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Growth Spaces and Growth Norm Estimates for $\bar\partial$ on Convex Domains of Finite Type

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Abstract. We consider the growth norm of a measurable function f defined by

$$\|f\|_{-\sigma} = \operatorname{ess\,sup}\{\delta_D(z)^\sigma |f(z)| : z \in D\},$$

where $\delta_D(z)$ denote the distance from z to ∂D. We prove some optimal growth norm estimates for $\bar\partial$ on convex domains of finite type.

1 Introduction and Statement of Results

Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. For $z \in D$ let $\delta_D(z)$ denote the distance from z to ∂D. For $\alpha > 0$ we define a measure dV_α on D by $dV_\alpha(z) = \delta_D(z)^{\alpha-1} dV(z)$ where $dV(z)$ is the volume element. For $0 < p, \alpha < \infty$ let $\|f\|_{p,\alpha}$ be the L^p-norm with respect to the measure dV_α and we define $L^{p,\alpha}(D) = \{f : \|f\|_{p,\alpha} < \infty\}$. Let $A^{p,\alpha}(D) = L^{p,\alpha}(D) \cap \mathcal{O}(D)$, where $\mathcal{O}(D)$ is the space of holomorphic functions on D. We will denote the usual Hardy space $H^p(D)$ by $A^{p,0}(D)$, and the associated norm by $\|f\|_{p,0}$. We can identify $A^{p,0}(D)$ in the usual way with a subspace of $L^p(\partial D : d\sigma)$. For $\alpha \ge 0$ and $0 < p < \infty$ we have (see Lemma 2.1)

$$\sup\{\delta_D(z)^{(n+\alpha)/p} |f(z)| : z \in D\} \lesssim \|f\|_{p,\alpha} \quad \text{for } f \in A^{p,\alpha}(D). \tag{1.1}$$

The estimate (1.1) motivated the author to consider the growth norm for general measurable functions. Let $0 < \sigma < \infty$. For a measurable function f on D we define the growth norm

$$\|f\|_{-\sigma} = \operatorname{ess\,sup}\{\delta_D(z)^\sigma |f(z)| : z \in D\}.$$

Let

$$L^{-\sigma}(D) = \{f : f \text{ measurable}, \|f\|_{-\sigma} < \infty\}.$$

For $\sigma = 0$ we let $L^{-0}(D) = L^\infty(D)$. Then growth spaces $L^{-\sigma}(D)$ are Banach spaces. Let $L^{-\sigma}_{(0,q)}(D)$ be the Banach space of $(0,q)$-forms whose coefficients belong to the $L^{-\sigma}(D)$ space.

We denote by $\Lambda_\sigma(D)$ the Lipschitz space of order $0 < \sigma < 1$ and by $\mathrm{BMO}(D)$ the BMO-space on D.

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Theorem 1.1 *Let* $D \Subset \mathbb{C}^n$ *be a convex domain with* C^∞*-smooth boundary of finite type* M. *Let* $0 \le \sigma < \infty$. *There are bounded linear operators* T_q *such that* $\bar\partial T_q f = f$ *for all* $f \in L^{-\sigma}_{(0,q+1)}(D) \cap C^1_{(0,q+1)}(D) \cap L^1_{(0,q+1)}(D)$ *with* $\bar\partial f = 0$ *and these operators satisfy the following estimates:*

(i) *For* $0 \le \sigma < \frac{1}{M}$,
$$\|T_q f\|_{\Lambda_{1/M-\sigma}(D)} \lesssim \|f\|_{-\sigma}.$$

(ii) *For* $\sigma = \frac{1}{M}$,
$$\|T_q f\|_{\mathrm{BMO}(D)} \lesssim \|f\|_{-1/M}.$$

(iii) *For* $\sigma > \frac{1}{M}$,
$$\|T_q f\|_{-(\sigma-1/M)} \lesssim \|f\|_{-\sigma}.$$

Let P be the Bergmen projection associated with D. Then we have

$$Pf(z) = \int_{z\in D} f(\zeta) B(z,\zeta)\, dV(\zeta),$$

where $f \in L^1(D)$ and $B(z,\zeta)$ is the Bergman kernel associated with D. Let $0 < \sigma < 1$. In [McSt] it was proved that

$$\|Pf\|_{\Lambda_\sigma(D)} \lesssim \|f\|_{\Lambda_\sigma(D)}. \tag{1.2}$$

Moreover, they proved that

$$\int_D \delta_D(\zeta)^{-\sigma} |B(z,\zeta)|\, dV(\zeta) \lesssim \delta_D(z)^{-\sigma}, \quad z \in D.$$

From this we can see that

$$\|Pf\|_{-\sigma} \lesssim \|f\|_{-\sigma}. \tag{1.3}$$

Corollary 1.2 *Under the assumptions of Theorem* 1.1, *the canonical solution* v *of the equation* $\bar\partial u = f$ *satisfies the following estimates:*

(i) *For* $0 \le \sigma < \frac{1}{M}$,
$$\|v\|_{\Lambda_{1/M-\sigma}(D)} \lesssim \|f\|_{-\sigma}.$$

(ii) *For* $\frac{1}{M} < \sigma < 1 + \frac{1}{M}$,
$$\|v\|_{-(\sigma-1/M)} \lesssim \|f\|_{-\sigma}.$$

Proof If u is the solution of the equation $\bar\partial u = f$ given by Theorem 1.1, the canonical solution v of the equation is $v = u - Pu$, where P is the Bergman projection associated to D. The results follow immediately from Theorem 1.1, (1.2), and (1.3). ∎

For more recent results about estimates for $\bar\partial$ on convex domains of finite type by means of integral kernels we refer the reader to [AhCh, Al, Cu, DFF, DiFi, DiFo1, DiFo2, Fi, He1, He2, Wa].

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2 Growth Spaces

In this section we can see that the growth space is a very convenient way to study embedding theorems among holomorphic function spaces.

Lemma 2.1 *Let $\alpha \geq 0$ and $0 < p < \infty$. Then we have*

$$\sup\{\delta_D(z)^{(n+\alpha)/p}|f(z)| : z \in D\} \lesssim \|f\|_{p,\alpha} \quad \text{for } f \in A^{p,\alpha}(D).$$

Proof For $p_0 \in D$ sufficiently near ∂D, we translate and rotate the coordinate system so that $z(p_0) = 0$ and the $\operatorname{Im} z_1$ axis is perpendicular to ∂D. Let $\mathcal{B}_\epsilon(p_0)$ denote the non-isotropic ball

$$\mathcal{B}_\epsilon(p_0) = \left\{ \frac{|z_1|^2}{(\epsilon\delta_D(p_0))^2} + \sum_2^n \frac{|z_j|^2}{\epsilon\delta_D(p_0)} < 1 \right\}.$$

Since ∂D is C^2, it follows that there is an $\epsilon_0 > 0$ such that for p_0 sufficiently near ∂D and $z \in \mathcal{B}_{\epsilon_0}(p_0)$ we have $z \in D$ and

$$\frac{\delta_D(p_0)}{2} \leq \delta_D(z) \leq 2\delta_D(p_0) \tag{2.1}$$

(see [Be]). Let $0 < p < \infty$ and $\alpha > 0$. Let $f \in A^{p,\alpha}(D)$. Since the plurisubharmonicity of $|f|^p$ is invariant by the affinity

$$(z_1, z_2, \dots, z_n) \to \left(\frac{z_1}{\epsilon_0\delta_D(p_o)}, \frac{z_2}{\sqrt{\epsilon_0\delta_D(p_0)}}, \dots, \frac{z_n}{\sqrt{\epsilon_0\delta_D(p_0)}} \right),$$

it follows that

$$\begin{aligned} |f(p_0)|^p &\lesssim \frac{1}{\operatorname{Vol}(\mathcal{B}_{\epsilon_0}(p_0))} \int_{\mathcal{B}_{\epsilon_0}(p_0)} |f(z)|^p \, dV(z) \\ &\lesssim \frac{1}{(\epsilon_0\delta_D(p_0))^{n+1}} \int_{\mathcal{B}_{\epsilon_0}(p_0)} |f(z)|^p \, dV(z). \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), it follows that

$$|f(p_0)| \lesssim \delta_D(p_0)^{-(n+\alpha)/p} \|f\|_{p,\alpha}.$$

Parameterizing ∂D locally by $x_1, z_2, \dots, z_n$ let $\tilde{\mathcal{B}}_{\epsilon_0}(p_0)$ denote the non-isotropic ball on ∂D

$$\tilde{\mathcal{B}}_{\epsilon_0}(p_0) = \left\{ z \in \partial D : \frac{|x_1|^2}{(\epsilon_0\delta_D(p_0))^2} + \sum_{j=2}^n \frac{|z_j|^2}{\epsilon_0\delta_D(p_0)} < 1 \right\}.$$

For any $u \in L^p(\partial D)$ we denote by Λu the Hardy–Littlewood maximal function of u:

$$\Lambda u(z) = \sup_{\epsilon > 0} \frac{1}{\sigma(\tilde{\mathcal{B}}_\epsilon(z))} \int_{\tilde{\mathcal{B}}_\epsilon(z)} |u| \, d\sigma.$$

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Let $f \in A^{p,0}(D)$. For $1 < p < \infty$ let f^* be a a boundary value function in $L^p(\partial D)$. For $z \in \mathcal{B}_{\epsilon_0}(p_0)$ let $\pi(z)$ denote the projection of z onto ∂D. Then it follows that

$$|f(z)| \leq C\Lambda f^*(\pi(z)) \quad \text{for } z \in \mathcal{B}_{\epsilon_0}(p_0).$$

From (2.2) we obtain that

$$\begin{aligned}
(2.3) \qquad |f(p_0)|^p &\lesssim \frac{1}{\delta_D(p_0)^{n+1}} \int_{\mathcal{B}_{\epsilon_0(p_0)}} |f(z)|^p \, dV(z) \\
&\lesssim \frac{1}{\delta_D(p_0)^{n+1}} \int_{\mathcal{B}_{\epsilon_0(p_0)}} \Lambda f^*(\pi(z))^p \, dV(z) \\
&\lesssim \frac{1}{\delta_D(p_0)^{n+1}} \int_{\tilde{\mathcal{B}}_{\epsilon_0(p_0)}} \int_{\delta_D(p_0)/2}^{2\delta_D(p_0)} \Lambda f^*(\zeta)^p \, dt d\sigma(\zeta) \\
&\lesssim \frac{1}{\delta_D(p_0)^n} \|\Lambda f^*\|^p_{L^p(\partial D)} \lesssim \frac{1}{\delta_D(p_0)^n} \|f\|^p_{p,0}.
\end{aligned}$$

If $0 < p \leq 1$, we apply the estimate (2.3) above to the function $|f|^{1/s}$, where s is a large positive number and with p replaced by sp. Then we get the required inequality. ■

For $0 < p < \infty$ and $\alpha \geq 0$ we define

$$L^{p,\alpha}_{-\sigma}(D) = L^{p,\alpha}(D) \cap L^{-\sigma}(D).$$

Then $L^{p,\alpha}_{-\sigma}(D)$ is a Banach space with the norm defined by

$$\|f\|_{p,\alpha,-\sigma} = \max\{\|f\|_{p,\alpha}, \|f\|_{-\sigma}\}.$$

Let $A^{p,\alpha}_{-\sigma}(D) = L^{p,\alpha}_{-\sigma}(D) \cap \mathcal{O}(D)$. By Lemma 2.1, we get

$$(2.4) \qquad A^{p,\alpha}_{-(n+\alpha)/p}(D) = A^{p,\alpha}(D), \quad 0 < p < \infty, \ \alpha \geq 0.$$

Let $\alpha > 0$ and $0 < p \leq q < \infty$. Then we have

$$\begin{aligned}
(2.5) \qquad \int_D |f|^q \, dV_{\alpha+\sigma(q-p)} &= \int_D |f|^p |f|^{q-p} \delta^{\alpha-1} \delta^{\sigma(q-p)} \, dV \\
&\leq \Big(\int_D |f|^p \, dV_\alpha \Big) (\sup \ \delta^\sigma |f|)^{q-p}.
\end{aligned}$$

By (2.5), we have that

$$\begin{aligned}
\|f\|_{q,\alpha+\sigma(q-p)} &\leq \|f\|^{p/q}_{p,\alpha} \|f\|^{1-p/q}_{-\sigma} \leq \|f\|_{p,\alpha} + \|f\|_{-\sigma} \\
&\lesssim \|f\|_{p,\alpha,-\sigma}.
\end{aligned}$$

Hence it follows that

$$(2.6) \qquad L^{p,\alpha}_{-\sigma}(D) \subset L^{q,\alpha+\sigma(q-p)}(D).$$

If we choose $\sigma = (n+\alpha)/p$, by (2.4) and (2.6), we have the following result.

Theorem 2.2 *Assume that* $0 < p \le q < \infty, \alpha, \beta > 0$, *and* $(n+\alpha)/p = (n+\beta)/q$. *Then* $A^{p,\alpha}(D) \subset A^{q,\beta}(D)$ *and the inclusion is continuous.*

Remark 2.3 The case $\alpha = 0$ in Theorem 2.2 is the embedding of Hardy spaces $H^p(D)$ into the weighted Bergman spaces $A^{q,\beta}(D)$. As expected, the embedding of the Hardy space is the most difficult one. Even though Beatrous [Be] proved the embedding $H^p(D) \subset A^{q,\beta}(D)$ for $0 < p < q < \infty$ with $n/p < (n+\beta)/q$, we cannot prove the optimal embedding of the case $n/p = (n+\beta)/q$ by using his method. The optimal embedding of the case $\alpha = 0$ was proved only in some model domains such as the unit disc [Du], the unit ball [BeBu], and the strongly pseudoconvex domain [Be]. Recently, the author proved the case $\alpha = 0$ in the convex domain of finite type [Ch]. The key point is the reproducing kernel with right estimate matching quasimetric on ∂D. However, in general domains not enough is known about the reproducing kernel with right estimate and so we must use a different approach. Stein [St] introduced the boundary behavior of H^p-functions in general bounded domains in $\mathbb{C}^n$ with C^2 boundary without making use of any assumption of pseudoconvexity. In [ChKw] we proved the case $\alpha = 0$ in general domains in $\mathbb{C}^n$. In our proofs we overcome the difficulty by using the growth space and Fatou's theorem for H^p-functions proved by Stein [St].

3 Construction of the Solution Formula for $\bar\partial$

From now on we denote by $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ a bounded convex domain with C^∞-smooth boundary of finite type M. We also define $D_\delta := \{z \in \mathbb{C}^n : \rho(z) < \delta\}$ for small absolute values $|\delta|$. The defining function ρ can be chosen in such a way that there exists a neighborhood U of ∂D such that $|d\rho(z)| > 1/2$ for all $\zeta \in U$ and all the domains $D_{\rho(\zeta)}$ are convex domains of finite type M. For details, see [Fi, §2].

If $\vec{n}_\zeta$ is the unit outward normal vector at ζ on the hypersurface $\{z : \rho(z) = \rho(\zeta)\}$, we define $w = \Phi(\zeta)(z-\zeta)$, where the unitary matrix $\Phi(\zeta)$ satisfies $\Phi(\zeta)\vec{n}_\zeta = (1, 0, \dots, 0)$ for all $\zeta \in U$. The following definitions are in [DiFo1]:

$$\rho_\zeta(w) = \rho(\zeta + (\overline{\Phi}(\zeta))^T w),$$

$$\tilde{S}_\zeta(w) = 3w_1 + Kw_1^2 - c\sum_{j=2}^{M} N^{2^j}\sigma_j \sum_{\substack{|\alpha|=j\\ \alpha_1=0}} \frac{1}{\alpha!}\frac{\partial^j \rho_\zeta}{\partial w^\alpha}(0)w^\alpha$$

for $N > 0$ suitably large and $c > 0$ suitably small (all independent of ζ). We define

$$\tilde{Q}^j_\zeta(w) = \int_0^1 \frac{\partial \tilde{S}_\zeta}{\partial w_j}(tw)\,dt, \quad j = 1, \dots, n,$$

and

$$\begin{aligned}\tilde{Q}(z,\zeta) &= (\tilde{Q}_1(z,\zeta), \dots, \tilde{Q}_n(z,\zeta))\\ &= \Phi(\zeta)^T\left(\tilde{Q}^1_\zeta(\Phi(\zeta)(z-\zeta)), \dots, \tilde{Q}^n_\zeta(\Phi(\zeta)(z-\zeta))\right).\end{aligned}$$

We put $\tilde S(z,\zeta) = \tilde S_\zeta(\Phi(\zeta)(z-\zeta))$. Then $\tilde S(z,\zeta)$ is a support function on D, holomorphic in $z \in \bar D$ and C^∞ in $\zeta \in U$ with the following estimates. Let $\vec v$ be a unit vector complex tangential to the level set $\{\rho = \rho(\zeta)\}$ at ζ. Define

$$a_{\alpha\beta}(\zeta, \vec v) = \frac{\partial^{\alpha+\beta}}{\partial\lambda^\alpha \partial\bar\lambda^\beta} \rho(\zeta + \lambda \vec v)|_{\lambda=0}.$$

Then there are constants $K, c, d > 0$, such that one has for all points z written as $z = \zeta + \mu \vec n_\zeta + \lambda \vec v$ with $\mu, \lambda \in \mathbb{C}$ the estimate

$$\begin{aligned} 2 \operatorname{Re} \tilde S(z,\zeta) \le &- |\operatorname{Re}\mu| - K(\operatorname{Im}\mu)^2 \\ &- c \sum_{j=2}^{M} \sum_{\alpha+\beta=j} |a_{\alpha\beta}(\zeta,\vec v)||\lambda|^j + d \sup\{0, \rho(z) - \rho(\zeta)\}. \end{aligned}$$

Since we want to define $\tilde Q$ for all ζ we choose two neighborhoods $\partial D \Subset U_1 \Subset U_2 \Subset U$ of the boundary and a smooth cut off function $0 \le \chi \le 1$ such that $\chi(\zeta) = 1$ for $\zeta \in U_1$ and $\chi(\zeta) = 0$ for $\zeta \in D \setminus U_2$. Using this we can define

$$\hat Q(z,\zeta) = \chi(\zeta) \tilde Q(z,\zeta).$$

Lemma 3.1 ([AhCh]) *There exists a constant C_1 such that for all $z, \zeta \in D$ we have*

$$-\operatorname{Re}\big(\langle \hat Q(z,\zeta), z-\zeta\rangle + C_1\rho(\zeta)\big) \gtrsim -\rho(z) - \rho(\zeta) + |z-\zeta|^M. \tag{3.1}$$

Now we define

$$s(z,\zeta) = \sum_{j=1}^{n} (\bar z_j - \bar\zeta_j) d\zeta_j,$$

$$Q(z,\zeta) = \frac{1}{C_1\rho(\zeta)} \sum_{j=1}^{n} \hat Q_j(z,\zeta) d\zeta_j$$

with the constant C_1 from Lemma 3.1 and $G(\lambda) = \lambda^{-N}$ for $\lambda \in \mathbb{C}$. For convenience we also introduce the notation

$$S(z,\zeta) = \langle \hat Q(z,\zeta), z-\zeta\rangle + C_1\rho(\zeta).$$

Using the above (1,0)-form $Q(z,\zeta)$, the Berndtsson–Andersson kernel [BeAn] becomes

$$K(z,\zeta) = \sum_{\nu=0}^{n-1} C_{n,\nu} G^{(\nu)}\big(1 + \langle Q(z,\zeta), z-\zeta\rangle\big) \frac{s \wedge (\bar\partial Q)^\nu \wedge (\bar\partial s)^{n-1-\nu}}{\langle s(\zeta,z), \zeta - z\rangle^{n-\nu}}.$$

Now we introduce the notation $K_q(z,\zeta)$ for the part of the kernel which is of degree $(0,q)$ with respect to z and define

$$T_q f(z) = \int_{\zeta\in D} f(\zeta) \wedge K_q(z,\zeta).$$

Since due to the weight function, the kernel vanishes for $\zeta \in \partial D$, the integral operators T_q are indeed solution operators for $\bar\partial$ in D. For convenience we write

$$K(z,\zeta) = \sum_{\nu=0}^{n-1} K^\nu(z,\zeta),$$

where

$$K^\nu(z,\zeta) = C_{n,\nu} G^{(\nu)}\big(1 + \langle Q(z,\zeta), z-\zeta\rangle\big) \frac{s \wedge (\bar\partial Q)^\nu \wedge (\bar\partial s)^{n-1-\nu}}{\langle s(\zeta,z), \zeta - z\rangle^{n-\nu}}. \tag{3.2}$$

4 Integral Estimates for the Bochner–Martinelli Kernel With Weight Factor

First we note that in (3.2) if $\nu = 0$, our kernel becomes the well-known Bochner–Martinelli kernel with some weight factor. Define

$$\mathbb{K}^0 f(z) = \int_{\zeta\in D} f(\zeta) \wedge K^0(z,\zeta).$$

Theorem 4.1 *Let $f \in L^{-\sigma}_{(0,q+1)}(D)$.*

(i) *For $0 < \sigma < 1$, $\|\mathbb{K}^0 f\|_{\Lambda_{1-\sigma}(D)} \lesssim \|f\|_{-\sigma}$.*

(ii) *For $\sigma > 1$, $\|\mathbb{K}^0 f\|_{-(\sigma-1)} \lesssim \|f\|_{-\sigma}$.*

Proof (i) In [AhCh], we proved that

$$\int_{\zeta\in D} |\rho(\zeta)|^{-\sigma} |\nabla_z K^0(z,\zeta)|\, dV(\zeta) \lesssim |\rho(z)|^{-\sigma} \quad \text{for all } z \in D.$$

Thus we get $|\nabla_z \mathbb{K}^0 f(z)| \lesssim \|f\|_{-\sigma} |\rho(z)|^{-\sigma}$ for all $z \in D$. By the Hardy–Littlewood lemma, we get the result.

(ii) We have

$$|K^0(z,\zeta)| \lesssim \Big|\frac{\rho(\zeta)}{S(z,\zeta)}\Big|^N \frac{1}{|z-\zeta|^{2n-1}}. \tag{4.1}$$

It follows from (3.1) of Lemma 3.1 that the support function satisfies the estimate

$$|S(z,\zeta)| \gtrsim |\operatorname{Im} S(z,\zeta)| - \rho(\zeta) - \rho(z) + |z-\zeta|^M \tag{4.2}$$

for every $z, \zeta \in D$. Moreover, we have

$$d\rho(\zeta) \wedge d(\operatorname{Im} S(z,\zeta)) \neq 0 \tag{4.3}$$

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for ζ close enough to ∂D and $|z-\zeta| < \epsilon_0$, sufficiently small $\epsilon_0 > 0$. Recall that $\operatorname{Im} S(z,\zeta) = \operatorname{Im} \tilde{S}(z,\zeta)$ for ζ close enough to ∂D. To see (4.3) it suffices to prove $\partial_z\rho(\zeta) \wedge \partial_z\tilde{S}(\zeta,\zeta) \neq 0$. By simple observation [DFF, (2)] we know that

$$\partial\tilde{S}(\zeta,\zeta) = 3\sum_{j=1}^{n}(\Phi(\zeta))_{1,j}\,dz_j,$$

where $(\Phi(\zeta))_{1,j}$ is the $(1,j)$-element of the unitary matrix $\Phi(\zeta)$ satisfying $\Phi(\zeta)\vec{n}_\zeta = (1,0,\dots,0)$. Since $\partial\rho(\zeta)$ and $\vec{n}_\zeta$ are same vectors up to a constant, we have $\partial\rho(\zeta) \wedge \partial\tilde{S}(\zeta,\zeta) \neq 0$. We fix $z \in D$ close to ∂D, and we use the coordinate system $t_1 = \rho(\zeta)$, $t_2 = \operatorname{Im}\tilde{S}(z,\zeta) = \operatorname{Im} S(z,\zeta)$, $t_3(z) = \cdots = t_{2n}(z) = 0$, $|t(\zeta) - t(\zeta')| \sim |\zeta - \zeta'|$ and $|t(\zeta)| < 1$ in a neighborhood $D \cap B(z,\gamma)$ of z. We prove the inequality for the first term in the right side of (4.1). By (4.2), we can see that the only singularity is of the form $|z-\zeta|^{-\alpha}$ and so everything is bounded if $|z-\zeta| > \gamma$. So it is enough to consider the case that z and ζ are in a neighborhood of ∂D and $\zeta \in B(z,\gamma)$.

We have

$$\begin{aligned}|\mathbb{K}^0 f(z)| &\lesssim \int_{\zeta\in D} |f(\zeta)||K^0(z,\zeta)|\,dV(\zeta)\\ &\lesssim \|f\|_{-\sigma}\int_{\zeta\in D}|\rho(\zeta)|^{-\sigma}|K^0(z,\zeta)|\,dV(\zeta).\end{aligned}$$

From (4.1) and (4.2) it follows that

$$\begin{aligned}I_0(z) &= \int_{\zeta\in D\cap B(z,\gamma)} |\rho(\zeta)|^{-\sigma}|K^0(z,\zeta)|\,dV(\zeta)\\ &\lesssim \int_{\zeta\in D\cap B(z,\gamma)} \frac{|\rho(\zeta)|^{N-\sigma}}{|S(z,\zeta)|^N|z-\zeta|^{2n-1}}\,dV(\zeta)\\ &\lesssim \int_{\zeta\in D\cap B(z,\gamma)} \frac{|\rho(\zeta)|^{N-\sigma}}{(|\operatorname{Im} S(z,\zeta)| + |\rho(\zeta)| + |\rho(z)|)^N|z-\zeta|^{2n-1}}\,dV(\zeta)\\ &\lesssim \int_{|(t_1,t_2,t')|<1} \frac{|t_1|^{N-\sigma}\,dt_1dt_2dt'}{(|t_1|+|t_2|+|\rho(z)|)^N|t|^{2n-1}}\\ &\lesssim \int_{|(t_1,t_2)|<1}\frac{|t_1|^{N-\sigma}\,dt_1dt_2}{(|t_1|+|t_2|+|\rho(z)|)^N(|t_1|+|t_2|)}.\end{aligned}$$

If we make the change of variables $t_1 = |\rho|t_1'$ and $t_2 = |\rho|t_2'$, and omit the primes, this becomes

$$\begin{aligned}I_0(z) &\lesssim |\rho(z)|^{-(\sigma-1)}\int_{(t_1,t_2)\in\mathbb{R}^2}\frac{|t_1|^{N-\sigma}\,dt_1dt_2}{(|t_1|+|t_2|+1)^N(|t_1|+|t_2|)}\\ &\lesssim |\rho(z)|^{-(\sigma-1)}\int_0^\infty \frac{t_1^{N-1-\sigma}}{(t_1+1)^{N-1}}\,dt_1 \lesssim |\rho(z)|^{-(\sigma-1)}.\end{aligned}$$

Thus we get the result (ii). ■

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5 Integral Estimates for the Kernels $K^\nu(z,\zeta)$

For $\zeta \in U$ and $\epsilon < \epsilon_0$ we define some sort of boundary distances by

$$\tau(\zeta, \vec{v}, \epsilon) = \sup\{r > 0 : |\rho(\zeta + \lambda\vec{v}) - \rho(\zeta)| < \epsilon, |\lambda| \le r, \lambda \in \mathbb{C}\}.$$

The quantity τ measures the size of the largest complex disc centered at ζ lying on the line spanned by $\vec{v}$ that fits in the domain $\{z : \rho(z) < \rho(\zeta)+\epsilon\}$. Next we define the ϵ-extremal basis $(\vec{v}_1, \dots, \vec{v}_n)$ centered at ζ of McNeal [Mc]. The first vector $\vec{v}_1$ is the unit vector in the direction of $\partial\rho(\zeta)$; having chosen $\vec{v}_1, \dots, \vec{v}_{i-1}$,let $\vec{v}_i$ be a unit vector orthogonal to $\vec{v}_1, \dots, \vec{v}_{i-1}$. In this way we obtain a basis $(\vec{v}_1, \dots, \vec{v}_n)$ depending on both ζ and $\epsilon > 0$. We denote the ν-th component of the coordinates with respect to this basis by $z_{\nu,\zeta,\epsilon}$. We call the coordinates by (ζ, ϵ)-extremal coordinates. We write $\tau_\nu(\zeta, \epsilon) := \tau(\zeta, \vec{v}_\nu, \epsilon)$. We can now define the non-isotropic polydiscs

$$AP_\epsilon(\zeta) = \{z \in \mathbb{C}^n : |z_{\nu,\zeta,\epsilon}| \le A\tau_\nu(\zeta, \epsilon), \nu = 1, \dots, n\}.$$

The following lemma can be found in [Mc].

Lemma 5.1

(i) *There are constants $C_1 > 1$ and $c_2 < 1$ (independent of ζ and ϵ) such that*

$$C_1 P_{\epsilon/2}(\zeta) \supset \tfrac{1}{2} P_\epsilon(\zeta) \quad \textit{for all } \zeta,\ \epsilon \tag{5.1}$$

$$C_1 P_t(\zeta) \subset P_\epsilon(\zeta) \quad \textit{for all } t < c_2\epsilon, \zeta, \epsilon. \tag{5.2}$$

(ii) *We have $\tau_1(\zeta, \epsilon) \approx \epsilon$ and $\epsilon^{1/2} \lesssim \tau_n(\zeta, \epsilon) \le \cdots \le \tau_2(\zeta, \epsilon) \lesssim \epsilon^{1/M}$. For $z \in P_\epsilon(\zeta)$ we have $|z - \zeta| \lesssim \epsilon^{1/M}$ and $z \notin P_\epsilon(\zeta)$ implies $|z - \zeta| \gtrsim \epsilon$.*

For integral estimates we define a family of polyannuli based on polydiscs from above. Using the constant C_1 from (5.1) of Lemma 5.1, we put

$$P^i_\epsilon(\zeta) = C_1 P_{2^i\epsilon}(\zeta) \setminus \tfrac{1}{2} P_{2^i\epsilon}(\zeta).$$

By (5.1) and (5.2) we see that

$$\bigcup_{i=0}^{\infty} P^{-i}_\epsilon(\zeta) \supset P_\epsilon(\zeta) \setminus \{\zeta\} \quad \text{and} \quad \bigcup_{i=0}^{\infty} P^{i}_\epsilon(\zeta) \supset P_{\epsilon_0}(\zeta) \setminus P_\epsilon(\zeta).$$

Lemma 5.2 ([DiMa]) *For integer i we have*

$$|S(z, \zeta)| \gtrsim 2^i \epsilon \tag{5.3}$$

uniformly in $z \in D \cap U,\ \zeta \in P^i_\epsilon(z) \cap D$.

Lemma 5.3 *Let $\rho = |\rho(z_0)|$. Then we get the estimate*

$$\int_{\zeta \in P^0_{c\rho}(z_0)} \frac{dV(\zeta)}{\tau_1(z_0, c\rho)^2 \prod_{j=n-\nu+2}^{n} \tau_j(z_0, c\rho)^2 |z_0 - \zeta|^{2n-2\nu-1}} \lesssim (c\rho)^{1/M}. \tag{5.4}$$

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Proof To estimate the integral (5.4) we make use of the $(c\rho)$-extremal coordinates at z_0. Then it is bounded by

$$\int_{|w_1|<\tau_1(z_0,c\rho)} \frac{du_1dv_1}{\tau_1(z_0,c\rho)^2} \times \prod_{j=n-\nu+2}^{n} \int_{|w_j|<\tau_j(z_0,c\rho)} \frac{du_jdv_j}{\tau_j(z_0,c\rho)^2} \prod_{j=2}^{n-\nu+1} \int_{|w_j|<\tau_j(z_0,c\rho)} \frac{du_jdv_j}{(\sum |w_j|)^{2n-2\nu-1}} \lesssim \int_0^{(c\rho)^{1/M}} \frac{r^{2n-2\nu-1}}{r^{2n-2\nu-1}}\,dr \lesssim (c\rho)^{1/M}.$$

■

For $\nu \geq 1$ we define

$$\mathbb{K}^\nu f(z) = \int_{\zeta\in D} f(\zeta) \wedge K^\nu(z,\zeta) \quad \text{for all } z \in D.$$

Theorem 5.4 *Let $\nu \geq 1$ and let $f \in L^{-\sigma}_{(0,q+1)}(D)$.*

(i) *For $0 \leq \sigma < \frac{1}{M}$, $\|\mathbb{K}^\nu f\|_{\Lambda_{1/M-\sigma}(D)} \lesssim \|f\|_{-\sigma}$.*

(ii) *For $\sigma = \frac{1}{M}$, $\|\mathbb{K}^\nu f\|_{\mathrm{BMO}(D)} \lesssim \|f\|_{-1/M}$.*

(iii) *For $\sigma > \frac{1}{M}$, $\|\mathbb{K}^\nu f\|_{-(\sigma-1/M)} \lesssim \|f\|_{-\sigma}$.*

Proof For the proof of (i) and (ii), it is enough to prove that

$$|\nabla \mathbb{K}^\nu f(z)| \lesssim \|f\|_{-\sigma} |\rho(z)|^{(1/M-\sigma)-1} \quad \text{for all } z \in D.$$

Thus we prove that

$$\int_{\zeta\in D} |\rho(\zeta)|^{-\sigma} |\nabla_z K^\nu(z,\zeta)|\, dV(\zeta) \lesssim |\rho(z)|^{(1/M-\sigma)-1} \quad \text{for all } z \in D. \tag{5.5}$$

For $\zeta \in P_\epsilon(z_0)$ we have

$$|\nabla_z K^\nu(z_0,\zeta)| \lesssim \Big|\frac{\rho(\zeta)}{S(z_0,\zeta)}\Big|^N \frac{\epsilon}{|\rho(\zeta)|^2 \tau_1(\zeta,\epsilon)^2 \prod_{n-\nu+2}^{n} \tau_j(\zeta,\epsilon)^2 |z_0-\zeta|^{2n-2\nu-1}}. \tag{5.6}$$

Let $\epsilon_0 > 0$ be sufficiently small. By (3.1), we can see that the only singularity is of the form $|z-\zeta|^{-\alpha}$ and so everything is bounded if $|z-\zeta| > \epsilon_0$. So it is enough to consider the case that ζ is in U and $\zeta \in P_{\epsilon_0}(z)$. For fixed z_0 we will define $\rho = |\rho(z_0)|$

and then split the polydisc $P_{\epsilon_0}(z_0)$ into the two parts $P_\rho(z_0)$ and $P_{\epsilon_0}(z_0)\setminus P_\rho(z_0)$. Recall that $P_\rho(z_0)$ can be covered by $\bigcup_{i=0}^{\infty} P_\rho^{-i}(z_0)$. By (5.6), it follows that

$$\begin{aligned} I_\nu^{-i}(z_0) &= \int_{\zeta\in P_\rho^{-i}(z_0)} |\rho(\zeta)|^{-\sigma}|\nabla_z K^\nu(z_0,\zeta)|\,dV(\zeta) \\ &\lesssim \int_{\zeta\in P_\rho^{-i}(z_0)} |\rho(\zeta)|^{-\sigma}\Big|\frac{\rho(\zeta)}{S(z_0,\zeta)}\Big|^N \\ &\qquad\qquad \times \frac{2^{-i}\rho\,dV(\zeta)}{|\rho(\zeta)|^2\tau_1(\zeta,2^{-i}\rho)^2\prod_{n-\nu+2}^{n}\tau_j(\zeta,2^{-i}\rho)^2|z_0-\zeta|^{2n-2\nu-1}}. \end{aligned}$$

It follows from (3.1) that $|\operatorname{Re} S(z_0,\zeta)| \gtrsim |\rho(z_0)| = \rho$ and so we have

$$\Big|\frac{\rho(\zeta)}{S(z_0,\zeta)}\Big|^N \lesssim \Big(\frac{|\rho(\zeta)|}{\rho}\Big)^{1+\sigma}\frac{|\rho(\zeta)|}{2^{-i}\rho}. \tag{5.7}$$

By (5.4) in Lemma 5.3, the integral can be estimated as follows

$$\begin{aligned} I_\nu^{-i}(z_0) &\lesssim \int_{\zeta\in P_\rho^{-i}(z_0)} \frac{2^{-i}\rho}{|\rho(\zeta)|^{2+\sigma}}\Big(\frac{|\rho(\zeta)|}{\rho}\Big)^{1+\sigma}\frac{|\rho(\zeta)|}{2^{-i}\rho} \\ &\qquad\qquad \times \frac{dV(\zeta)}{\tau_1(z_0,2^{-i}\rho)^2\prod_{j=n-\nu+2}^{n}\tau_j(z_0,2^{-i}\rho)^2|z_0-\zeta|^{2n-2\nu-1}} \\ &\lesssim \rho^{-1-\sigma}(2^{-i}\rho)^{1/M} = (2^{-i})^{1/M}\rho^{(1/M-\sigma)-1}. \end{aligned}$$

This implies that

$$\begin{aligned} \int_{\zeta\in P_\rho(z_0)} |\rho(\zeta)|^{-\sigma}|\nabla_z K^\nu(z_0,\zeta)| &\lesssim \sum_{i=0}^{\infty}(2^{-i})^{1/M}\rho^{(1/M-\sigma)-1} \\ &\lesssim \rho^{(1/M-\sigma)-1} = |\rho(z_0)|^{(1/M-\sigma)-1}. \end{aligned} \tag{5.8}$$

To estimate the integral over $P_{\epsilon_0}(z_0)\setminus P_\rho(z_0)$ we use the covering $\bigcup_{i=0}^{\infty} P_\rho^{i}(z_0)$. Then we have

$$\begin{aligned} I_\nu^{i}(z_0) &= \int_{\zeta\in P_\rho^{i}(z_0)} |\rho(\zeta)|^{-\sigma}|\nabla_z K^\nu(z_0,\zeta)|\,dV(\zeta) \\ &\lesssim \int_{\zeta\in P_\rho^{i}(z_0)} |\rho(\zeta)|^{-\sigma}\Big|\frac{\rho(\zeta)}{S(z_0,\zeta)}\Big|^N \\ &\qquad\qquad \times \frac{2^{i}\rho\,dV(\zeta)}{|\rho(\zeta)|^2\tau_1(\zeta,2^{-i}\rho)^2\prod_{n-\nu+2}^{n}\tau_j(\zeta,2^{-i})^2|z_0-\zeta|^{2n-2\nu-1}}. \end{aligned}$$

It follows from (5.3) that for $\zeta\in P_\rho^{i}(z_0)$ we have

$$\Big|\frac{\rho(\zeta)}{S(z_0,\zeta)}\Big|^N \lesssim (2^i)^{-(1+\sigma)}\Big(\frac{|\rho(\zeta)|}{\rho}\Big)^{1+\sigma}\frac{|\rho(\zeta)|}{2^{i}\rho}.$$

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By (5.4), the integral can be estimated as follows

$$I_\nu^i(z_0) \lesssim \int_{\zeta\in P_\rho^i(z_0)} (2^i)^{-(1+\sigma)} \left(\frac{|\rho(\zeta)|}{\rho}\right)^{1+\sigma} \frac{|\rho(\zeta)|}{2^i\rho} 2^i\rho$$
$$\times \frac{dV(\zeta)}{|\rho(\zeta)|^{2+\sigma}\tau_1(z_0,2^i\rho)^2\prod_{j=n-\nu+2}^n \tau_j(z_0,2^i\rho)^2|z_0-\zeta|^{2n-2\nu-1}}$$
$$\lesssim (2^i)^{(1/M-\sigma)-1}\rho^{(1/M-\sigma)-1}.$$

Since $(1/M-\sigma)-1<0$, it follows that

$$\text{(5.9)}\qquad \int_{\zeta\in P_{\epsilon_0}(z_0)\setminus P_\rho(z_0)} |\rho(\zeta)|^{-\sigma}|\nabla_z K^\nu(z_0,\zeta)| \lesssim \sum_{i=0}^\infty (2^i)^{(1/M-\sigma)-1}\rho^{(1/M-\sigma)-1}$$
$$\lesssim \rho^{(1/M-\sigma)-1} = |\rho(z_0)|^{(1/M-\sigma)-1}.$$

From (5.8) and (5.9) we get (5.5).

Now we prove (iii). For $\zeta\in P_\epsilon(z_0)$ we have

$$|K^\nu(z_0,\zeta)| \lesssim \Big|\frac{\rho(\zeta)}{S(z_0,\zeta)}\Big|^N \frac{\epsilon}{|\rho(\zeta)|\tau_1(\zeta,\epsilon)^2\prod_{n-\nu+2}^n\tau_j(\zeta,\epsilon)^2|z_0-\zeta|^{2n-2\nu-1}}.$$

Let $\epsilon_0>0$ be sufficiently small. For fixed z_0 we will define $\rho=|\rho(z_0)|$ and then split the polydisc $P_{\epsilon_0}(z_0)$ into the two parts $P_\rho(z_0)$ and $P_{\epsilon_0}(z_0)\setminus P_\rho(z_0)$. Recall that $P_\rho(z_0)$ can be covered by $\bigcup_{i=0}^\infty P_\rho^{-i}(z_0)$. By (5.6), it follows that

$$J_\nu^{-i}(z_0) = \int_{\zeta\in P_\rho^{-i}(z_0)} |\rho(\zeta)|^{-\sigma}|K^\nu(z_0,\zeta)|\,dV(\zeta)$$
$$\lesssim \int_{\zeta\in P_\rho^{-i}(z_0)} |\rho(\zeta)|^{-\sigma}\Big|\frac{\rho(\zeta)}{S(z_0,\zeta)}\Big|^N$$
$$\times \frac{2^{-i}\rho\,dV(\zeta)}{|\rho(\zeta)|\tau_1(\zeta,2^{-i}\rho)^2\prod_{n-\nu+2}^n\tau_j(\zeta,2^{-i}\rho)^2|z_0-\zeta|^{2n-2\nu-1}}.$$

It follows from (3.1) that $|\operatorname{Re} S(z_0,\zeta)|\gtrsim|\rho(z_0)|=\rho$ and so we have

$$\Big|\frac{\rho(\zeta)}{S(z_0,\zeta)}\Big|^N \lesssim \Big(\frac{|\rho(\zeta)|}{\rho}\Big)^\sigma\frac{|\rho(\zeta)|}{2^{-i}\rho}.$$

By (5.4), the integral can be estimated as follows

$$J_\nu^{-i}(z_0) \lesssim \int_{\zeta\in P_\rho^{-i}(z_0)} \frac{2^{-i}\rho}{|\rho(\zeta)|^{1+\sigma}}\Big(\frac{|\rho(\zeta)|}{\rho}\Big)^\sigma\frac{|\rho(\zeta)|}{2^{-i}\rho}$$
$$\times \frac{dV(\zeta)}{\tau_1(z_0,2^{-i}\rho)^2\prod_{j=n-\nu+2}^n\tau_j(z_0,2^{-i}\rho)^2|z_0-\zeta|^{2n-2\nu-1}}$$
$$\lesssim \rho^{-\sigma}(2^{-i}\rho)^{1/M} = (2^{-i})^{1/M}\rho^{-(\sigma-1/M)}.$$

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This implies that

$$\int_{\zeta\in P_\rho(z_0)} |\rho(\zeta)|^{-\sigma}|K^\nu(z_0,\zeta)| \lesssim \sum_{i=0}^{\infty}(2^{-i})^{1/M}\rho^{-(\sigma-1/M)} \lesssim \rho^{-(\sigma-1/M)} = |\rho(z_0)|^{-(\sigma-1/M)}. \tag{5.10}$$

To estimate the integral over $P_{\epsilon_0}(z_0)\backslash P_\rho(z_0)$ we use the covering $\bigcup_{i=0}^{\infty} P_\rho^i(z_0)$. Then we have

$$\begin{aligned} J_\nu^i(z_0) &= \int_{\zeta\in P_\rho^i(z_0)} |\rho(\zeta)|^{-\sigma}|K^\nu(z_0,\zeta)|\,dV(\zeta) \\ &\lesssim \int_{\zeta\in P_\rho^i(z_0)} |\rho(\zeta)|^{-\sigma}\Big|\frac{\rho(\zeta)}{S(z_0,\zeta)}\Big|^N \\ &\qquad\times \frac{2^i\rho\,dV(\zeta)}{|\rho(\zeta)|\tau_1(\zeta,2^{-i}\rho)^2\prod_{n-\nu+2}^{n}\tau_j(\zeta,2^{-i})^2|z_0-\zeta|^{2n-2\nu-1}}. \end{aligned}$$

It follows from (5.3) that for $\zeta\in P_\rho^i(z_0)$ we have

$$\Big|\frac{\rho(\zeta)}{S(z_0,\zeta)}\Big|^N \lesssim \Big(\frac{|\rho(\zeta)|}{2^i\rho}\Big)^{1+\sigma}.$$

By (5.4), the integral can be estimated as follows

$$\begin{aligned} J_\nu^i(z_0) &\lesssim \int_{\zeta\in P_\rho^i(z_0)} \Big(\frac{|\rho(\zeta)|}{2^i\rho}\Big)^{1+\sigma} 2^i\rho \\ &\qquad\times \frac{dV(\zeta)}{|\rho(\zeta)|^{1+\sigma}\tau_1(z_0,2^i\rho)^2\prod_{j=n-\nu+2}^{n}\tau_j(z_0,2^i\rho)^2|z_0-\zeta|^{2n-2\nu-1}} \\ &\lesssim (2^i)^{-(\sigma-1/M)}\rho^{-(\sigma-1/M)}. \end{aligned}$$

Since $\sigma-1/M>0$, it follows that

$$\begin{aligned} \int_{\zeta\in P_{\epsilon_0}(z_0)\backslash P_\rho(z_0)} &|\rho(\zeta)|^{-\sigma}|K^\nu(z_0,\zeta)| \\ &\lesssim \sum_{i=0}^{\infty}(2^i)^{-(\sigma-1/M)}\rho^{-(\sigma-1/M)} \lesssim \rho^{-(\sigma-1/M)} \\ &= |\rho(z_0)|^{-(\sigma-1/M)}. \end{aligned} \tag{5.11}$$

From (5.10) and (5.11) we get (iii). ■

By Theorem 4.1 and Theorem 5.4, we get Theorem 1.1.

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6 Sharpness of the Estimates

In this section we give an example to show that the estimates in Corollary 1.2 are optimal in some sense. We restrict ourselves to the complex ellipsoid in $\mathbb{C}^2$.

Let $D = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^M < 1\}$ where M is an even positive integer.

Sharpness of the case $\sigma = 0$

This was proved in [Ra1].

Let $\sigma > 0$. Define $v\colon D \to \mathbb{C}$ by $v(z) = \bar z_2/(1 - z_1)^\sigma$, where we use the principal branch for the $(1 - z_1)^\sigma$. It follows that the (0,1)-form

$$f = \bar\partial v = d\bar z_2/(1 - z_1)^\sigma$$

is $\bar\partial$-closed on D. Let $\rho(z) = |z_1|^2 + |z_2|^M - 1$. We have

$$|\rho(z)|^\sigma |f(z)| \lesssim (1 - |z_1|^2 - |z_2|^M)^\sigma \frac{1}{|1 - z_1|^\sigma} \lesssim 1.$$

Thus we have $f \in L^{-\sigma}_{(0,1)}(D)$.

Sharpness of the case $0 < \sigma < 1/M$

Proposition 6.1 *Suppose $u \in \Lambda_\alpha(D)$ satisfies $\bar\partial u = f$ on D. Then $\alpha \le 1/M - \sigma$.*

Proof For $0 < d < 1/2$, the integral

$$A(d) = \int_{|z_2| = d^{1/M}} [u(1 - d, z_2) - u(1 - 2d, z_2)]\, dz_2 \tag{6.1}$$

is well defined, and if $u \in \Lambda_\alpha(D)$, one obtains

$$|A(d)| \lesssim d^\alpha \cdot d^{1/M}$$

by direct estimation. On the other hand, $\bar\partial(u - v) = 0$, so $u = v + h$, with $h \in \mathcal{O}(D)$. By Cauchy's theorem we can replace u by v in the integral (5.1). Therefore

$$\begin{aligned} A(d) &= \Big[\frac{1}{d^\sigma} - \frac{1}{(2d)^\sigma}\Big] \int_{|z_2| = d^{1/M}} \bar z_2\, dz_2 \\ &= \Big[\frac{1}{d^\sigma} - \frac{1}{(2d)^\sigma}\Big] 2\pi i\, d^{2/M}. \end{aligned} \tag{6.2}$$

If $\alpha > 1/M - \sigma$, (6.1) and (6.2) lead to a contradiction as $d \to 0$. ■

Sharpness of the case $\sigma = 1/M$

We have

$$|\nabla v| \lesssim \frac{|z_2|}{|1-z_1|^{1/M+1}} + \frac{1}{|1-z_1|^{1/M}}.$$

Since $|z_2| < (1-|z_1|^2)^{1/M} \leq |1-z_1|^{1/M}$, it follows that

$$|\nabla v| \lesssim \frac{1}{|1-z_1|}.$$

We note that

$$|\rho(z)| = 1 - |z_1|^2 - |z_2|^M \leq 1 - |z_1| \leq |1 - z_1|.$$

Thus we have

$$|\nabla v| \lesssim \frac{1}{|\rho(z)|}.$$

By the Hardy–Littlewood lemma, $v \in \mathrm{BMO}(D)$ (see [Ra2]).

For $0 < d < 1/2$ we consider the integral

$$B(d) = \int_{|z_2|=d^{1/M}} [v(1-d, z_2) - v(1-2d, z_2)]\, dz_2.$$

If $v \in \Lambda_\epsilon(D)$, we see that

$$|B(d)| \lesssim d^\epsilon \cdot d^{1/M}. \tag{6.3}$$

However, we have

$$B(d) = \Big[\frac{1}{d^{1/M}} - \frac{1}{(2d)^{1/M}}\Big] 2\pi i\, d^{2/M}. \tag{6.4}$$

We can see that (6.3) and (6.4) lead to a contradiction as $d \to 0$. Thus $v \notin \Lambda_\epsilon(D)$ for any $\epsilon > 0$.

Proposition 6.2 *Suppose u satisfies $\bar\partial u = f$ on D. Then $u \notin \Lambda_\epsilon(D)$ for any $\epsilon > 0$.*

Proof

Since $v \in \mathrm{BMO}(D)$, $v \in L^2(D)$. Let $r_{z_1} = (1-|z_1|^2)^{1/M}$. We consider the inner product $\langle h, v\rangle$ for every $h \in L^2(D) \cap \mathcal{O}(D)$. By Fubini's theorem, we have

$$\begin{aligned}\langle h, v\rangle &= \int_D h(z)\, \overline{v(z)}\, dV(z)\\ &= \int_{|z_1|<1} \frac{dA(z_1)}{(1-z_1)^{1/M}} \int_{|z_2|<r_{z_1}} z_2 h(z_1, z_2)\, dA(z_2).\end{aligned}$$

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Putting $z_2 = re^{i\theta}$, we see

$$\begin{aligned}\int_{|z_2|<r_{z_1}} z_2\, h(z_1,z_2)\, dA(z_2) &= \int_0^{r_{z_1}} \int_0^{2\pi} r^2 e^{i\theta} h(z_1, re^{i\theta})\, d\theta dr \\ &= \int_0^{r_{z_1}} r^2 \Big(\int_0^{2\pi} e^{i\theta} h(z_1, re^{i\theta})\, d\theta \Big)\, dr \\ &= \int_0^{r_{z_1}} r^2 \cdot 0 \cdot\, dr = 0,\end{aligned}$$

since $h(z_1,\cdot)$ is holomorphic. Thus v is orthogonal to $L^2(D) \cap \mathcal{O}(D)$, *i.e.*, v is the canonical solution for $\overline{\partial} u = f$.

Assume that $\overline{\partial} u = f$ and $u \in \Lambda_\epsilon(D)$. By [McSt], it follows that $Pu \in \Lambda_\epsilon(D)$, where P is the Bergman projection on D. Thus $v = u - Pu \in \Lambda_\epsilon(D)$. This is a contradiction. Hence there is no solution u in $\Lambda_\epsilon(D)$ to the equation $\bar\partial u = f$ for $\epsilon > 0$. ∎

Sharpness of the case $\sigma > 1/M$

Note that

$$|v| \lesssim \frac{|z_2|}{|1-z_1|^\sigma} \lesssim \frac{1}{|1-z_1|^{\sigma-1/M}} \lesssim \frac{1}{|\rho(z)|^{\sigma-1/M}}.$$

Thus $v \in L^{-(\sigma-1/M)}(D)$.

Proposition 6.3 *Suppose $u \in L^{-\alpha}(D)$ satisfies $\bar\partial u = f$ on D. Then $\alpha \geq \sigma - 1/M$.*

Proof For $0 < d < 1/2$, we consider the integral

$$C(d) = \int_{|z_2|=d^{1/M}} u(1-d, z_2)\, dz_2.$$

If $u \in L^{-\alpha}(D)$, then

$$\begin{aligned}|C(d)| &\lesssim \int_{|z_2|=d^{1/M}} |u(1-d,z_2)|\, |dz_2| \\ &\lesssim \int_{|z_2|=d^{1/M}} \frac{1}{|\rho(1-d,z_2)|^\alpha}\, |dz_2|.\end{aligned}$$

We have

$$|\rho(1-d,z_2)| = 1-(1-d)^2 - |z_2|^M \geq 1-(1-d)^2 \geq d.$$

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Thus we have

$$|C(d)| \lesssim \frac{1}{d^{\alpha}} \cdot d^{1/M}. \tag{6.5}$$

On the other hand, $\bar{\partial}(u - v) = 0$, so $u = v + h$, with $h \in \mathcal{O}(D)$. By Cauchy's theorem we can replace u by v in the integral (5.1). Therefore

$$C(d) = \frac{1}{d^{\sigma}} \cdot 2\pi i\, d^{2/M}. \tag{6.6}$$

If $\alpha < \sigma - 1/M$, (6.5) and (6.6) lead to a contradiction as $d \to 0$. ■

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