, then the standard basis for $H^*(SP^\infty S^3; \mathbf{Z}/p)$, p odd, consists of the set of monomials

1.4 $\{u_I v_J{}^N| \quad 0 \leqq i_0 < i_1 < \dots < i_s, 1 \leqq j_1 < j_2 < \dots < j_t, n_k \geqq 0\}$

while the standard basis for $H^*(SP^\infty S^3; \mathbf{Z}/2)$ is

1.5 $\{u_I{}^N | 0 \leqq i_0 < i_1 < \dots < i_s, n_k \geqq 0\}$.

Since $\beta_2 u_i = u^2_{i-1}$, $i \geqq 1$, 1.4 also describes the standard basis for $H^*(SP^\infty S^3; \mathbf{Z}/2)$ if we set $v_i = u^2_{i-1}$, $i \geqq 1$.

Let

$$S = \{J = (j_1, j_2, \dots, j_t) | \quad 1 \leqq j_i < j_2 < \dots < j_t, t \geqq 0\}.$$

$t = 0$ refers to the empty sequence. Define

$$u_{(I)} = \beta_{(p)} u_I, \quad I \in S$$

$$u_{(I, J(l))} = \beta_{(p)} (u_I u_{J(l)}), \quad J \in S.$$

Here $\beta_{(p)}$ is the Bockstein associated to the short exact sequence

$$\mathbf{Z}_{(p)} \xrightarrow{p} \mathbf{Z}_{(p)} \to \mathbf{Z}/p,$$

and $J(l)$ is the sequence obtained from J by omitting j_l.

PROPOSITION. *$H^*(SP^\infty S^3; \mathbf{Z}_{(p)})$ is the ring generated by u_0 (the 3-dimensional fundamental class), $u_{(I)}$, v_j, $I \in S - \{\emptyset\}$, $j \geqq 1$, with relations*

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(i) $pu_{(I)} = 0 = pv_j$,

(ii) $\sum_{l=1}^{s} (-1)^{l-1} u_{J(l)} v_{j_l} = 0, \quad J \in S - \{\emptyset\}$,

(iii) $u_{(I)} u_{(J)} = \sum_{l=1}^{s} (-1)^{l-1} u_{(I, J(l))} v_{j_l}, \ I, J \in S - \{\emptyset\}$.

Comments. (i) is clear since reduction mod p is monic in dimensions > 3. (ii) is easy from $\beta_p \beta_p u_I = 0$. (iii) can be proven via a straightforward induction (on $s, t \geqq 2$). A proof of this proposition (which we omit) can be given by a counting argument based on the following observation.

1.6. *The* $\mathbf{Z}/p$*-vector space* $\sum_{i>3} H^i(SP^\infty S^3; \mathbf{Z}_{(p)})$ *has a basis given by*

$$\{u_o^{\epsilon} u_{(I)}^{\epsilon'} v_J^N\},$$

where $\epsilon, \epsilon' = 0$ *or* 1 *and* $I, J \in S$ *satisfy one of* a) $I = \emptyset$ b) $J = \emptyset$ c) $I \neq \emptyset \neq J$ *and there exists* $i \in I$ *with* $i \leqq j$ *for all* $j \in J$.

The crucial point is that relation (ii) enables one to express $u_{(I)} v_J$ as a sum of terms $u_{(K)} v_L$, K, L satisfying c) when I, J fail to satisfy c).

Of course, one obtains the ring $H^*(SP^{p^r} S^3; \mathbf{Z}_{(p)})$ by truncating the ideal $\mathfrak{a}_{p^r}$.

2. $K^*(SP^{p^r} S^3; G)$.

THEOREM 2.1. *For any* $r \geqq 0$ and any prime p, $\tilde{K}^i (SP^{p^r} S^3; \mathbf{Z}_{(p)}) \simeq \mathbf{Z}_{(p)}$ *or* 0 *according as* $i = 1$ *or* 0.

THEOREM 2.2. *For any* $r \geqq 0$ *and any prime* p, $\tilde{K}^i (SP^{p^r} S^3; \mathbf{Z}/p) \simeq \mathbf{Z}/p$ *or* 0 *according as* $i = 1$ *or* 0.

The equivalence of these two results is an immediate consequence of the short exact sequence (a Universal Coefficient Theorem [**3**])

$$0 \to \tilde{K}^i (X; \mathbf{Z}_{(p)}) \otimes \mathbf{Z}/p \to \tilde{K}^i(X; \mathbf{Z}/p) \to \tilde{K}^{i+1}(X; \mathbf{Z}_{(p)}) * \mathbf{Z}/p \to 0.$$

2.1 and 2.2 admit equivalent statements in terms of the corresponding Atiyah-Hirzebruch spectral sequences. We treat the two cases, p odd and $p = 2$, separately. Also we shall write $d_r x = y$ when in fact $d_r x = Ny$ for some integer $N \not\equiv 0 \bmod p$, with the single exception

$$d_{2(p-1)+1} u_0 = -v_1;$$

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then u_2' in 2.3′ below becomes a cycle.

THEOREM 2.3′. *Let* $r \geqq 0$ *and* p *be an odd prime. Set*

$$u_1' = u_1,\ u_2', = u_2 + u_0 v_1{}^{p-1} \quad \text{and} \quad u_R' = u_{k-1} v_{k-1}^{p-1}, \qquad 3 \leqq k \leqq r.$$

Then the nontrivial differentials of the AHSS

$$H^*(Y_r;\mathbf{Z}/p) \Rightarrow K^*(Y_r;\mathbf{Z}/p), \quad Y_r = SP^{p^r}S^3,$$

are completely determined by

$$d_{2(p-1)+1}u_i = \begin{cases} -v_1 & i = 0 \\ 0 & i = 1 \\ v_{i-1}^p & 2 \leqq i \leqq r \end{cases}$$

$$d_{k(i)}u_i' = v_{i+1} \quad 1 \leqq i \leqq r - 1$$

where

$$k(1) = 2p(p - 1) + 1, \quad k(2) = 2p^2(p - 1) + 1$$

and for $j \geqq 1$,

$$k(2j + 1) = 2(p^{2j+1} + p^{2j-1} + \dots + p - (j + 1))(p - 1) + 1,$$

$$k(2j + 2) = 2(p^{2j+2} + p^{2j} + \dots + p^2 - (j + 1))(p - 1) + 1.$$

THEOREM 2.3″. *Let* $r \geqq 0$ *and* $p = 2$. *Set*

$$u_2' = u_2 + u_0{}^3, \quad u_3' = u_1{}^3 \quad \text{and} \quad u_k' = u_{k-2}'u_{k-2}{}^2, \qquad 4 \leqq k \leqq r.$$

Then the nontrivial differentials of the AHSS

$$H^*(Y_r;\ \mathbf{Z}/2) \Rightarrow K^*(Y_r;\ \mathbf{Z}/2), \quad Y_r = SP^{2^r}S^3,$$

are completely determined by

$$d_3 u_i = \begin{cases} u_0^2 & i = 0 \\ 0 & i = 1 \\ u_{i-2}{}^4 & i \geqq 2 \end{cases}$$

$$d_5 u_1 = u_1{}^2,\ d_9 u_2' = u_2{}^2 \text{ and } d_{2(10k-21)+1}u_k' = u_k{}^2, \qquad 3 \leqq k \leqq r - 1.$$

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The result $d_5 u_1 = u_1^2$ is due to Hodgkin [**7**, Proposition 3.1].

Inspection of the various E_j levels in 2.3′ and 2.3″ reveals that E_j is multiplicatively generated by elements whose p-rank exceeds $N(j)$, where $N(j) \to \infty$ as $j \to \infty$. Hence

COROLLARY 2.4. *The inverse limit groups*

$$\mathscr{K}^i(\mathbf{Z}, 3; \mathbf{Z}/p) = \varprojlim K^i(SP^{p^r}S^3; \mathbf{Z}/p)$$

vanish for $i = 0, 1$.

THEOREM 2.5. *Let* $r \geqq 0$ *and* p *be any prime. Set*

$$u'_{(1,k)} = u_{(1,k)} + u_0 v_{k-1}{}^p, \quad k > 1, \quad \textit{and}$$

$$u'_{(j,k)} = u'_{(j-1,k-1)} v_{k-1}^{p-1}, \quad 2 \leqq j \leqq k.$$

Then the nontrivial differentials of the AHSS $H^*(Y_r; \mathbf{Z}_{(p)}) \Rightarrow K^*(Y_r; \mathbf{Z}_{(p)})$, $Y_r = SP^{p^r}S^3$, *are given by*

(i) $d_{2(p-1)+1} u_0 = -v_1$

$$d_{2(p-1)+1} u_{(j,k)} = \begin{cases} v_j v_{k-1}^p - v_{j-1}^p v_k & \textit{if } j > 1 \\ v_1 v_{k-1}^p & \textit{if } j = 1 \end{cases}$$

$$d_{2(p-1)+1} u_{(I)} = \sum_{l=1}^{s} c_{i_l} v_{i_l-1}^p \, u_{(I\{i_l\})}, \quad \textit{where } I = (i_1, i_2, \ldots, i_s)$$

and

$$c_{i_1} = 1 \textit{ or } 0 \textit{ according as } i_1 > 1 \textit{ or } i_1 = 1,\ c_{i_l} = (-1)^l \textit{ for } l > 1.$$

(ii) $d_{2(p^k-1)+1} p^{k-1} u_0 = v_k, \quad 1 < k \leqq r.$

(iii) $d_{2n(j)(p-1)+1} u'_{(j-1,k)} = v_j v_k, \quad 2 \leqq j \leqq k < r,$

where

$$n(2) = p \textit{ and } n(j) = p^{j-1} + p^{j-2} + \ldots + p - (2j - 5) \quad \textit{when } j > 2.$$

The remark following 2.3″ about E_j levels and p-rank applies here save in dimension 3 (where in the limit $r \to \infty$ no class survives either) and so we have

COROLLARY 2.6. *The inverse limit groups* $\tilde{\mathcal{K}}^i(Z, 3; Z_{(p)}) = \varprojlim \tilde{K}^i(SP^{p^r}S^3; \mathbf{Z}_{(p)})$ *vanish for* $i = 0, 1$.

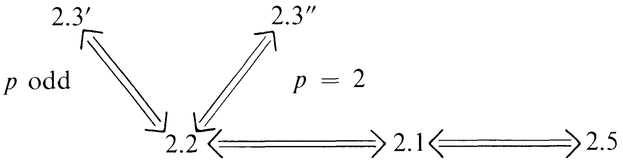
The above cited equivalences (after 2.2) are 2.2 ⇔ 2.3′ if p is odd, 2.2 ⇔ 2.3″ if $p = 2$, and 2.1 ⇔ 2.5. We shall prove by induction on r the first equivalence (i.e., p odd). The remaining equivalences are proved analogously.

The case $r = 1$ is a simple verification, so assume 2.2 ⇔ 2.3′ for p odd and all Y_k, $k \leqq r$. The implication 2.3′ ⇒ 2.2 for Y_{r+1} is clear, but the converse is more interesting. Each of the differential graded algebras $E(u_0, u_2) \otimes \mathbf{Z}/p[v_1]$, $E(u_1)$, $E(u_{k+1}) \otimes \mathbf{Z}/p[v_k]$, $2 \leqq k \leqq r$, $\mathbf{Z}/p[v_{r+1}]$, has $d_{2(p-1)+1}$ homology $E(u_2')$, $E(u_1)$, $\mathbf{Z}/p[v_k]/v_k^p$, $2 \leqq k \leqq r$, $\mathbf{Z}/p[v_{r+1}]$, respectively. Hence, by the Kunneth formula,

$$E_{2(p-1)+2} \simeq E(u_1, u_2') \otimes \mathbf{Z}/p[v_2, \dots, v_{r+1}]$$

modulo the ideal generated by the p^{th} powers $v_2^p, \dots, v_r^p$ and all elements of p-rank $> p^{r+1}$. Naturality with respect to the inclusion $Y_r \to Y_{r+1}$ and the induction hypothesis produces an E-level $E(u_r', u_{r+1}') \otimes \mathbf{Z}/p[v_{r+1}]$ modulo the elements of p-rank $> p^{r+1}$. The argument is completed by noting that the behaviour of the last differential $u_r' \to v_{r+1}$ is determined by the size of $\tilde{K}^i(Y_{r+1}; \mathbf{Z}/p)$.

In Sections 3 and 4 we sketch an inductive proof of 2.5, using the equivalences



In particular we shall show 2.5 for r ⇒ 2.3′, 2.3″ for r ⇒ 2.5 for $r + 1$, where the last implication requires the additional result 3.2.

3. An auxiliary space. Recall if M_2 denotes the 2^{nd} stage of the Milnor construction for $K(Z, 3)$, there is a map $i:M_2 \to K(Z, 3)$. M_2 is homeomorphic to the adjunction space $X = \sum CP^\infty \cup_q C(A)$, where A is the join $CP^\infty * CP^\infty$ and q is the Hopf construction of the standard H-space structure $m:CP^\infty \times CP^\infty \to CP^\infty$. If we view S^3 as the suspension $\sum S^2$ and recall that CP^∞ is homeomorphic to $SP^\infty S^2$, then the analysis of [**12**, Section 2] shows that there is a commutative diagram of the form

$$\begin{array}{ccc} X^k & \subset & X \\ \downarrow & & \downarrow \\ SP^k S^3 & \subset & SP^\infty S^3 \end{array}$$

where X^k is the finite approximation $\Sigma\, CP^k \cup {}_qC(A^{(2k+1)})$, $A^{(2k+1)}$ the $2k + 1$-skeleton of A (q above can be taken cellular). In this section we compute the AHSS $H^*(X_r;\, G) \Rightarrow K^*(X_r;\, G)$, $X_r = X^{p^r}$ for all r and $G = \mathbf{Z}_{(p)}$ or $\mathbf{Z}/p$.

LEMMA 3.1. (Atiyah-Hirzebruch [**4**]). *Let X be a finite* CW *complex, X^q its q-skeleton and let $v \in H^k(X;\, \mathbf{Z})$.* (i) *Then $d_s v = 0$ for all $s < r$ if and only if there exist $u \in H^k(X^{(k+r-1)}, X^{(k-1)};\, \mathbf{Z})$ and $\xi \in K^*(X^{(k+r-1)}, X^{(k-1)})$ such that $\sigma(u) = v$* and *$\mathrm{ch}\xi = \rho * u$ + higher terms, where ρ is the natural homomorphism*

$$H^k(X^{(k+r-1)}, X^{(k-1)};\, \mathbf{Z}) \simeq H^k(X, X^{(k-1)};\, \mathbf{Z}) \to H^k(X;\, \mathbf{Z})$$

and *$\rho *$ is induced by the coefficient homomorphism $\rho: Z \to \mathbf{Q}$.* (ii) *Suppose that $d_s v = 0$ for all $s < r$ and that α is a cochain representative for $(\mathrm{ch}\xi)_{k+r-1}$. Then $\delta\alpha$ is an integral cochain and is a representative for $d_r v$, where δ is essentially the rational cochain coboundary for $(X^{(k+r)}, X^{(k+r-1)}, X^{(k-1)})$.*

The usefulness of this lemma clearly rests on the availability of the desired element ξ.

PROPOSITION 3.2. *In the* AHSS *$H^*(X;\, \mathbf{Z}_{(p)}) \Rightarrow K^*(X;\, \mathbf{Z}_{(p)})$ for p any prime and X our adjunction space, the nontrivial differentials are completely given by*

$$d_{2(p^t-1)+1}(p^{t-1}Su) \neq 0, \quad t \geqq 1.$$

In particular

(i) *for X_r there are r nontrivial differentials*

$$d_{2(p^t-1)+1}(p^{t-1}Su) \neq 0, \quad 1 \leqq t \leqq r$$

and $p^r Su \in H^3(X_r;\, \mathbf{Z}_{(p)})$ survives to a generator of $K^1(X_r;\, \mathbf{Z}_{(p)}) \simeq \mathbf{Z}_{(p)}$;

(ii) *the induced homomorphism*

$$i^*: K^1(X_{r+1};\, \mathbf{Z}_{(p)}) \to K^1(X_r;\, \mathbf{Z}_{(p)})$$

is multiplication by p, and the inverse limit group $\mathscr{K}^1(X;\, \mathbf{Z}_{(p)}) = 0$.

Proof. The s^{th} partial sum of $\ln(1 + x)$, when $x = x_1 + x_2 + x_1x_2$, is

$$\sum_{k=1}^{s} \frac{(-1)^{k+1}(x_1 + x_x + x_1x_2)^k}{k};$$

it has no mixed term $x_1^i x_2^j$, $i, j \geqq 1$ when $i + j \leqq s$. This is true because

$$1 + x = 1 + x_1 + x_2 + x_1x_2 = (1 + x_1)(1 + x_2),$$

thus

$$\ln(1 + x) = \ln(1 + x_1) + \ln(1 + x_2),$$

i.e., no mixed terms appear in the limiting case, and the tail

$$\sum_{k=s+1}^{\infty} \frac{(-1)^{k+1}(x_1 + x_2 + x_1x_2)^k}{k}$$

has no mixed term for $i + j \leqq s$.

Using the formula $q^*(Su^i) = S(u * 1 + 1 * u)^i$, we may compute the kernel and cokernel of the homomorphism

$$H^*(\Sigma CP^\infty; \mathbf{Z}_{(p)}) \xrightarrow{\delta} H^*(X, \Sigma CP^\infty; \mathbf{Z}_{(p)})$$

and find $\ker \delta \simeq \mathbf{Z}_{(p)}$ with generator Su and $(\operatorname{coker} \delta)^i$ has exactly the torsion summand $\mathbf{Z}/p$ when $i = 2_{p^t} + 2$ with generator given by

$$(1/p) \sum_{k=1}^{p^t-1} \binom{p^t}{k} S(u^k * u^{p^t-k}).$$

Since

$$P^{p^{t-1}} P^{p^{t-2}} \dots P^p P^1 r_p(Su) = r_p Su^{p^t},$$

this element is also

$$\beta_{(p)} P^{p^t} P^{p^{t-1}} \dots P^1 r_p(Su), \quad p \text{ odd}, \quad \text{or}$$

$$\beta_{(2)} Sq^{2^t} Sq^{2^{t-1}} \dots Sq^2 r_2(Su); \quad p = 2.$$

Since the induced homorphism

$$i^*: H^k(X_r; \mathbf{Z}_{(p)}) \to H^k(X_{r-1}; \mathbf{Z}_{(p)})$$

is an isomorphism for $k \leqq 2p^{r-1} + 2$, naturality with respect to i^* reduces an induction on r to the claims

(1) for $r = 1$, $d_{2(p-1)+1}Su \neq 0$ for X_1;

(2) for any r, $d_{2(p^r-1)+1}(p^{r-1}Su) \neq 0$ for X_r.

(1) is clear from the explicit

$$d_{2(p-1)+1}Su = -\beta_{(p)}P^1(Su).$$

For (2) we shall apply the Atiyah-Hirzebruch Lemma 3.1 using the opening observation about mixed terms in the expansion of $\ln(1 + x)$. Naturality with respect to $i:X_{r-1} \to X_r$ implies

$$d_s(p^{t-1}Su) = 0 \quad \text{for all } s < 2(p^r - 1) + 1.$$

From the diagram

$$\begin{array}{ccccc} K^1(X, \Sigma CP^{p^r}) & \to & K^1(X) \to K^1(\Sigma CP^{p^r}) & \to & K^0(X, \Sigma CP^{p^r}) \\ \wr\wr & & \wr\wr & & \\ 0 & & K^1(X, B) & & \end{array}$$

$B = (X_r)^2 \sim$ point, $X = (X_r)^m$, $m = 3 + 2(p^t - 1) + 1 - 1$, we see that $K^1(X, B) \simeq \ker \delta$. To show

$$d_{2(p^r-1)+1}(p^{r-1}Su) = v_r$$

we look for a suitable element ξ (as required by 3.1 (ii)) in $\ker \delta$.

The element

$$\xi = p^{r-1} \sum_{k=1}^{p^r-1} \frac{(-1)^{k+1}Sx^k}{k}$$

is an element of $K^1(\Sigma CP^{pr}; \mathbf{Z}_{(p)})$ i.e., the coefficients $p^{r-1} \cdot 1/k$, $k \leqq p^r - 1$, have denominators prime to p, and is in $\ker \delta$ by virtue of our initial observation.

We claim

$$(\text{ch}\xi)_3 = p^{r-1}Su \quad \text{and} \quad (\text{ch}\xi)_{2p^r+1} = (1/p)Su^{p^r}.$$

That $(\text{ch}\xi)_3 = p^{r-1}Su$ is clear since $\text{ch}Sx = Su$. We wish to compute the Chern character of ξ. Since $\text{ch}Sx^k = S(e^u - 1)^k$, we introduce the variable $y = e^u - 1$. Then

$$\frac{d}{du}\text{ch}\xi = \frac{dy}{du} \cdot \frac{d}{dy}\text{ch}\xi = S\left(p^{r-1}e^u \sum_{k=1}^{p^r-1} (-1)^{k+1}y^{k-1}\right)$$

$$= S\left(p^{r-1}(1 + y) \sum_{k=1}^{p^r-1} (-1)^{k+1}y^{k-1}\right)$$

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$$= S(p^{r-1}(1 + y^{p^{r-1}}))$$
$$= S(p^{r-1}(1 + (e^u - 1)^{p^{r-1}}))$$
$$= S(p^{r-1}(1 + u^{p^{r-1}} + \text{higher terms})).$$

Hence the coefficient of Su^{p^r} in chξ must have been $1/p$. But then

$$\delta(1/pSu^{p^r}) \neq 0$$

and so $d_{2(p^t-1)+1}p^{r-1}Su$ is nonzero.

The Universal Coefficient Theorem [**3**] implies that the preceding Proposition 3.2 is equivalent to the following generalization of Hodgkin's result [**7**, Lemma 3.5]. One can also give an independent proof along the lines of Hodgkin's original proof.

PROPOSITION 3.3. (i) $K^1(X_r; \mathbf{Z}/p) \simeq \mathbf{Z}/p$;

(ii) *In the* AHSS $H^*(X_r; \mathbf{Z}/p) \Rightarrow K^*(X_r; \mathbf{Z}/p)$, *p any prime, the nontrivial differentials are given by*

$$d_{2p^t(p-1)+1}(Su^{p^t}) \neq 0 \quad \textit{for all } t \leqq r - 1.$$

(iii) *The induced homorphism* $i^*:K^1(X_{r+1}; \mathbf{Z}/p) \to K^1(X_r; \mathbf{Z}/p)$ *is the zero homomorphism and the inverse limit group* $\mathscr{K}^1(X; \mathbf{Z}/p) = 0$.

4. Sketch proof of 2.5. We begin with a description of the homology of the initial differential $d_{2(p-1)+1}$ (which we abbreviate to d). As d is stable on the set of elements of p-rank $\leqq m$ when $p|m$, we may study our problem for the limiting space $K(\mathbf{Z}, 3)$. Our standing assumptions in 4.1 – 4.5 below are that the space is $K(\mathbf{Z}, 3)$ and that p is an odd prime.

LEMMA 4.1. ker *d contains the elements* pu_0, $u'_{(1,i)}$, $i \geqq 2$, v_j, $j \geqq 1$, *and all products*

$$u_{(1,i_1)} \dots u'_{(1,i_s)} v_{j_1}^{n_1} v_{j_2}^{n_2} \dots v_{j_t}^{n_t}.$$

LEMMA 4.2. ker *d contains the subalgebra generated by all* $u'_{(1,i)}$, v_j, $2 \leqq i$, $1 \leqq j$ *and this subalgebra is isomorphic to*

$$E(u'_{(1,i)})_{i \geqq 2} \otimes Z/p\,[v_j]_j \geqq 1.$$

LEMMA 4.3. *Let K be the subalgebra generated by* pu_0, $u_{(1,i)}$, v_j $2 \leqq i$ *and* $1 \leqq j$ *(so K is isomorphic to*

$$\mathbf{Z}_{(p)} \oplus (E(u'_{(1,i)})_{i\geqq 2} \otimes \mathbf{Z}/p\,[v_j]_{j\geqq 1})$$

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with pu_0 a generator of the first summand; note $pu_{(1,i)} = pv_j = 0$ implies $(pu_0)u_{(1,i)} = (pu_0)\, v_j = 0$). Then the image of d contains the ideal of K generated by

(i) v_1;

(ii) $v_i v_{j-1}^p - v_{i-1}^p v_j$, $2 < i < j$ *and* $v_2 v_{j-1}^p$, $2 < j$;

(iii) $v_i u'_{(1,j)} - v_j u'_{(1,i)}$, $2 \leqq i < j$;

(iv) $v_{i-l}^p u'_{(1,j)} - v_{j-1}^p u'_{(1,i)}$, $2 \leqq i < j$ *and* $v_{j-1}^p u'_{(1,2)}$, $3 \leqq j$;

(v) $u'_{(1,i)} u'_{1,j)}$, $2 \leqq i < j$.

PROPOSITION 4.4. *Let I be the image of d restricted to the subring generated by u_0, $u_{(i,j)}$, $1 \leqq i < j$; v_j, $1 \leqq j$; $u_{(1,i,j)}$, $2 \leqq i < j$. Then*

$$K/I \simeq \mathbf{Z}_{(p)} \oplus S_1$$

where $\mathbf{Z}_{(p)}$ is generated by pu_0, and S_1 is a $\mathbf{Z}/p$-vector space with basis $B_1 \cup B_2$, B_1 consisting of all v_J^N satisfying (1)–(3) below, and B_2 consisting of all $u_{(1,i)} v_J{}^N$ satisfying (4)–(5) below.

(1) $1 \notin J$, *i.e.*, $v_J^N = v_{j_1}{}^{n_1} \dots v_{j_t}{}^{n_t}$ *has no v_1 factor.*

(2) $v_J{}^N$ *has no $v_2 v_j{}^P$ factor*, $2 \leqq j$.

(3) $v_J{}^N$ *has no $v_i{}^P v_j$ factor*, $2 < i < j - 1$.

(4) $u'_{(1,i)} v_J{}^N$ *has no factor $u'_{(1,i)} v_j$ for $j < i$.*

(5) $u'_{(1,i)} v_J{}^N$ *has no factor $u'_{(1,i)} v_j^P$ for $i \leqq j$.*

Lemmas 4.1 and 4.3 are easy verifications. For 4.2 first show

$$u'_{(1,i_1)} u'_{(1,i_2)} = -u_0 u_{(1,i_1)} v_{i_2-1}^p + u_0 u_{(1,i_2)} v_{i_1-1}^p + u_{(1,i_1,i_2)} v_1,$$

and then use induction on s to obtain a similar expression for $u_{(1,i_1)} u_{(1,i_2)} \dots u_{(1,i_s)}$. Lemma 4.2 then follows from 1.6 in paragraph 1. Lemma 4.4 is immediate from the preceding lemmas.

PROPOSITION 4.5. ker d/im$d$ *is isomorphic to K/I.*

Proof. We have already examined the differential d on the summands of (1.6) having as factor an element of $\{u_0, u_{(i,j)}, u_0 u_{(i,j)}, u_0 u_{(1,i,j)}\}$ or having all factors v_j's. As $K \subset \ker d$, we must show that any element in $\ker d - K$ also is an element of image d.

Set $\alpha_k = v_{k-1}^p$. Fix an s-tuple

$$I_s = (i_1, i_2, \dots, i_s), \quad 2 \leqq i_1 < i_2 < \dots < i_s, \; s \geqq 3.$$

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Let C be the set of all $(s - 1)$-tuples and R the set of all $(s - 2)$-tuples, both selected from I_s. Order C (resp. R) by increasing p-rank (e.g. for $s = 4$, we have

$$C = \{ (i_1, i_2, i_3), (i_1, i_2, i_4), (i_1, i_3, i_4), (i_2, i_3, i_4) \} \quad \text{and}$$

$$R = \{ (i_1, i_2), (i_1, i_3), (i_2, i_3), (i_1, i_4), (i_2, i_4), (i_3, i_4) \}.$$

We inductively define matrices A_s, $3 \leqq s$ with entries α_{i_k}, 0 where $i_k \in I_s$. (Blanks are zeroes.)

$$A_3 = \begin{bmatrix} \alpha_{i_2} & \alpha_{i_3} & \\ -\alpha_{i_1} & & \alpha_{i_3} \\ & -\alpha_{i_1} & -\alpha_{i_2} \end{bmatrix}$$

Then A_s will be the $\binom{s}{s-2} \times \binom{s}{s-1}$-matrix

$$A_s = \left[\begin{array}{c|c} \multicolumn{2}{c}{B_s} \\ \hline 0 & A_{s-1} \\ & \end{array}\right]$$

where

$$B_s = (-1)^{s-1} \left[\begin{array}{c|c} \alpha_{i_{s-1}} & \\ -\alpha_{i_{s-2}} & \\ -\alpha_{i_{s-3}} & \alpha_{i_s} I \\ \vdots & \\ \pm\alpha_{i_1} & \end{array}\right]$$

The relation the matrices A_s bear to our question about the kernel and image of d can be seen as follows. Consider the first matrix A_3. Let C_3 be the set of all pairs (i_1, i_2), $2 \leqq i_1 < i_2 < \infty$, R_3 the set $\{2, 3, 4 \ldots\}$. Since the summand $W_3 = \oplus\, u_{(i_1,i_2)} v_J{}^N$, summed over all $2 \leqq i_1 < i_2$, is additively generated over $\mathbf{Z}/p[v_j]_{j \geqq 2}$ by the set $\{u_I | I \in C_3\}$, and the image $d(W)$ is generated by $\{v_j\}_{j \geqq 2}$, d is represented by the infinite matrix

$$\begin{bmatrix}
\alpha_3 & \alpha_4 & & \alpha_5 & & \dots & \alpha_k & & & & \dots \\
-\alpha_2 & & \alpha_4 & & \alpha_5 & & & \alpha_k & & & \\
& -\alpha_2 & -\alpha_3 & & & \alpha_5 & & & \ddots & & \\
& & & -\alpha_2 & -\alpha_3 & -\alpha_4 & & & & \alpha_k & \\
& & & & & & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{k-1} & \\
\vdots & & & & & & & & & \vdots &
\end{bmatrix}$$

The matrix A_3 arises by deletion of all but $3 = \binom{3}{1}$ and rows and $3 = \binom{3}{2}$ columns. Now rank $A_3 = 2$, so from this infinite matrix we see that ker $(d|W_3)$ is generated by all the elements

$$-\alpha_{i_3}u_{(i_1,i_2)} + \alpha_{i_2}u_{(i_1,i_3)} - \alpha_{i_1}u_{(i_2,i_3)}.$$

However from the description of d on the generators $\{u_{(i_1,i_2,i_3)}\}$, we observe that this generating set for ker $(d|W_3)$ is also a generating set for the image $d(W_4)$, where W_4 is the summand $\oplus\, u_{(i_1,i_2,i_3)}v_J{}^N$, summed over all $2 \leqq i_1 < i_2 < i_3$, of (1.6).

A completely analogous discussion applies to the kernel of d on the summand

$$W_s = \oplus\, u_{(i_1,i_2,\dots,i_{s-1})}v_J{}^N, \quad 2 \leqq i_1 < i_2 < \dots < i_{s-1},$$

with the matrix A_s replacing A_3. The argument works because

$$\text{rank } A_s = \text{rank } A_{s-1} + 1 = s - 1$$

(whence every s columns of the corresponding infinite matrix has singly generated kernel).

Set

$$W_s' = \oplus\, u_{(1,i_1,\dots,i_{s-1})}\, v_J{}^N.$$

Then it remains to consider d on the summands W_s', u_0W_s,u_0W_s'. We have $d:W_s' \to W_{s-1}'$ and the corresponding matrix describing the kernel of d is identical to the case $d:W_s \to W_{s-1}$ already considered. Furthermore d on $\oplus\,(u_0W_s \oplus u_0W_s')$ is actually monic.

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PROPOSITION 4.6. *Assume* (2.3)′ *and* (2.3)″ *for a fixed integer r. Then for any prime p*

(i) $\widetilde{K}^i(SP^{p^r+p^s}S^3; \mathbf{Z}/p) \simeq \mathbf{Z}/p \oplus \mathbf{Z}/p, \quad i = 0, 1, 1 \leqq s < r;$

(ii) $\widetilde{K}^i(SP^{2p^r}S^3; \mathbf{Z}/p) \simeq 0, \mathbf{Z}/p, \quad i = 0, 1.$

Proof. Let p be odd. Then

$$H^*(SP^mS^3; \mathbf{Z}/p) \simeq E(u_i)_{0\leqq i\leqq r} \otimes \mathbf{Z}/p[v_j]_{1\leqq j\leqq r}/\mathfrak{a}_m$$

where $p^r \leqq m < p^{r+1}$ and $\mathfrak{a}_m$ is the ideal of elements of p-rank $> m$. The multiplicative generators u_i, v_j restricted from SP^mS^3 to $SP^{p^r}S^3$ also generate multiplicatively $H^*(SP^{p^r}S^3; \mathbf{Z}/p)$, hence both spaces have the same set of nontrivial differentials in their respective spectral sequences (as described by (2.3)′). The permanent cycles however will not in general be the same because of the different truncations.

Let $m = p^r + p^s$, $1 \leqq s < r$. By (2.3)′ and naturality the E levels of the spectral sequence for SP^mS^3 are given by

(4.1) $\quad E(u'_i, u'_{i+1}) \oplus \mathbf{Z}/p[v_j]_{i<j\leqq r}/(\mathfrak{a}_m, v_{i+1}{}^p, \ldots, v_{r-1})$

before the E level when $i = s$. At this level u'_sv_r becomes a permanent cycle since $v_{s+1}v_r$ has p rank $> m = p^r + p^s$. For $s < i < r - 1$ the E-levels have the form $S \oplus \mathbf{Z}/p$ where S is the usual form (4.1) for that level and $\mathbf{Z}/p$ is generated by u'_sv_r. When $i = r - 1$, u'_r and $u'_{r-1}u'_r$ become permanent cycles, the latter since $u'_rv_r \in \mathfrak{a}_m$. (Note the p-rank of $u'_{r-1}u'_r$ is only $p^r + p$, so $u'_{r-1}u'_r \neq 0$. Secondly u'_sv_r is not killed by $u'_su'_{r-1}$ because there are no elements of the form u'_sx, p-rank $x < p^r$, beyond the E-level when $i = s$.)

For $m = 2p^r$ only u'_r survives all the differentials. u'_sv_r is now $u_{r-1}v_r$ and this dies via

$$u'_{r-1}v_r \mapsto v_r{}^2 \neq 0.$$

($v_r{}^2 \neq 0$ when $m = 2p^r$.) Similarly

$$u'_{r-1}u'_r \mapsto u'_rv_r \neq 0.$$

For $p = 2$ there really is no need to modify the above argument. Instead of

$$E(u_i)_{0\leqq i\leqq r} \otimes \mathbf{Z}/p[v_j]_{1\leqq j\leqq r}/\mathfrak{a}_m$$

we have

$$\mathbf{Z}/2\,[u_i]_{0 \leqq i \leqq r}/\mathfrak{a}_m.$$

The introduction of variables $v_j = u_{j-1}^2$ however is enough to make the above proof work.

Proof of 2.5 *for p an odd prime*. When $r = 1$ the result follows from the formula

$$d_{2p-1} = -\beta_{(p)}P^1.$$

So now assume 2.5 for $r > 1$ and consider the initial differential d_{2p-1} for $Y_{r+1} = SP^{p^{r+1}}S^3$. A straightforward calculation using Nakaoka's description of $H^*(Y_{r+1}; \mathbf{Z}/p)$ and an Adem relation gives d_{2p-1} as described in 2.5 (i), with $r + 1$ replacing r.

The homology of d_{2p-1} is $\mathbf{Z}_{(p)} \oplus V$ with pu_0 generating the summand $\mathbf{Z}_{(p)}$, and the set

$$(4.2) \quad u'^{\epsilon}_{(1,i)} v_i^{n_i} v_{i+1}^{n_{i+1}} \dots v_{r+1}^{n_{r+1}},$$

$$\epsilon = 0, 1;\ 2 \leqq i \leqq r,\ n_i \geqq 0, \text{ of } p\text{-rank} \leqq p^{r+1}$$

satisfying also $n_2 < p + 1$; $n_k < p$ for $3 \leqq k \leqq r$ whenever $n_2 > 1$, (see 4.4 and 4.5) a basis for the $\mathbf{Z}/p$-vector space V.

Decompose E_{2p} as $I \oplus D$, where $I \simeq \mathbf{Z}_{(p)} \oplus (\mathbf{Z}/p)^r$ with generators pu_0, $v_2, v_3 \dots, v_{r+1}$ and D the span of the complement of $\{v_2, v_3, \dots, v_{r+1}\}$ in (4.2). In I the nontrivial differentials are described by

$$d_{2p^t-1}p^{t-1}u_0 = v_t, \quad 2 \leqq t \leqq r + 1$$

(note that $2p^t - 1 = 2(p^{t-1} + p^{t-1} + \dots + p + 1)(p - 1) + 1$). Assuming these classes survive to be the required E levels, we shall have this result by naturality, first for $2 \leqq t \leqq r$ via the inclusion $Y_r \to Y_{r+1}$, and second, for $t = r + 1$ via the inclusion $X_{r+1} \to Y_{r+1}$.

The remaining nontrivial differentials all live on the summand D and are all described by the statement

$$u'_{(k,l)} \to v_{k+1}v_l \quad \text{for all } 1 \leqq k < l \leqq r + 1.$$

Observe that the same differential sends all $u_{(k,l)} \mapsto v_{k+1}v_l$ for fixed k but varying l, while for different values of k, different differentials are involved. One can verify the implication

$$\{u'_{(k,k+1)} \mapsto v_{k+1}^2\} \Rightarrow \{u'_{(k,l)} \mapsto v_{k+1}v_l, \text{ all } l > k\},$$

using the 'exchange relation'

$$u'_{(k,l)}v_{l-1} = u'_{(k,l-1)}v_l.$$

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By iteration we have

$$u'_{(k,l)}v_{l-1}v_{l-2}\dots v_{k+1} = u'_{(k,k+1)}v_{k+2}v_{k+3}\dots v_l.$$

Let d now denote the differential corresponding to the k in the statement of our observation. As

$$d(v_{k+2}\dots v_l) = 0$$

by naturality (and the easy $d(v_{r+1}) = 0$) and

$$d(u'_{(k,k+1)}) = v_{k+1}^2$$

by the assumption, then using the derivation property of d

$$d(u'_{(k,l)}v_{l-1}v_{l-2}\dots v_{k+1}) = d(u'_{(k,k+1)}v_{k+2}\dots v_l)$$
$$= d(u'_{(k,k+1)})v_{k+2}\dots v_l)$$
$$= v_{k+1}^2v_{k+2}\dots v_l = d(u'_{(k,l)})v_{l-1}\dots v_{k+1}.$$

As multiplication by $v_{k+1}\dots v_{l-1}$ is monic, $d(u'_{(k,l)}) = v_{k+1}v_l$.

Our task is thus reduced to showing

$$u'_{(k,k+1)} \mapsto v_{k+1}^2 \quad \text{for each } k.$$

Actually by induction and naturality via $Y_r \to Y_{r+1}$ we need only show

$$u'_{r-1,r} \mapsto v_r^2.$$

Briefly, this is a consequence of 4.6 and the Universal Coefficient Theorem: one can show that the failure of $u'_{(r-1,r)} \mapsto v_r^2$ for some r contradicts the order of the groups given by 4.6.

The proof of 2.5 for $p = 2$ is similar with only minor alterations needed in 4.2 and 4.3(v).

5. $BP\langle 1\rangle^*(K(\mathbf{Z}, 3); \mathbf{Z}/p)$. Recall that localized, periodic K-theory $K^*{}_{\text{per}}(\text{—})_{(p)}$ splits naturally as a direct sum of theories

$$K^*{}_{\text{per}}(\text{—})_{(p)} = E_0^*K^*(\ \ )_{(p)} \oplus E_1^*K^*(\ \ )_{(p)} \oplus \dots$$
$$\oplus E_{p-2}^*K^*(\ \ )_{(p)}.$$

Each $E_i^*K^*(\ \ )_{(p)}$ is a cohomology theory, and $E_0^*K^*(\ \ )_{(p)}$ is moreover a multiplicative, periodic cohomology theory of period $2(p - 1)$ with coefficients

$$E_0^*K^*(pt)_{(p)} \simeq \mathbf{Z}_{(p)}[x_1, x_1^{-1}].$$

Here $x_1 = u^{p-1}$, $u \in K^{-2}(pt)$ the Bott element.

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Localized, connective K-theory $bu^*_{(p)}$ also splits naturally into a direct sum of $p - 1$ cohomology theories, one of which, G^*, is also multiplicative and has coefficients

$$G^*(pt) \simeq \mathbf{Z}_{(p)}[x_1], \quad \deg x_1 = 2p - 2.$$

Johnson and Wilson [**8**] have identified G^* as $BP{<}1{>}^*$. There is a natural transformation

$$BP{<}1{>}^* \to E_0^* K^*(\quad)_{(p)},$$

which on coefficient rings is just the obvious inclusion

$$\mathbf{Z}_{(p)}[x_1] \hookrightarrow \mathbf{Z}_{(p)}[x_1, {x_1}^{-1}].$$

The preceding discussion applies to the associated mod p theories and provides a natural transformation

$$BP{<}1{>}^*(\quad; \mathbf{Z}/p) \to E_0^* K^*(\quad; \mathbf{Z}/p),$$

which on coefficients is the usual inclusion

$$\mathbf{Z}/p[x_1] \hookrightarrow \mathbf{Z}/p[x_1, {x_1}^{-1}].$$

Consider localized, periodic mod p K-theory for the spaces $Y_r = SP^{p^r}S^3$. For the associated AHSS's we have

$$d_{2k(p-1)+1}a = x_1^k b$$

as given in Theorem 2.3′ and Theorem 2.3″. Therefore, as the differentials for

$H^*(Y_r; \mathbf{Z}/p[x_1, x_0$f $E_0^* K^*(Y_r; \mathbf{Z}/p)$ rests solely on the calculation of the homology of each differential.

From the existence of the natural transformation

$$BP{<}1{>}^*(\quad; \mathbf{Z}/p) \to E_0^* K^*(\quad; \mathbf{Z}/p)$$

we see that the differentials have the same description (modulo some coefficient a power of x_1). However, as ${x_1}^{-1}$ does not exist for $BP{<}1{>}$, the descriptions of $E_0^* K^*(Y_r; \mathbf{Z}/p)$ and $BP{<}1{>}^*(Y_r; \mathbf{Z}/p)$ differ radically.

To simplify the description of $BP{<}1{>}^*(Y_r; \mathbf{Z}/p)$ we define a function

$$l(k) = p^{k-1}, \quad k = 1, 2, 3 \quad \text{and}$$

$$l(2k + \epsilon) = \left(\sum_{h=0}^{k-1} p^{2h+1+\epsilon}\right) - (k-1), \quad \epsilon = 0, 1; \ k \geqq 2.$$

THEOREM 5.1. (i) $E_0^*K^*(Y_r; \mathbf{Z}/p) \simeq E(u_r')$, *the exterior algebra over* $\mathbf{Z}/p[x_1, x_1^{-1}]$ *on one generator* u_r'.

(ii) *Let p be an odd prime.*

$$BP\langle 1\rangle^*(Y_r; \mathbf{Z}/p) \simeq E(v_k u_j', v_t^p u_h', u_r')_{\substack{1 \leqq k \leqq j < r \\ 1 \leqq h < t \leqq r-2}} \otimes \mathbf{Z}/p[x_1][v_i]_{1 \leqq i \leqq r}/R_p,$$

the tensor product over $\mathbf{Z}/p[x_1]$ *of the exterior algebra* $E(v_k u_j', v_t^p u_h', u_r')$ *and the polynomial algebra* $\mathbf{Z}/p[x_1][v_i]$ *modulo the relations*

$$R_p: x_1^{l(i)} v_i = 0, \quad x_1 v_i^p = 0, \quad 2 \leqq i < r, \quad x_1^{l(k)} v_k u_j' = 0 = x_1 v_t^p u_h'.$$

and the relations that the generators $v_k u_j'$, $v_t^p u_h'$, u_r', v_i *inherit from the ring structure of* $H^*(Y_r; \mathbf{Z}/p)$, *namely, if* $x = x' u_{j_1}'$, $y = y' u_{j_2}' \in \{v_k u_j', v_t^p u_h' \mid k, j, th$ *as above*$\}$, *then*

$$xy = 0 \text{ if } j_1 \equiv j_2 \bmod 2$$

$$(v_k u_j')u_r' = 0 \quad \text{if } r \equiv j \bmod 2,$$

$$(v_{k_1} u_{j_1}')(v_{k_2} u_{j_2}') = 0 \quad \text{if } j_1 \equiv j_2 \bmod 2$$

and all elements of p-rank $> p^r$ *equal* 0.

(iii) $BP\langle 1\rangle^*(Y_r; \mathbf{Z}/2)$ *is the polynomial algebra over* $\mathbf{Z}/2[x_1]$ *on the generators given in* (ii) *modulo the* $\mathbf{Z}/2[x_1]$*-module relations given in* (ii) *together with the relations that the generators inherit from the ring structure of* $H^*(Y_r; \mathbf{Z}/2)$ (*remembering that* $v_i = u_{i-1}^2$).

Proof. (i) is a straightforward consequence of Theorems 2.3′ and 2.3″. The existence of x_1^{-1} implies the annihilation of every even dimensional class (they are all images of the various differentials), and all odd dimensional classes save those of the form $x_1^i u_r'$, $i \in \mathbf{Z}$.

(ii) We calculate the homology of each differential. The initial differential $d_{2(p-1)+1}$ is given by

$$u_0 \mapsto -x_1 v_1, \quad u_1 \mapsto 0, \quad \text{and}$$

$$u_i \mapsto x_1 v_{i-1}^p, \quad 2 \leqq i < r,$$

and its homology is

$$E(u_1, u_2') \otimes \mathbf{Z}/p[x_1][v_i]_{1 \leqq i \leqq r} \text{ modulo } x_1 v_1 = x_1 v_{i-1}^p = 0, \quad 2 \leqq i \leqq r.$$

The next nontrivial differential $d_{2p(p-1)+1}$, given by

$$u_1 \mapsto x_1^p v_2, \; u_2' \mapsto 0,$$

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has homology

$$E(v_1u_1, v_t^pu_1, u_2', u_3')_{2\leqq t\leqq r-2} \otimes \mathbf{Z}/p[x_1]\,[v_i]_{1\leqq i\leqq r}$$

modulo the earlier relations plus the new relations

$$x_1v_1u_1 = x_1v_t^pu_1 = 0,\ x_1{}^pv_2 = 0 \quad \text{and} \quad (v_1u_1)u_3' = (v_t^pu_1)u_3' = (v_{t_1}^pu_1)(v_{t_2}^pu_1) = 0.$$

Note that $x_1v_1u_1 = 0$ since this is the case at $E_{2(p-1)+1}$:

$$d_{2(p-1)+1}(-u_0u_1) = x_1v_1u_1.$$

Also since $u_1^2 = 0$ and $u_3' = u_1v_2^{p-1}$, $(v_1u_1)u_3' = 0$. Finally v_1u_1 is a $d_{2p(p-1)+1}$-cycle because

$$d_{2p(p-1)+1}v_1u_1 = x_1{}^pv_1v_2,$$

but $x_1v_1v_2 = 0$ already at $E_{2(p-1)+2}$. Similarly for $v_t^pu_1$.

Continuing in this way we may suppose we have arrived at an E-level which is described by

$$E(v_ku_j', v_t^pu_h', u_{s-1}', u_s')_{\substack{1\leqq k\leqq j\leqq s-1<r\\ 1\leqq h<t\leqq r-2}} \otimes \mathbf{Z}/p[v_i]_{1\leqq i\leqq r}$$

modulo the relations

$$x_1{}^{l(i)}v_i = 0,\quad i < s,\quad x_1v_j^p = 0,\quad 2 \leqq j < r,$$

$$x_1{}^{l(k)}v_ku_j' = x_1v_t^pu_h' = 0,\quad 1 \leqq k \leqq j < s-1,\ 1 \leqq h < t \leqq r-2,$$

and the product structure inherited from $H^*(Y_r; \mathbf{Z}/p)$. The next nontrivial differential sends

$$u_{s-1}' \mapsto x_1{}^{l(s)}v_s,\quad u_s', v_ku_{s-1}', v_t^pu_{s-1}' \mapsto 0,\quad k \leqq s-1.$$

($x_1{}^{l(k)}v_k = 0$ holds before this E level since $k < s$, hence $x_1{}^{l(k)}v_ku_{s-1}' = 0$.) Hence we have reproduced the analogous description for the next nontrivial E-level. This establishes (ii) by induction.

(iii) The proof for $p = 2$ is identical to the odd prime case when one replaces the exterior relation $u_{i-1}^2 = 0$ by the defining relation $u_{i-1}^2 = v_i$.

The differentials for the AHSS

$$H^*(K(Z, 3); \mathbf{Z}_{(p)}) \Rightarrow BP\langle 1\rangle^*(K(\mathbf{Z}, 3))$$

are again completely determined by those for

$$H^*(\quad; \mathbf{Z}_{(p)}) \Rightarrow K^*_{\text{per}}(\quad; \mathbf{Z}_{(p)}).$$

However, some nontrivial group extensions and a large amount of

p-torsion make the problem of describing $BP\langle 1\rangle^*(K(\mathbf{Z}, 3))$ somewhat more difficult.

While we do not calculate $BP\langle 1\rangle^*(K(\mathbf{Z}, 3))$ here, we can provide some information about it by considering the finite approximation spaces X_r of Section 3.

THEOREM 5.2. (i) *$BP\langle 1\rangle^*(X_r; \mathbf{Z}/p)$ contains a subring which is generated as a $BP\langle 1\rangle^*(pt; \mathbf{Z}/p)$-module by $v_1, v_2, \ldots, v_r, u_r$ with module structure*

$$x_1^{p^i-1} v_1 = 0, \quad 1 \leqq i \leqq r,$$

and trivial ring structure

$$v_i v_j = 0, \quad v_i u_r = 0, \quad 1 \leqq i, j \leqq r.$$

(ii) *$BP\langle 1\rangle^*(X_r)$ contains a subring with $BP\langle 1\rangle^*(pt)$-module generators $p^r u_0, v_1, v_2, \ldots, v_r$,* relations

$$x_1^{p^i-1} v_1 = p v_{i-1}, \; 2 \leqq i \leqq r,$$

and trivial ring structure

$$v_i v_j = 0, \quad (p^r u_0)^2 = 0, \quad (p^r u_0) \cdot v_i = 0, \quad 1 \leqq i, j \leqq r.$$

Proof. (i) The nontrivial differentials are

$$u_{i-1} \mapsto x_1^{p^i-1} v_i, \quad 1 \leqq i \leqq r.$$

Thus u_r and $v_1, v_2, \ldots, v_r$ generate a submodule with

$$x_1^{p^i-1} v_i = 0.$$

The ring structure is induced from that of $H^*(X_r; \mathbf{Z}/p)$.

(ii) In this case the nontrivial differentials are

$$p^{i-1} u_0 \mapsto x_1^{N_i} v_i, \quad N_i = p^{i-1} + p^{i-2} + \ldots + p + 1, \; 1 \leqq i \leqq r,$$

and so $p^r u_0, v_1, \ldots, v_r$ survive to E_∞ to generate a subring. In the exact sequence

$$BP\langle 1\rangle^3(X_r) \xrightarrow{\rho} BP\langle 0\rangle^3(X_r) \xrightarrow{\Delta_1} BP\langle 1\rangle^{2p+2}(X_r)$$

the homomorphism ρ is multiplication by p^r, $\mathbf{Z}_{(p)} \to \mathbf{Z}_{(p)}$, since $p^r u_0$ is the generator for $BP\langle 1\rangle^3(X_r)$. As order $(BP\langle 1\rangle^{2p+2} X_r) = p^r$, with E_∞ generators $v_1, x_1^p v_2, x_1^{p^2+p} v_3, \ldots, x_1^{N_r} v_r$, we must have

$$BP\langle 1\rangle^{2p+2} X_r \simeq \mathbf{Z}/p^r.$$

In particular, for $r = 2$, $x_1{}^p v_2 = pv_1$.

For arbitrary r consider the exact sequence

$$BP<0>^a(X_s) \xrightarrow{\Delta_1} BP<1>^b(X_r) \xrightarrow{\cdot x_1} BP<1>^c(X_r)$$

where $b = a + 2p - 1$ and $c = b - 2p + 2$. By the above argument the right most group is cyclic of order p^r when $c = 2p + 2$. As the group $BP<0>^a(X_r) = 0$ for all odd $a > 3$, $(\cdot x_1)$ is monic for all

$$b = 2p^r + 2 - k(2p - 2),$$
$$0 \leqq k \leqq (p - 1)^{-1}(p^r - 2p + 1);$$

the largest value of k occurs when $c = 2p + 2$. By examining the groups $BP<1>^b(X_r)$ as $k = 1, \ldots, (p - 1)^{-1} \cdot (p^r - 2p + 1)$, and the classes $x_1{}^i v_j$ surviving to E_∞ in these dimensions, we see that these groups are all cyclic and, as a result, the desired relations

$$x_1{}^{p^{i-1}} v_i = pv_{i-1}, \quad 2 \leqq i \leqq r,$$

must hold. The trivial ring structure is implied by that of $H^*(X_r; \mathbf{Z}_{(p)})$.

While we do not attempt a description of $BP<1>^*(Y_r)$, the generators given in 5.2 (ii) are easily seen to be present for Y_r with the same $\mathbf{Z}_{(p)}[x_1]$-module structure. However, their products are no longer zero. Also, $BP<1>^*(Y_r)$ has elements not in the subring multiplicatively generated by the generators in 5.2 (ii). For them we would have to give a complete description of the images of all differentials of the AHSS

$$H^*(Y_r; \mathbf{Z}_{(p)}) \Rightarrow K^*(Y_r; \mathbf{Z}_{(p)}).$$

So while the AHSS

$$H^*(Y_r; \mathbf{Z}_{(p)}[x_1]) \Rightarrow BP<1>^*(Y_r)$$

is determined out to E_∞, more book keeping is needed as well as more care in resolving the group extension problem.

Let E^* be one of the cohomology theories in 5.1. Then passage to the inverse limit over r from the rings $E^*(Y_r; \mathbf{Z}/p)$ and induced homomorphisms

$$i_r^*: E^*(Y_{r+1}; \mathbf{Z}/p) \to E^*(Y_r; \mathbf{Z}/p)$$

yields statements about the corresponding inverse limit theory $\mathscr{E}^*(K(\mathbf{Z}, 3); \mathbf{Z}/p)$. For example, when $E^* = E_0^* K^*(\ \ )_{(p)}$ the homomorphisms i_r^* are all zero. Thus the inverse limit group vanishes. On the other hand, the inverse limit

$$\lim_{\substack{\leftarrow \\ r}} BP\langle 1\rangle^*(Y_r; \mathbf{Z}/p)$$

is quite large. For this we formally construct rings S_r isomorphic to $BP\langle 1\rangle^*(Y_r; \mathbf{Z}/p)$, with generators e_{kj}, e_{th}, f_i in place of $v_k u_j'$, $v_t^p u_h'$, u_r' and v_i. S_r is given the $\mathbf{Z}/p[x_1]$-module structure, induced by this isomorphism. Let R_r be the ring (and $\mathbf{Z}/p[x_1]$-module) obtained from S_r by forgetting the $\mathbf{Z}/p[x_1]$-module relations satisfied by e_{kj}, e'_{th}, e_r, f_i in S_r (i.e., R_r is free as a $\mathbf{Z}/p[x_1]$-module), and let T_r be the subring of R_r generated by these relations. Then the obvious homomorphisms $T_{r+1} \to T_r$ (similarly for the rings R_r, S_r), which act as the identity on e_{kj}, e'_{th}, f_i common to T_{r+1}, T_r, and which send the additional generators e_{r+1}, v_{r+1}, e_{kr}, e'_{th} to zero, form a surjective system. Hence the short exact sequence of inverse systems

$$0 \to \{T_r\} \to \{R_r\} \to \{S_r\} \to 0$$

induce an exact sequence

$$0 \to T = \lim_{\leftarrow} T_r \to R = \lim_{\leftarrow} R_r \to \lim_{\leftarrow} S_r \to 0.$$

Thus

$$\mathcal{BP}\langle 1\rangle^*(K(Z, 3); \mathbf{Z}/p) = \lim_{\leftarrow} S_r \simeq R/T.$$

One can further identify R as a ring of formal power series in the e_{kj}, e'_{th}, v_l, $1 \leqq k \leqq j < \infty$, $1 \leqq h < t < \infty$, $1 \leqq l < \infty$ mod the relations induced from the ring structure of $H^*(Y_r; \mathbf{Z}/p)$, and T as the subring generated by all the $\mathbf{Z}/p[x_1]$-module relations gotten by setting $r = \infty$.

6. Remarks. 1. An interesting restatement of our results for the AHSS

$$H^*(SP^{p^r}S^3; \mathbf{Z}/p) \Rightarrow K^*(SP^{p^r}S^3; \mathbf{Z}/p)$$

is given in

THEOREM 6.1. *Let $1 \leqq r \leqq \infty$. Filter $Y_r = SP^{p^r}S^3$ by the subspaces $Y_{-1} = pt$, $Y_i = SP^{p^i}S^3$, $0 \leqq i \leqq r$. Then in the spectral sequence associated to the filtration*

$$E_1^{s,t} = K^{s+t}(Y_s, Y_{s-1}; \mathbf{Z}/p) \simeq \begin{cases} \mathbf{Z}/p & 1 \leqq s \leqq r. \\ 0 & s = 0,\ t = 2k;\ s > r. \\ \mathbf{Z}/p & s = 0,\ t = 2k + 1. \end{cases}$$

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$d_1^{s,t}:E_1^{s,t} \to E_1^{s+1,t}$ *is an isomorphism* $\mathbf{Z}/p \to \mathbf{Z}/p$ *for* $s \equiv (t-1) \bmod 2$ *and the zero homomorphism for* $s \equiv t \bmod 2$. *Hence the spectral sequence collapses* $E_2^{*,*} \simeq E_\infty^{*,*}$.

Recall the elementary fact that the initial differential $d_1^{s,t}$ is just the composite δj^* in the commutative diagram

(6.1)
$$\begin{array}{ccc} K^{s+t}(Y_s, y_{s-1}; \mathbf{Z}/p) & \xrightarrow{\;d_1^{s,t}\;} & K^{s+t+1}(Y_{s+1}, Y_s; \mathbf{Z}/p) \\ & {\scriptstyle j^*}\searrow \quad \nearrow{\scriptstyle \delta} & \\ & K^{s+t}(Y_s; \mathbf{Z}/p) & \end{array}$$

Thus we have used the AHSS to evaluate inductively (on s) the coboundary homomorphism δ in (6.1). The groups $E_1^{s+1,t}$ have to be computed (as well as the initial differential $d_1^{s,t}$). But as good fortune would have it, the AHSS

$$H^*(X, A; \mathbf{Z}/p) \Rightarrow K^*(X, A; \mathbf{Z}/p), \quad (X, A) = (Y_s + 1, Y_s)$$

has no additional nontrivial differentials beyond those for $(X, A) = (Y_s, pt)$. Hence an induction on s is possible. In particular, for (Y_{s+1}, Y_s) the only possibly additional nontrivial differential would be that which kills v_{s+1} (in Y_{s+1}). But as u_s' has p-rank $\leqq p^s$ and so does not exist in $H^*(Y_{s+1}, Y_s; \mathbf{Z}/p)$, v_{s+1}, to provide generators for $E_1^{s+1,t}$. But now $d_1^{s,t}$ is described above in 6.1 if and only if

$$K^i(Y_{s+1}; \mathbf{Z}/p) \simeq \mathbf{Z}/p, 0 \quad \text{for } i = 1, 0,$$

so the argument of Sections 3 and 4 thus proves that the behaviour of $d_1^{s,t}$ is correctly stated in 6.1. In terms of the cited generators $d_1^{s,t}u_s' = v_s + 1$.

If we replace the cohomology theory $K^*(\ ; \mathbf{Z}/p)$ by $BP{<}1{>}^*(\ ; \mathbf{Z}/p)$ the spectral sequence of 6.1 also collapses at E_2. Although the $E_1^{s,t}$ groups are now more complicated, the differential $d_1^{s,t}$ is simply given by $d_1^{s,t}u_s' = x_1^{l(s+1)}v_{s+1}$. The collapse at E_2 again occurs because the differentials defined on nontrivial elements land in zero groups; a verification which requires a closer look at the dimensions of survivors of $d_1^{*,*}$.

2. We could have studied the K-homology groups of Y_r instead of its K-cohomology groups. In fact, using some Universal Coefficient Theorems of D. W. Anderson [**1**] or certain spectral sequence pairings [**11**], we can see that the homology AHSS is determined by the cohomology AHSS

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for the spaces Y_r.

3. The referee has suggested that the additive structure of $K\tilde{U}^*(CP_r^p S^{2n+1}; Z_{(p)})$ could be obtained as for the case $p = 2$ by V. P. Snaith in Math. Scand. *38* (1976). A note on symmetric maps for spheres, 78-80. ($CP_r^p X$ is the r^{th} iterated cyclic product of X.) Assuming the existence of a transfer t for $CP_r^p X \to SP^{p^r} X$, one might be able to obtain the AHSS for $SP^{p^r} S^3$ from that of $CP_r^p S^3$ (also hoping that the latter is obtainable from its K-theory). However, the details of this approach might be as long as those given in the present paper. On the other hand, it may be of independent interest since it applies to all odd spheres S^{2n+1}, not just S^3.

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