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EXTREMAL PROBLEMS IN H^p

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Abstract

Let $1 \leq p < \infty$ and $1/p + 1/q = 1$. If $\phi \in L^q$, we denote by T_ϕ the functional defined on the Hardy space H^p by

$$T_\phi^p(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\phi(e^{i\theta})\, d\theta/2\pi.$$

A function f in H^p, which satisfies $T_\phi^p(f) = \|T_\phi^p\|$ and $\|f\|_p \leq 1$, is called an extremal function. Also, ϕ is called an extremal kernel when $\|\phi\|_q = \|T_\phi^p\|$. In this paper, using the results in the case of $p = 1$, we study extremal kernels and extremal functions for $p > 1$.

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1. Introduction

Let U be the open unit disc in the complex plane and ∂U the boundary of U. A function f which is analytic in U is said to belong to the class H^p $(0 < p < \infty)$ if

$$\|f\|_p = \lim_{r \to 1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p} < \infty.$$

The class of bounded analytic functions is denoted by H^∞ and $\|f\|_\infty = \lim_{r\to 1} \max_{0\leq\theta\leq 2\pi} |f(re^{i\theta})|$. Each $f \in H^p$ has a radial limit $f(e^{i\theta})$ almost

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everywhere. If $h \in H^p$ has the form

$$h(z) = \exp\left\{\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log|h(e^{it})|\,dt/2\pi + i\alpha\right\} \quad (z \in U)$$

for some real α, then h is called an outer function. We call $Q \in H^\infty$ an inner function if $|Q(e^{i\theta})| = 1$ a.e. on ∂U. Let h be a nonzero function in H^1. Then h is an outer function if and only if u is constant whenever $uh \in H^1$ for some $u \in L^\infty$ with $u \geq 0$ a.e.

DEFINITION. Let g be a nonzero function in H^1. We say g is a strong outer function if it has the following property: if $ug \in H^1$ for some Lebesgue measurable u with $u \geq 0$ a.e., then u is constant.

For $k \in L^q$, put

$$\|k + zH^q\| = \inf\{\|k + z\ell\|_q : l \in H^q\}.$$

Then $\|T_k^p\| = \|k + zH^q\|$ where $1/p + 1/q = 1$. Also ϕ is an extremal kernel if and only if $\|\phi\|_q = \|\phi + zH^q\|$. When $q < \infty$ and $1/p + 1/q = 1$, if $k \in L^q$ there exists a unique extremal kernel ϕ in $k + zH^q$ and a unique extremal function f in H^p such that

$$f(e^{i\theta})\phi(e^{i\theta}) \geq 0 \text{ a.e. } \theta \tag{1}$$

and

$$|f(e^{i\theta})|^p = \|\phi\|_q^{-q}|\phi(e^{i\theta})|^q \text{ a.e. } \theta \tag{2}$$

(cf. [2, pages 132–133]). When $q = \infty$ and $p = 1$, if $k \in L^\infty$ there exists an extremal kernel ϕ in $k + zH^\infty$, but there may not exist any extremal functions in H^1. Moreover, if an extremal function exists then the extremal kernel is unique. In general, there may exist many extremal functions. In this paper S_k denotes the set of extremal functions of H^1. If S_k is weak-* compact in H^1, then S_k consists of functions f in H^1 which have the following form:

$$f = \gamma \prod_{j=1}^{n}(z - a_j)(1 - \bar{a}_j z)g, \tag{3}$$

where γ is positive constant, $|a_j| \leq 1$ $(1 \leq j \leq n)$ and g is a strong outer function and the extremal function ϕ has the form

$$\phi = \bar{z}^n \frac{|g|}{g}. \tag{4}$$

This has been shown by the author [4, Theorem 2]. In this paper, using this description of S_k, we describe extremal kernels and extremal functions for

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$1 < p < \infty$ and $1/p + 1/q = 1$. Hence extremal functions for $1 < p < \infty$ have similar forms to those for $p = 1$. Previously these have been described in the case of rational kernels in different forms and by a different method (cf. [3]).

2 General kernels

The following simple theorem, which gives a relation between extremal problems of H^1 and H^p $(1 < p < \infty)$, is essential in this paper.

THEOREM 1. *For each* p $(1 < p < \infty)$ *and for each function*

$$k \in L^q(1/p + q/q = 1)$$

with $k \notin zH^q$, *if* ϕ *is a unique extremal kernel in* $k + zH^q$ *and* f *is a unique extremal function of* T_k^p *then*

$$\phi = \phi_0 h, \qquad f = \|\phi\|_q^{-q} Q h^{q/p}$$

and

$$\|\phi\|_q^{-q} Q h^q \in S_{\phi_0}, \qquad \phi_0 = \overline{Q}|h|^q h^{-q},$$

where h *is an outer function with* $|\phi| = |h|$, *and* Q *is the inner part of* f. *Conversely, if* ϕ *and* f *have the forms above, then* ϕ *is an extremal kernel and* f *is an extremal function of* T_ϕ^p.

PROOF. If ϕ is an extremal kernel, then by (2) in the introduction, $\log|\phi|$ is integrable and hence there exists an outer function h with $|\phi| = |h|$. By (2), $f = \|\phi\|_q^{-q/p} Q h^{q/p}$, where Q is the inner part of f. Put $\phi_0 = \phi/h$. Then by (1) in the introduction

$$\|\phi\|_q^{-q/p} Q h^{q/p+1} \phi_0 \geq 0 \text{ a.e. on } \partial U.$$

The L^1-norm of

$$\|\phi\|_q^{-q/p} Q h^{q/p+1} = \|\phi\|_q^{-q/p} Q h^q$$

is $\|\phi\|_q$. Hence

$$\|(\|\phi\|_q^{-q} Q h^q \phi_0)\|_1 = 1 \qquad \text{and} \qquad \|\phi\|_q^{-q} Q h^q \phi_0 \geq 0 \text{ a.e.}$$

By [2, page 133], $\|\phi\|_q^{-q} Q h^q$ belongs to S_{ϕ_0} and $\phi_0 = \overline{Q}|h|^q h^{-q}$.

When $p = 2$, ϕ is an extremal kernel for H^2 if and only if ϕ belongs to $\overline{H}^2$. This trivial result is also a corollary of the following.

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COROLLARY 1. *Let ϕ be a function in L^q with $\phi \notin zH^q$, $1 < q < \infty$ and $1/p + 1/q = 1$. Then ϕ is a unique extremal kernel of T_ϕ^p if and only if*

$$\phi = \overline{Q}|h|^q h^{-q/p},$$

where Q is an inner function and h is an outer function in H^q.

PROOF. If ϕ is a unique extremal kernel then by Theorem 1, $\phi = \phi_0 h$ and $\phi_0 = \overline{Q}|h|^q h^{-q}$. Hence the 'only if' part follows. Conversely, if $\phi = \overline{Q}|h|^q h^{-q/p}$ then $|\phi| = |h|$. Put $f = \|\phi\|_q^{-q} Q h^{q/p}$. Then $\phi f \geq 0$ a.e. on ∂U and $\|\phi\|_q^{-q} Q h^q \in S_{\phi_0}$ if $\phi_0 = \phi/h$.

COROLLARY 2. *Let $g = Bh$ be a nonzero function in H^q $(1 < q < \infty)$, where B is an inner function and h is an outer function. Then $\|\overline{g} + zH^q\| = \|\overline{g}\|_q$ if and only if $Bh^{2-q}/|h|^{2-q}$ is an inner function.*

PROOF. If $\|\overline{g} + zH^q\| = \|\overline{g}\|_q$ then $\overline{g}$ is a unique extremal kernel and $|\overline{g}| = |h|$ by the definition of g. By Corollary 1, $\overline{Bh} = \overline{Q}|h|^q h^{-q/p}$ and hence $\overline{B}|h|^2 h^{-2} = \overline{Q}|h|^q h^{-q}$. This implies the 'only if' part. The 'if' part is also clear by Corollary 1.

COROLLARY 3. (i) *If g and g^{-1} are in H^∞ and nonconstant, then for any q with $1 < q < \infty$, and $q \neq 2$, $\|\overline{g} + zH^q\| \neq \|\overline{g}\|_q$.*

(ii) *If g is an inner function, then for any q with $1 < q < \infty$, $\|\overline{g} + zH^q\| = \|\overline{g}\|_q$.*

(iii) *Suppose $2 < q < \infty$. If $g \in H^q$ is a nonconstant outer function, then $\|\overline{g} + zH^q\| \neq \|\overline{g}\|_q$.*

(iv) *If $1 < q \leq 2$, then there exists an outer function g in H^q such that $\|\overline{g} + zH^q\| = \|\overline{g}\|_q$.*

(v) *If $2 < q < \infty$, then there exists a nonzero function g such that g is not an inner function in H^q and $\|\overline{g} + zH^q\| = \|\overline{g}\|_q$.*

PROOF. (i) If g and g^{-1} are i H^∞, then for any q with $1 < q < \infty$, $g \in H^q$ and $g^{2-q}/|g|^{2-q}$ is not an inner function if $q \neq 2$. For, if $g^{2-q}/|g|^{2-q} = Q$ is inner, then $g^{q-2}Q$ is a nonnegative function in H^∞ and hence it is a nonzero constant. Thus g is a constant. This contradiction and Corollary 2 imply (i). Part (ii) is clear from Corollary 2.

(iii) When $2 < q < \infty$, if g is outer then $g^{q-2} \in H^{q/q-2} \subset H^1$. If $\|\overline{g} + zH^q\| = \|\overline{g}\|_q$, then by Corollary 2 there exists an inner function Q

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such that

$$Q = \frac{g^{2-q}}{|g|^{2-q}} = \frac{|g|^{q-2}}{g^{q-2}}.$$

Hence Qg^{q-2} is a non-negative function in H^1 and hence Qg^{q-2} is outer (see [4, Proposition 5]). Thus Q and g are constants. This contradiction implies (iii).

(iv) When $1 < q < 2$, put $g = \{-(z-1)^2\}^{1/2-q}$. Then $g^{2-q}/|g|^{2-q} = z$. Now part (iv) follows from Corollary 2.

(v) Put $B = z^2$, $Q = z$ and $g = \{-(z-1)^2\}^{1/q-2}$. Then $Bg^{2-q}/|g|^{2-q} = Q$ and $g \in H^q$ if $q > 2$.

3. Special kernels

Let C denote the space of continuous functions on ∂U and set $A = H^\infty \cap C$. Then $H^1 = (C/zA)^*$. If $\phi_0 \in C$, then S_{ϕ_0} is weak-* compact (cf. [4, page 225]). If $k \in L^q$ is a good function, then the ϕ_0 in Theorem 1 may satisfy the condition that S_{ϕ_0} is weak-* compact.

THEOREM 2. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Suppose $\phi = \phi_0 h$ is a unique extremal kernel of T_ϕ^p and S_{ϕ_0} is weak-* compact, where h is an outer function in H^q with $|\phi| = |h|$. If f is an extremal function of T_ϕ^p, then*

$$\phi = \bar{z}^n \frac{|g|}{g} \prod_{j=1}^{s} (1 - \bar{a}_j z)^{2/q} \left\{ \prod_{j=s+1}^{n} (z - a_j)(1 - \bar{a}_j z) \right\}^{1/q} g^{1/q}$$

and

$$f = \|\phi\|_q^{-q} \prod_{j=1}^{s} (z - a_j)(1 - \bar{a}_j z)^{2/p} \left\{ \prod_{j=s+1}^{n} (z - a_j)(1 - \bar{a}_j z) \right\}^{1/p} g^{1/p},$$

where $|a_j| < 1$ if $1 \le j \le s$, $|a_j| = 1$ if $s+1 \le j \le n$ and g is a strong outer function.

PROOF. By hypothesis S_{ϕ_0} is weak-* compact. Hence by (3) in the introduction and Theorem 2, we have

$$\|\phi\|_q^{-q} Q h^q = \gamma \prod_{j=1}^{n} (z - \alpha_j)(1 - \bar{\alpha}_j z) g_1$$

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where γ is a positive constant, $|\alpha_j| \le 1$ $(1 \le j \le n)$ and g_1 is a strong outer function, and Q is the inner part of f. Put $g = \gamma g_1$. Then g is also a strong outer function and $|g|g^{-1} = |g_1|g_1^{-1}$ and we can write the right hand as the following:

$$\|\phi\|_q^{-q} Q h^q = \prod_{j=1}^{n} (z - a_j)(1 - \overline{a}_j z) g$$

where $|a_j| < 1$ if $1 \le j \le s$ and $|a_j| = 1$ if $s+1 \le j \le n$. Hence

$$Q = \prod_{j=1}^{s} \frac{z - a_j}{1 - \overline{a}_j z} \quad \text{and} \quad h^q = \|\phi\|_q^q \prod_{j=1}^{s} (1 - \overline{a}_j z)^2 \prod_{j=s+1}^{n} (z - a_j)(1 - \overline{a}_j z) g .$$

By (4) in the introduction, $\phi_0 = \overline{z}^n |g| g^{-1}$. Now Theorem 1 implies the theorem.

Corollary 4. *Let* $1 < p < \infty$ *and* $1/p + 1/q = 1$. *If* $k = k_2/k_1 \in L^q$ *with* $k \notin zH^q$ *and* k_j *is a nonzero function* H^∞ *for* $j = 1, 2$, *then the extremal kernel* ϕ *of* T_k^p *has the form*

$$\phi = \overline{Q}_1 Q_2 h_2 / h_1 = \phi_0 h, \ \phi_0 = \overline{Q}_1 Q_2 \quad \textit{and} \quad h = h_2/h_1$$

where $k_1 = Q_1 h_1$, $k_1 \phi = Q_2 h_2$, Q_j *is inner and* h_j *is outer. If* Q_1 *is a finite Blaschke product then* Q_2 *is also a finite Blaschke product and degree of* $Q_2 \le$ *degree of* Q_1. *Suppose* $\{\beta_j\}_{j=1}^n$ *are the zeros of* Q_1, $\{\alpha_j\}_{j=1}^t$ *are the zeros of* Q_2 *and* $t \le n$. *If* f *is a unique extremal kernel of* T_k^p, *then*

$$\phi = \prod_{j=t+1}^{s} (1 - \overline{a}_j z)^{2/q} \left\{ \prod_{j=s+1}^{n} (z - a_j)(1 - \overline{a}_j z) \right\}^{1/q} \times \gamma^{1/q} \prod_{j=1}^{t} \frac{(z - \alpha_j)(1 - \overline{\alpha}_j z)^{2/q-1}}{(z - \beta_j)(1 - \overline{\beta}_j z)^{2/q-1}} \prod_{j=t+1}^{n} \frac{1}{(z - \beta_j)(1 - \overline{\beta}_j z)^{2/q-1}}$$

and

$$f = \|\phi\|_q^{-q} \prod_{j=t+1}^{s} (z - a_j)(1 - \overline{a}_j z)^{2/p-1} \left\{ \prod_{j=s+1}^{n} (z - a_j)(1 - \overline{a}_j z) \right\}^{1/p} \times \gamma^{1/p} \prod_{j=1}^{t} \frac{(1 - \overline{\alpha}_j z)^{2/p}}{(1 - \overline{\beta}_j z)^{2/p}} \prod_{j=t+1}^{n} (1 - \overline{\beta}_j z)^{-2/p},$$

where $|a_j| < 1$ *if* $t+1 \le j \le s$, $|a_j| = 1$ *if* $s+1 \le j \le n$ *and* γ *is a positive constant.*

PROOF. The first part is clear. For the second part, since $\phi_0 = \overline{Q}_1 Q_2$ and S_{ϕ_0} is non-empty by Theorem 2, Q_2 is a finite Blaschke product of degree at most degree Q_1. We will prove only the third part. Since $\phi_0 = \overline{Q}_1 Q_2$ is a continuous function, S_{ϕ_0} is weak-* compact and hence we can apply Theorem 2 to the corollary. Since

$$\phi_0 = \prod_{j=1}^{n} \frac{1-\overline{\beta}_j z}{z-\beta_j} \prod_{j=1}^{t} \frac{z-\alpha_j}{1-\overline{\alpha}_j z},$$

putting $S^1 =$ the unit sphere of H^1, we obtain that

$$S_{\phi_0} = \left\{ \gamma \prod_{j=t+1}^{n} (z-a_j)(1-\overline{a}_j z) g_1 \in S^1 : |a_j| < 1 \right.$$
$$\left. \text{if } t+1 \le j \le s \text{ and } |a_j| = 1 \text{ if } \quad s+1 \le j \le n \right\}$$

and

$$g = \gamma g_1 = \gamma \prod_{j=1}^{t} \frac{(1-\overline{\alpha}-jz)^2}{(1-\overline{\beta}_j z)^2} \prod_{j=t+1}^{n} (1-\overline{\beta}_j z)^{-2}.$$

This can be proved from the fact that

$$\left(\prod_{j=1}^{t} \frac{1-\overline{\beta}_j z}{z-\beta_j} \prod_{j=1}^{t} \frac{z-\alpha_j}{1-\overline{\alpha}_j z} \right) \prod_{j=1}^{t} \frac{(1-\overline{\alpha}_j z)^2}{(1-\overline{\beta}_j z)^2} \ge 0 \text{ a.e. on } \partial U$$

and

$$S_{\phi_1} = \left\{ \gamma_1 \prod_{j=t+1}^{n} (z-a_j)(1-\overline{a}_j z) \prod_{j=1t+1}^{n} (1-\overline{\beta}_j z)^{-2} \in S^1 : |a_j| < 1 \right.$$
$$\left. \text{if } t+1 \le j \le s \text{ and } |a_j| = 1 \quad \text{if } s+1 \le j \le n \right\}.$$

where $\phi_1 = \prod_{j=t+1}^{n} \frac{1-\overline{\beta}_j z}{z-\beta_j}$.

Thus Theorem 2 implies the corollary if the concrete forms of ϕ_0 and g are used.

A. J. Macintyre and W. W. Rogosinski [3] described completely the extremal functions and the kernels in the case of rational kernels, by a different method. Their result follows from Corollary 4. When a kernel k is analytic on ∂U, the extremal kernel [5, page 141] and the extremal function [1] had been described.

If a kernel k is analytic on ∂U, then the extremal kernel ϕ of k is of the form $\phi = Q^{2/p} G$ where Q is a trigonometric polynomial and G is holomorphic on ∂U (cf. [5, page 141]). It is not difficult to see that S_{ϕ_0} is weak-* compact. Hence we can apply Theorem 2 to ϕ. This describes an extremal kernel and an extremal function a little differently from [5] and [1].

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