Notes on the method of Orthogonal Projection.

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This paper is designed to call attention to certain extensions of this method, which do not seem to be generally recognised. Most writers consider the ellipse alone as amenable to orthogonal projection, whereas all the conic sections are so.

With the hyperbola we may use this method in three ways—

(1) By projecting a given diameter in the figure to be a transverse axis of the projected figure, when the result will follow by symmetry.

(2) By projecting the hyperbola into a rectangular one.

(3) By an *imaginary orthogonal projection* from the circle.

Of the first two, which are obvious enough, I shall give one or two examples, although I think it is scarcely necessary to do so, especially as I believe all are amenable to the third mode of treatment. I shall omit figures for the most part as they can readily be drawn.

I. P′CP is any diameter, P′Q, P′R are drawn parallel to the asymptotes, meeting tangent at P, in Q and R. Show that PQ = PR.

Project the figure so that P′CP becomes the transverse axis, then the latent symmetry of the figure becomes *explicit*, and it is manifest that PQ = PR. So for a multitude of other examples.

II. CP and CD are conjugate; from P a tangent PT is drawn to the conjugate hyperbola, and from D, DQ, to the original hyperbola. Show that CQ and CT are conjugate. (*Aberdeen Senior Mathematical Examination* 1884.)

Project the hyperbola into a rectangular one. Then the whole figure has the asymptote between CP and CD for an axis of symmetry, and if we made it revolve half a revolution round it as an axis, we should get the same figure again. Hence CQ and CT are equally inclined to the asymptote, and by property of rectangular hyperbola are therefore conjugate. Here again, a latent symmetry has been made explicit.

I now pass to the third method, which has the widest scope. Following the *principle of continuity*, we argue that as the ellipse and hyperbola shade into one another by insensible gradations, and are conics of co-ordinate importance, that is, each requires five conditions, therefore propositions which are true for the one remain true for the other. This is reasoning from analogy rather than imaginary projection, but it comes to the same thing. Let it be granted then, that from a circular cylinder we can cut any hyperbola, just as we can cut any ellipse—which of course involves planes inclined at imaginary angles, whose sines are greater than unity. [Throughout this paper I shall use "*concentric*" as an abbreviation for the three attributes,—concentric, similar, and similarly situated.] Now we know that the asymptotes of a hyperbola may themselves be considered a hyperbola, for which the ratio of similitude $=0$. This may easily be shown, for on any radius vector OP take p so that $Op=\mu OP$, then when μ is very small we get a hyberbola lying very close to the asymptotes, and in the limit coinciding with them. This enables us to prove a well-known proposition, namely, that

III. If the tangent at P meet the asymptotes in Q and R, then triangle CQR has constant area.

For take two concentric circles—then any tangent to the inner, cuts off from the outer circle, a segment of constant area, by symmetry. Areas are projective properties, and we infer that the same truth holds with regard to any two *concentric* conics, whether ellipses or hyperbolas. Hence any tangent to the inner of two *concentric* hyperbolas cuts off a constant area from the outer. And they have the same asymptotes. Now let the outer hyperbola (the one nearer the asymptotes) shrink up till it becomes the asymptotes, then the proposition still remains true, and we get the result we set out to prove, the curvilinear segment degenerating into the triangle CQR. So the facts that QR is bisected in P, and that if we draw any chord RQQ′R′ across the hyperbola and asymptotes, then $RQ=Q'R'$ follow at once from considering two concentric circles. Many other examples might be given. The following, as also Ex. I, can be treated both by this method and by the first.

IV. PQ is any chord; parallels to the asymptotes are drawn through P and Q. Show that the other diagonal of the parallelogram thus formed passes through C.

Let PQ be any chord of a circle. From P and Q as centres, describe two equal circles; their common chord bisects PQ at right angles, and therefore passes through the centre.

Now let the two circles become very small; their points of crossing are now imaginary, but their radical axis or common chord will still pass through the centre. Project the big circle orthogonally

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into a hyperbola, then the two very small circles or points (as they are in the limit) become pairs of straight lines parallel to the asymptotes. For we have already seen that any pair of straight lines is a very small hyperbola, and is therefore projected into a point, or very small circle. Hence the result follows.

I next proceed to the relation between the hyperbola and its conjugate. Since both are *concentric* with the asymptotes, it follows that they are *concentric* with each other. Hence, as before, we can argue from two concentric circles to the hyperbola and conjugate. It is important to note that two circles are required. For instance, if in the ellipse CP and CD are conjugate, the parallelogram contained by CP, CD, and the tangents at P and D, has constant area. The same thing appears to be the case in the hyperbola, but there is a slight difference—one fact is not the projection of the other. In fact the latter case is the projection of the following—(Fig. 30).

P and D are points on two concentric circles such that CP and CD are at right angles; tangents PR, DR are drawn. Show that rectangle CR has constant area. As it has of course by symmetry.

The ratio of similitude for the hyperbola and conjugate is $\sqrt{-1}$, as may be established thus.

V. Let us have two concentric circles, and let $CP = \mu CD$, and be at right angles to CD always. (Fig. 30). Draw the tangents PR, DR, then $CR^2 = (1+\mu^2)CD^2$;
therefore R traces out a circle whose radius $= \sqrt{1+\mu^2}CD$.

Hence for two *concentric* hyperbolas, the ratio of similitude being μ, the locus of intersection of tangents drawn at points where they are met by conjugate diameters, is another *concentric* hyperbola, whose ratio of similitude is $\sqrt{1+\mu^2}$.

Now for the hyperbola and its conjugate, the locus is the asymptotes—that is

$$\sqrt{1+\mu^2} = 0$$

therefore $$\mu = \sqrt{-1},$$

the ratio of similitude being 0 for the asymptotes.

As another example take the following.

VI. Show that the hyperbola is its own reciprocal with regard to the conjugate.

From any point P (Fig. 31) in the outer of two concentric circles, draw tangents to the inner. The chord of contact will, by symmetry, touch a concentric circle, and if $CR = \mu CP$, then $CW = \mu^2 CP$.

In the case we are considering, $\mu = \sqrt{-1}$, and therefore we infer that if P be any point on the hyperbola its polar with regard to the conjugate, touches the original hyperbola at P′, the other extremity of diameter CP, since $CP' = -CP = CP \times (\sqrt{-1})^2$.

As a final example, I take one involving focal properties. The projective property of the focus may be readily established geometrically, for we know that

1. A revolving right angle, makes with the circular asymptotes a harmonic pencil, and with no pair of *fixed* lines *but* the circular asymptotes.

2. If OL, OM be tangents to a conic, and the tangent at P meet LM in R, then OL, OP, OM, OR is harmonic.

3. If the tangent at P meet directrix in Z, then PSZ is a right angle. Combining these we see that the tangents to a conic from a focus, must pass through the circular points at infinity.

VII. A number of parabolas pass through a fixed point, and touch two lines. Show that the chord of contact envelopes a hyperbola. (*Aberdeen Senior Mathematical Examination* 1884).

Project orthogonally any hyperbola asymptotic to OP, OQ, (Fig. 32) into a circle, then OP, OQ become the circular asymptotes, and the parabolas remain parabolas, while O becomes their focus S. Hence the problem becomes—Given a point and a focus of a parabola, find the envelope of polar of the focus, or the directrix. And this is of course a circle, whose centre is the fixed point, and which passes through S. Returning to the original figure, we find that the envelope is a *hyperbola whose centre is* A, *which passes through* O, *and whose asymptotes are parallel to* OP, OQ.

On a number of concurrent spheres.

By R. E. Allardice, M.A.

This paper included an analytical proof of the following theorem:

If five spheres 1, 2, 3, 4, 5, pass through the same point, and if the four points in which the four spheres 1, 2, 3, 4, intersect in sets of three be coplanar; and if the same be true for the sets 1, 2, 3, 5; 1, 2, 4, 5; 1, 3, 4, 5; it will also be true for the remaining set 2, 3, 4, 5.

The same theorem is true for similar and similarly situated quadric surfaces.