BULL. AUSTRAL. MATH. SOC. 08C05, 08A60
VOL. 65 (2002) [491-502]

NONDUALISABLE SEMIGROUPS

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An infinite family of finite semigroups is studied. It is shown that most of them do not generate a quasivariety which admits a natural duality.

The theory of *natural dualities* has its roots in the duality for Boolean Algebras due to Stone (see [**21**]) and in Pontryagin's duality for Abelian Groups (see [**15, 16**]). The discovery by Priestley of a duality for Distributive Lattices (see [**17, 18**]) provided additional momentum to the field. After this, many similar dualities were rapidly discovered. Then in [**11**], Davey and Werner provided a general framework within which most dualities could be placed. Their theory of *natural dualities* is applicable to any quasivariety generated by a finite algebra, and shows how such dualities arise naturally from the structure of the quasivariety.

There is now a large body of results on natural dualities. The recent book [**3**] by Clark and Davey is the definitive work on the subject. The *dualisability problem* is basic to the theory of natural dualities. It asks which finite algebras in a given class are *dualisable*, that is, generate a quasivariety which admits a natural duality. This problem is subtly different from that of finding which quasivarieties have dualities, since a natural duality is produced from a particular generator of a quasivariety. The *Independence Theorem* (jointly discovered by Davey and Willard in [**12**] and by Saramago in [**20**]) simplifies the situation somewhat. It states that when two finite algebras generate the same quasivariety, it is dualisable with respect to one if and only if it is dualisable with respect to the other.

The dualisability problem has been solved piecemeal for various well-known classes of algebras. Since the quasivariety of Boolean Algebras is generated by any one of its members, every finite Boolean Algebra is shown to be dualisable by a combination of Stone's original result and the Independence Theorem. Similarly, every finite Distributive Lattice is dualisable. In [**11**], Davey and Werner built on Pontryagin's duality for Abelian

Received 6th November, 2001
I wish to thank David Clark, who gave a very lucid seminar talk which started my investigations. He was also very helpful in his explanations of the history and present status of natural duality theory. I also wish to thank Brian Davey and Jane Pitkethly, who graciously provided me with some of the TEX macros used in this paper.

Groups, and showed that every cyclic group is dualisable. This was later extended by Davey, who showed in [**7**] that every finite Abelian Group is dualisable.

In the three classes above, the dualisability problem was solved by showing that every finite algebra in the class was dualisable. It is of course possible that some of the finite algebras in a class are dualisable while some are not. This happens in the class of congruence distributive algebras. Davey and Werner proved in [**11**] that a finite algebra in a congruence distributive variety is dualisable if it has an n-ary near-unanimity term for some n. Davey, Heindorf and McKenzie proved the converse in [**8**]. Together, these two results reduce the dualisability problem for an algebra in a congruence distributive variety to the problem of deciding whether or not it has an n-ary near-unanimity term for some n. This is not exactly a *solution* of the dualisability problem for this class, since there is no known recursive method to determine whether or not such a near-unanimity term exists. Indeed, recent unpublished work by Ralph McKenzie and by Miklos Maroti shows that several problems related to the existence of a near-unanimity term are undecidable.

In broad generality, the "simplest" algebras in a class of algebras are dualisable, while the others are not. For example, the dualisability problem has been solved for finite Commutative Rings with Identity by Clark, Idziak, Sabourin, Szabó and Willard in [**6**]. They show that a finite commutative ring is dualisable if and only if its Jacobsen radical annihilates itself. On the other hand, the results of Clark, Davey and Pitkethly in [**5**] show how messy things can be. They completely solve the dualisability problem for the class of three-element unary algebras. While one might expect such a simple class to have a correspondingly simple dividing line between its dualisable and nondualisable members, their results show that this is not at all the case. For more examples and for background, the reader is again referred to the Clark and Davey book [**3**].

An area that is presently being worked on is the class of Semigroups. As mentioned above, finite Abelian Groups are dualisable. Davey and Quackenbush show in [**10**] that the finite dihedral groups D_n for odd n are dualisable as well. On the other hand, Quackenbush and Szabó prove in [**19**] that finite nonabelian nilpotent groups are not dualisable.

While the dualisability problem is nowhere near solved for the class of Groups, even less is known for other Semigroups. It is known that Rectangular Bands are dualisable (see [**3**]). Davey and Knox show in [**9**] that certain small commutative semigroups are dualisable. Among other results, their method shows that any semigroup produced by adjoining a zero to a dualisable group is dualisable. Their method also applies to left-zero semigroups (which satisfy $x \cdot y \approx x$) and to right-zero semigroups, both of which are rectangular bands. It yields that semigroups produced by adjoining a zero to a finite left-zero or right-zero semigroup are dualisable.

Apart from the work of Quackenbush and Szabó mentioned above, no examples of nondualisable semigroups were known. The role of this paper is to provide a new class of such examples, the first which are not groups. The new results are in the second section, after a preliminary section devoted to background material and notation. A section of examples and remarks concludes this paper.

1. Preliminaries

We let $\mathbb{N}$ denote $\{0,1,2,\ldots\}$, the set of natural numbers, and we use $\mathbb{N}^+$ for the set $\{1,2,3,\ldots\}$. Our notation will be standard for Universal Algebra—either of the books [**2**] or [**14**] may be used as a reference.

We shall briefly review the key concepts and definitions of natural duality theory, and then state the theorems which we shall borrow from other sources. A more detailed exposition may be found in Clark and Davey [**3**].

We let $\underline{\mathbf{M}}$ be a finite algebra, and attempt to construct a duality for the quasivariety $\mathcal{A} := \mathbb{ISP}\underline{\mathbf{M}}$ by using an *alter ego* of $\underline{\mathbf{M}}$. This alter ego will have the same universe M as $\underline{\mathbf{M}}$, but will have new operations and relations on it. So we define an *algebraic relation on* $\underline{\mathbf{M}}$ to be a set $R \subseteq M^n$ which is the universe of a subalgebra of $\underline{\mathbf{M}}^n$. The arity of R can be any $n \in \mathbb{N}^+$. For each $n \in \mathbb{N}^+$, we could also define an *algebraic operation* (or *algebraic partial operation*) on $\underline{\mathbf{M}}$ to be a homomorphism from $\underline{\mathbf{M}}^n$ (or a subalgebra of $\underline{\mathbf{M}}^n$) to $\underline{\mathbf{M}}$. Since it will be enough for our purposes to represent any such algebraic (partial) operation by the relation that is its graph, we shall restrict ourselves to considering algebraic relations on the set M.

We say that an *alter ego of* $\underline{\mathbf{M}}$ is a topological structure $\underset{\sim}{\mathbf{M}} = \langle M; R, \mathcal{T}\rangle$ where R is a set of algebraic relations on $\underline{\mathbf{M}}$, and $\mathcal{T}$ is the discrete topology on M. If we can find the correct alter ego $\underset{\sim}{\mathbf{M}}$ of $\underline{\mathbf{M}}$, we may be able to represent the algebras in the quasivariety $\mathcal{A}$ as algebras of continuous homomorphisms.

So let $\underset{\sim}{\mathbf{M}} = \langle M; R, \mathcal{T}\rangle$ be some fixed alter ego of $\underline{\mathbf{M}}$. Given any algebra $\mathbf{A} \in \mathcal{A}$, we define its *dual*, $\mathrm{D}(\mathbf{A})$, to be the set $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$, of all homomorphisms from $\mathbf{A}$ to $\underline{\mathbf{M}}$, viewed as a substructure of $\underset{\sim}{\mathbf{M}}^A$. (We give $\underset{\sim}{\mathbf{M}}^A$ the product topology.) Now $\mathrm{D}(\mathbf{A})$ is a substructure of $\underset{\sim}{\mathbf{M}}^A$, and it is closed since all of the relations of $\underset{\sim}{\mathbf{M}}$ are finitary.

Similarly, the set of continuous homomorphisms from $D(\mathbf{A})$ into $\underset{\sim}{\mathbf{M}}$ forms a subalgebra $ED(\mathbf{A})$ of $\underline{\mathbf{M}}^{D(\mathbf{A})}$. For E and D to give a duality, we need that $\mathbf{A}$ is isomorphic to $ED(\mathbf{A})$. Note that the *evaluation map* $e_{\mathbf{A}} : \mathbf{A} \to ED(\mathbf{A})$ defined by $e_{\mathbf{A}}(a)(x) = x(a)$ is an embedding. These considerations lead us to the following definition.

We say that $\underset{\sim}{\mathbf{M}}$ *yields a duality* for $\mathbf{A}$ if and only if $e_{\mathbf{A}}$ is surjective. This is equivalent to having every continuous homomorphism $\alpha : D(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ equal to evaluation at some $a \in A$ (so $\alpha(x) = x(a)$ for this a). We then say that the algebra $\underline{\mathbf{M}}$ is *dualisable* if and only if there is some choice of $\underset{\sim}{\mathbf{M}}$ which yields a duality on each member of $\mathcal{A}$. And of course $\underline{\mathbf{M}}$ is *nondualisable* if and only if it has no alter ego which yields a duality for every algebra $\mathbf{A} \in \mathcal{A}$. Thus to show that the algebra $\underline{\mathbf{M}}$ is nondualisable, it is enough to produce an algebra $\mathbf{A}$ in $\mathcal{A}$ such that no alter ego $\underset{\sim}{\mathbf{M}}$ yields a duality for it.

When $\alpha : \mathrm{D}(\mathbf{A}) \to M$, a subset B of A is a *support* for α if and only if for all $x \in \mathrm{D}(\mathbf{A})$ the value of $\alpha(x)$ is determined by $x{\restriction}_B$. For each $Y \subseteq \mathrm{D}(\mathbf{A})$ and $a \in A$, a map $\alpha : \mathrm{D}(\mathbf{A}) \to M$ is *given by evaluation at a on Y* if $\alpha{\restriction}_Y = e_{\mathbf{A}}(a){\restriction}_Y$. For each $n \in \mathbb{N}^+$, let R_n denote the set of n-ary algebraic relations on $\underline{\mathrm{M}}$. When $\alpha : \mathrm{D}(\mathbf{A}) \to M$, we say that α *preserves* R_n if and only if α preserves every relation in R_n. The following lemma was first explicitly proved in [**4**].

Lemma 1.1. *Let $n \in \mathbb{N}^+$, let $\mathrm{A} \in \mathbb{ISP}\underline{\mathrm{M}}$ and let $\alpha : \mathrm{D}(\mathbf{A}) \to M$. Then α preserves R_n if and only if α agrees with an evaluation on each n-element subset of* $\mathrm{D}(\mathbf{A})$.

2. A Condition for Nondualisability

We shall use the *ghost element* method to prove that certain semigroups are nondualisable. This method was introduced in [**11**] and has been used extensively since then. For a good survey of its use, the interested reader is referred to [**3**].

The basic idea of this method is to use a lemma such as the following to obtain a contradiction. The nicely polished lemma below is quoted from [**5**].

For an algebra $\mathbf{A} \leqslant \underline{\mathbf{M}}^S$ and $s \in S$, we define $\rho_s : \mathbf{A} \to \underline{\mathbf{M}}$ to be the restriction to A of the projection onto s.

Lemma 2.1. *Let $\underline{\mathrm{M}}$ be a finite algebra. Suppose that S is a nonempty set, $\mathbf{A}$ is a subalgebra of $\underline{\mathbf{M}}^S$ and that $\alpha : \mathrm{D}(\mathbf{A}) \to M$. Assume that*

(i) *α has finite support in A, and*

(ii) *α is an evaluation on each finite subset of* $\mathrm{D}(\mathbf{A})$.

Define $g_\alpha \in M^S$ by $g_\alpha(s) := \alpha(\rho_s)$, for all $s \in S$. Then if $\underline{\mathrm{M}}$ is dualisable, g_α must be in A.

This lets us show that a finite algebra $\underline{\mathbf{M}}$ is nondualisable by finding an algebra $\mathbf{A}$ and a map $\alpha : \mathrm{D}(\mathbf{A}) \to M$ which satisfies (i) and (ii) of Lemma 2.1, such that $g_\alpha \notin A$. The element g_α is then called a *ghost element of* $\mathbf{A}$.

We shall use the following consequence of the previous lemma for all our ghost element proofs. This lemma is due to David Clark, and is a special case of Lemma 5.2 of [**5**]. As a newcomer to the field of Natural Dualities, I admire it for its simplicity and ease of application.

Lemma 2.2. *Let a finite algebra $\underline{\mathrm{M}}$ be given. Let S be a nonempty set, let $\mathbf{A} \leqslant \mathrm{M}^S$, and let A_0 be an infinite subset of A. Assume that for every homomorphism $x : \mathbf{A} \to \underline{\mathbf{M}}$, there is an element a_x of A_0 such that x has the same constant value $\alpha(x)$ on $A_0 - \{a_x\}$. As in the previous lemma, we define the element g_α of M^S by setting $g_\alpha(s) := \alpha(\rho_s)$ for all $s \in S$. If $\underline{\mathrm{M}}$ is dualisable, g_α must be in A.*

https://doi.org/10.1017/S0004972700020542 Published online by Cambridge University Press

PROOF: Pick any three distinct elements of A_0, $\{a_1, a_2, a_3\}$. For any homomorphism x, $\alpha(x)$ is the majority of $x(a_1)$, $x(a_2)$ and $x(a_3)$. Thus $\{a_1, a_2, a_3\}$ is a finite support for α.

Given any finite subset X of $\mathrm{D}(\mathbf{A})$, choose $b \in A_0 - \{a_x \mid x \in X\}$. Then α is given by evaluation at b on X. It now follows from Lemma 2.1 that $g_\alpha \in A$ if $\underline{\mathrm{M}}$ is dualisable. $\square$

For various semigroups $\underline{\mathrm{M}}$ we shall be working with elements of the Cartesian power $\mathrm{M}^{\mathbb{N}}$ that are almost everywhere constant. It will be convenient to modify some notation from [5] to refer to these sequences. So let $k, n_1, \ldots, n_k \in \mathbb{N}$ and let $a, b_1, \ldots, b_k \in \mathrm{M}$. We define the sequence $a^{b_1 \ldots b_k}_{n_1 \ldots n_k}$ in $\mathrm{M}^{\mathbb{N}}$ by

$$a^{b_1 \ldots b_k}_{n_1 \ldots n_k}(i) = \begin{cases} b_j & \text{if } i = n_j, \text{ for some } j \in \{1, \ldots, k\}, \\ a & \text{otherwise.} \end{cases}$$

Ghost element arguments in the literature tend to be *ad hoc*, with new approaches needed for every new class of algebras. It does indeed seem to be hard to prove a nice general result for semigroups by this method as well. The next lemma carves out a large class of semigroups in which one particular kind of ghost element argument shows nondualisability.

LEMMA 2.3. *Let $\underline{\mathrm{M}}$ be a finite semigroup. Then $\underline{\mathrm{M}}$ is nondualisable if there are three distinct elements a, b and c of M such that the following conditions hold.*

(i) *For all $x \in M$, $b \cdot x = b$.*
(ii) *We have $a \cdot a = a$, $a \cdot b = a$, $c \cdot a = a$, and $c \cdot c = c$.*
(iii) *Whenever φ is an endomorphism of $\underline{\mathrm{M}}$ that does not send a and b to the same element, we have that $\varphi(a) = a$ and that for all y in M, $\varphi(c) \cdot y = \varphi(c)$ implies $y \cdot a = a$.*

PROOF: Suppose that $\underline{\mathrm{M}}$ is a finite semigroup that satisfies the conditions of the lemma. We shall use Lemma 2.2, taking S to be $\mathbb{N}$ and $g \in \mathrm{M}^{\mathbb{N}}$ to be $a^b_0 = \langle b, a, a, a, \ldots \rangle$. The set A_0 will be $\{a^{bb}_{0j} \mid j \geqslant 1\}$, and we sgall take $\mathbf{A}$ to be the subalgebra of $\underline{\mathrm{M}}^{\mathrm{N}}$ generated by $\{h \in \mathrm{M}^{\mathbb{N}} \mid h(0) = b \text{ and there exists } j \geqslant 1 \text{ with } h(j) = b\}$ together with A_0. Note that condition (i) implies that g is not in $\mathbf{A}$. Let $x : \mathbf{A} \to \underline{\mathrm{M}}$ be such that $x\restriction_{A_0}$ is not constant. So there are $m, n \geqslant 1$ with $x(a^{bb}_{0m}) \neq x(a^{bb}_{0n})$. Now consider $x(a^{bbb}_{0mn})$. It must be different from at least one of $x(a^{bb}_{0m})$ and $x(a^{bb}_{0n})$. Without loss of generality, suppose it differs from $x(a^{bb}_{0m})$, and consider $\mathbf{B} = \{a^{bdb}_{0mn} \mid d \in \mathrm{M}\}$. We have that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, and that $f : \underline{\mathrm{M}} \to \mathbf{B}$ given by $f(d) = a^{bdb}_{0mn}$ is an isomorphism.

Then $(x\restriction_B) \circ f$ is an endomorphism of $\underline{\mathrm{M}}$ that takes a and b to different elements, so condition (iii) applies to it. Thus $x(a^{bb}_{0n}) = x(a^{bab}_{0mn}) = a$, and $x(a^{bcb}_{0mn}) = c'$, where $c' \in M$ is such that for all y, $c' \cdot y = c'$ implies $y \cdot a = a$.

Now consider $x(a^{bb}_{0k})$, where k is a number different from 0, m and n. We shall be done if we can show that $x(a^{bb}_{0k})$ must always be a, for then a^{bb}_{0m} will be the only

https://doi.org/10.1017/S0004972700020542 Published online by Cambridge University Press

element of A_0 that x does not send to a. By (ii), we have that $a^{b\,c\,b}_{0mn} = a^{b\,c\,b}_{0mn} \cdot a^{b\,c\,b}_{0mk}$ and $a^{b\,c\,b}_{0mk} \cdot a^{bb}_{0n} = a^{bb}_{0k}$. Applying the homomorphism x to these equations gives us $c' = c' \cdot x(a^{b\,c\,b}_{0mk})$ and $x(a^{b\,c\,b}_{0mk}) \cdot a = x(a^{bb}_{0k})$, so $x(a^{bb}_{0k}) = a$ by condition (iii). ☐

It should be noted that the above lemma does not capture the full power of the ghost element method. In different drafts of this paper, I used a number of similar statements. Each one was better suited to showing that a particular class of semigroups was nondualisable. The above lemma just happens to be the best one for proving Lemma 2.4.

It seems natural to define semigroups by representing them as sets of functions under composition, for this way it is not necessary to check associativity. By the analogue of Cayley's Theorem, nothing is lost by doing this.

If $p \geqslant 1$, we define the following functions on $\{0, 1, 2, \dots, p\}$. Let $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{p}$ be the constant functions with values $0, 1, 2, \dots, p$ respectively. If T is any subset of $\{1, 2, 3, \dots, p\}$, we say that T is *oned* if and only if it contains 1. Then for any such oned subset T, we define the function f_T on $\{0, 1, 2, \dots, p\}$ by setting $f_T(n)$ to be 1 if $n \in T$, and setting it to be 0 otherwise. (So f_T can be thought of as the characteristic function of T.) We shall call functions of the form f_T *2-idempotent*, since they are idempotent functions into a two-element set. For consistency, we shall always take this two-element set to be $\{0, 1\}$.

Observe that whenever T and U are oned subsets of $\{1, 2, 3, \dots, p\}$, that $\mathbf{0} \circ f_T = f_T \circ \mathbf{0} = \mathbf{0}$, $\mathbf{1} \circ f_T = f_T \circ \mathbf{1} = \mathbf{1}$ and that $f_T \circ f_U = f_U$. Thus it is easy to see that whenever $p \geqslant 1$, $k \geqslant 1$ and $T_1, T_2, \dots, T_k$ are distinct oned subsets of $\{1, 2, 3, \dots, p\}$ that the set $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{p}, f_{T_1}, \dots, f_{T_k}\}$ is closed under composition and forms a semigroup. We shall call semigroups produced by this construction *2-idempotent derived*, since their nonconstant functions are all idempotent functions into a common two element set.

Note that many features of the representation of a 2-idempotent derived semigroup can be recovered from that semigroup. Every element of the underlying set is the range of a constant function, and these constant functions are definable in the semigroup as $\{u \mid u \circ v = u \text{ for all } v\}$. The f_{T_i} are definable since they are the only non-constant functions. We may use each constant function to access its value, and vice versa. Thus we shall sometimes identify constant functions with their values, where convenient. For example, we say that we can define $\{0, 1\}$ as the range of any of the non-constant functions. What this actually means is that $\mathbf{0}$ and $\mathbf{1}$ are the only functions g such that $g = h \circ g$ for some h such that there are functions u and v with $h \circ u \neq h \circ v$.

So which 2-idempotent derived semigroups can be shown by Lemma 2.3 to be nondualisable? To answer this question, we need to examine the structure of the *hypergraph* with edges $T_1, \dots, T_k$. Hypergraphs are the natural generalisation of (undirected) graphs, where we allow edges to be sets of cardinality other than 2. So we shall consider a *hypergraph* to consist of a set V of vertices, together with a set E of subsets of V. Elements of E will be called *edges* of the hypergraph $\langle V, E\rangle$. We say that two hypergraphs $\langle V, E\rangle$

and $\langle W, F\rangle$ are *isomorphic* if and only if there is a bijection $\psi : V \cup E \to W \cup F$ such that V goes to W, E goes to F and for all $v \in V$ and $e \in E$, we have $v \in e$ if and only if $\psi(v) \in \psi(e)$. Note that we are forced to view hypergraphs as multi-sorted relational structures, and that our definition of isomorphism reflects this. (If one imposes orders on the edge sets E and F, and demands that ψ take each element of E to the corresponding element of F, the resulting notion is called *strong isomorphism.*) Observe that when it is specialised to graphs, our definition of hypergraph isomorphism agrees with graph isomorphism.

We define the *complement* of the hypergraph $\langle V, E\rangle$ to be $\langle V, \{\bar{e} \mid e \in E\}\rangle$, where $\bar{e}$ denotes $V - e$, the complement of e. Note that this is not the hypergraph with edgeset $\mathcal{P}(V) - E$, although this would be another possible way to define the complement of a hypergraph. We shall not need it here, but more material on hypergraphs can be found in [**1**].

When $\mathbf{S} = \left\langle\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}, f_{T_1}, f_{T_2}, \ldots, f_{T_k}\}, \circ\right\rangle$ is a 2-idempotent derived semigroup, we define its *derived hypergraph* $H(\mathbf{S})$ to be $\left\langle\{0, 1, 2, \ldots, p\}, \{T_1, T_2, \ldots, T_k\}\right\rangle$.

We shall also need to consider the equivalence relation β on $\{0, 1, 2, \ldots, p\}$, defined by $\beta = \left\{\langle u, v\rangle \mid \text{for all } i,\ f_{T_i}(u) = f_{T_i}(v)\right\}$.

Lemma 2.4. *Let $p \geqslant 1$, let $k \geqslant 1$, and let $\mathbf{S} = \left\langle\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}, f_{T_1}, f_{T_2}, \ldots, f_{T_k}\}, \circ\right\rangle$ be 2-idempotent derived. Then if the following conditions are met, $\mathbf{S}$ is not dualisable.*

- (i) Separability *The equivalence relation β is equal to Δ on $\{0, 1, \ldots, p\}$.*
- (ii) Rigidity *The derived hypergraph $H(\mathbf{S})$ is not isomorphic to its complement.*

Proof: Let $\mathbf{S} = \left\langle\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}, f_{T_1}, \cdots f_{T_k}\}, \circ\right\rangle$ satisfy the separability and rigidity conditions of this lemma. We shall use Lemma 2.3, with $a = 0$, $b = 1$ and $c = f_{T_1}$. Checking the conditions of that lemma, we see that (i) and (ii) are true.

For (iii), we first show that $\mathbf{S}$ is subdirectly irreducible with monolith $\{\mathbf{0}, \mathbf{1}\}^2 \cup \Delta$. So let distinct q and r in S be given. If q and r are both in $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}\}$, then (i) gives a T_j such that one of $f_{T_j}(q)$ and $f_{T_j}(r)$ is 0 and the other is 1. This shows that $\langle \mathbf{0}, \mathbf{1}\rangle \in \mathrm{Cg}\left(\langle q, r\rangle\right)$. Next suppose that one of the two elements, say q, is in $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}\}$, while r is some f_{T_j}. Then $r \circ \mathbf{0} = f_{T_j}(0) = \mathbf{0}$, while $q \circ \mathbf{0} = q$. Thus $\langle q, \mathbf{0}\rangle \in \mathrm{Cg}\left(\langle q, r\rangle\right)$. Similarly, $r \circ \mathbf{1} = f_{T_j}(1) = \mathbf{1}$ and $q \circ \mathbf{1} = q$ yield $\langle q, \mathbf{1}\rangle \in \mathrm{Cg}\left(\langle q, r\rangle\right)$. Combining, we get $\langle \mathbf{0}, \mathbf{1}\rangle \in \mathrm{Cg}\left(\langle q, r\rangle\right)$. Lastly, suppose that q is f_{T_i} and r is f_{T_j}, for some i and j. Since $q \neq r$, $T_i \neq T_j$ and there is some $s \leqslant p$ with $f_{T_i}(s) \neq f_{T_j}(s)$. That is, $f_{T_i} \circ s \neq f_{T_j} \circ s$. So one is $\mathbf{0}$, the other is $\mathbf{1}$, and $\langle \mathbf{0}, \mathbf{1}\rangle \in \mathrm{Cg}\left(\langle q, r\rangle\right)$. It is easy to verify that $\{\mathbf{0}, \mathbf{1}\}^2 \cup \Delta$ is a congruence, so it is the monolith of $\mathbf{S}$.

Letting ψ be any endomorphism of $\mathbf{S}$ that does not send $\mathbf{0}$ and $\mathbf{1}$ to the same place, we have that ψ must be an automorphism. Now ψ permutes the elements of $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p}\}$, since they are characterised as the values y such that $y \circ z = y$ for all z. Thus ψ also permutes the set $F = \{f_{T_1}, \ldots, f_{T_k}\}$, and $\{\mathbf{0}, \mathbf{1}\} = \{y \mid f \circ y = y \text{ for all } f \in F\}$.

We claim that ψ fixes both $\mathbf{0}$ and $\mathbf{1}$. For if not, then ψ must switch them. Assuming this is so, we look at any $u \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{p}\}$ and at any $f_{T_i} \in F$. Let j be the unique element of $\{1, \dots, k\}$ such that $f_{T_j} = \psi(f_{T_i})$. Then $u \in T_i$ if and only if $f_{T_i} \circ u = \mathbf{1}$ if and only if $\psi(f_{T_i}) \circ \psi(u) = \mathbf{0}$ if and only if $f_{T_j} \circ \psi(u) = \mathbf{0}$ if and only if $\psi(u) \notin T_j$ if and only if $\psi(u) \in \overline{T_j}$. This shows that ψ induces an isomorphism between the hypergraph $\langle \{0, 1, 2, \dots, p\}, \{T_1, \dots, T_k\} \rangle$ and its complement, a contradiction.

So we have that ψ takes a to a, and takes c to some element c' of the set $F = \{f_{T_1}, \dots, f_{T_k}\}$. Now if $c' \circ y = c'$, y must equal c', in which case $y \circ a = a$, and (iii) is proved. $\square$

The *separability* condition of the lemma ensures that the semigroup is subdirectly irreducible with monolith containing $\langle 0, 1 \rangle$. This implies that any endomorphism that does not identify 0 and 1 is an automorphism. The *rigidity* condition imposes additional restrictions by ruling out any automorphism that exchanges 0 and 1. In the context of 2-idempotent derived semigroups, this is enough to allow a ghost element argument of the form used here to work.

We can generalise the previous lemma to show that more 2-idempotent derived semigroups are nondualisable. The key to doing this is to get around the restriction of the separability condition of the lemma. We shall use the *Independence Theorem* of Davey, Saramago and Willard (see [**12**] or [**20**]), which states that whenever two finite algebras generate the same quasivariety, one is dualisable if and only if the other one is.

Lemma 2.5. *Let* $\mathbf{S}$ *be a finite 2-idempotent derived semigroup containing at least one 2-idempotent function. Then there is a 2-idempotent derived semigroup* $\mathbf{S}'$ *which satisfies the separability condition of Lemma 2.4 and generates the same quasivariety as* $\mathbf{S}$.

Proof: Let $\mathbf{S} = \langle \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{p}, f_{T_1}, f_{T_2}, \dots, f_{T_k}\}, \circ \rangle$ be a given 2-idempotent derived semigroup, with $p \geqslant 1$ and $k \geqslant 1$.

We shall also use β to denote the corresponding equivalence relation on the constant functions of the semigroup $\mathbf{S}$. Since k is nonzero, β has at least two blocks. The new semigroup $\mathbf{S}'$ is now formed as a subalgebra of $\mathbf{S}$ produced by removing all but one member from each block of β. If $\mathbf{0}$ or $\mathbf{1}$ is in a block of β of size greater than one, they are chosen to be the remaining constant function from that block. It is easy to check that $\mathbf{S}'$ is in fact a subalgebra of $\mathbf{S}$ containing $\mathbf{0}$ and $\mathbf{1}$, and that it now satisfies the separability condition.

To complete the proof, we shall show that $\mathbf{S}$ is in the quasivariety generated by $\mathbf{S}'$. It suffices to give a method using products and subalgebras that constructs a semigroup with the size of all the blocks of β doubled. For this construction can then be used repeatedly, to make all the blocks of β at least as large as they are in $\mathbf{S}$. After this, $\mathbf{S}$ is obtained as a subalgebra of the constructed semigroup.

So let $\mathbf{B} = \langle \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{p}, f_{T_1}, f_{T_2}, \dots, f_{T_k}\}, \circ \rangle$ be any 2-idempotent derived semigroup, and consider $C = \big(\{\mathbf{0},\mathbf{1},\mathbf{2},\dots,\mathbf{p}\} \times \{\mathbf{0},\mathbf{1}\}\big) \cup \big(\{f_{T_1}, f_{T_2},\dots,f_{T_k}\} \times \{\mathbf{0}\}\big)$, a subset of $B \times B$. It is easily proved that it is the universe of a subalgebra $\mathbf{C}$ of $\mathbf{B} \times \mathbf{B}$, and that $\langle f_{T_1}, \mathbf{0}\rangle, \dots, \langle f_{T_k}, \mathbf{0}\rangle$ are the idempotent functions into the 2-element set $\{\langle \mathbf{0},\mathbf{0}\rangle, \langle \mathbf{1},\mathbf{0}\rangle\}$. Observe that for all $j \leqslant k$ and for all $u \in \{0,1,2,\dots,p\}$, that $\langle f_{T_j}, \mathbf{0}\rangle\big(\langle u, 0\rangle\big)$ and $\langle f_{T_j}, \mathbf{0}\rangle\big(\langle u, 1\rangle\big)$ are equal to $\langle 1, 0\rangle$ if and only if $u \in T_j$. So the idempotent functions $\langle f_{T_j}, \mathbf{0}\rangle$ produce an equivalence relation β on $\mathbf{C}$ that has the same structure as the original β on $\mathbf{B}$, except that every block of the new β has twice as many elements. ∎

We can sum all this up in the following.

Theorem 2.6. *Let $p \geqslant 1$, let $k \geqslant 1$, and let $T_1, T_2, \dots, T_k$ be distinct subsets of $\{1,2,3,\cdots p\}$ which all contain 1. Consider the hypergraph constructed from $\big\langle\{0,1,2,\dots,p\}, \{T_1,\dots,T_k\}\big\rangle$ by removing extra vertices where needed so that no two vertices ever lie in exactly the same set of edges. If this hypergraph is not isomorphic to its complement, the semigroup $\big\langle\{\mathbf{0},\mathbf{1},\mathbf{2},\dots,\mathbf{p}, f_{T_1}, f_{T_2},\dots,f_{T_k}\}, \circ\big\rangle$ is nondualisable.*

Proof: This follows easily from Lemma 2.4, Lemma 2.5, and the Independence Theorem. ∎

The use of hypergraphs in the above theorem seems to be unavoidable, since they give a simple framework in which the theorem can be easily stated. It also seems unlikely that the condition that a hypergraph is not isomorphic to its complement can be replaced with a nicer equivalent condition. It can be shown that the problem of deciding whether or not a hypergraph is isomorphic to its complement is equivalent to the Graph Isomorphism Problem, for which no polynomial-time algorithm is known. (I omit the proof of this, although it does not appear in the literature. The interpretations used are fairly obvious, and the issue is only tangentially related to the results of this paper. For background, the interested reader is referred to [**13**].)

3. Examples and Remarks

The smallest 2-idempotent derived semigroups do not meet the conditions of Lemma 2.4. When p is 1, then $\{1,2,3,\dots,p\}$ has a unique oned subset, $\{1\}$. The derived hypergraph is $\big\langle\{0,1\}, \{\{1\}\}\big\rangle$, which is isomorphic to its complement. The corresponding 2-idempotent derived semigroup has 3 elements. This interesting semigroup will be considered in this section.

When $p = 2$, the only possible 2-idempotent functions are $f_{\{1\}}$ and $f_{\{1,2\}}$. Separability fails unless both are included in the semigroup, but rigidity fails if they are.

When $p = 3$, there are four 2-idempotent functions: $f_{\{1\}}, f_{\{1,2\}}, f_{\{1,3\}}, f_{\{1,2,3\}}$. If not enough of them are included in a semigroup with constant functions $\{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\}$, then it will not meet the separability condition of Lemma 2.4. For instance, in the semigroup $\big\langle\{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}, f_{\{1,3\}}, f_{\{1,2,3\}}\}, \circ\big\rangle$, we have that $\{1,3\}$ is a block of β, and separability fails. To illustrate Lemma 2.5, we consider the subalgebra formed by removing $\mathbf{3}$. It is isomorphic

$\circ$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$f_{\{1\}}$	$f_{\{1,2\}}$	$f_{\{1,3\}}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$
$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$
$f_{\{1\}}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$f_{\{1\}}$	$f_{\{1,2\}}$	$f_{\{1,3\}}$
$f_{\{1,2\}}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$f_{\{1\}}$	$f_{\{1,2\}}$	$f_{\{1,3\}}$
$f_{\{1,3\}}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	$f_{\{1\}}$	$f_{\{1,2\}}$	$f_{\{1,3\}}$

Table 1: Smallest nondualisable semigroup known

to $\langle\{\mathbf{0},\mathbf{1},\mathbf{2},f_{\{1\}},f_{\{1,2\}}\},\circ\rangle$, and the lemma gives us that the two semigroups generate the same quasivariety. Since rigidity fails for this new semigroup, we can not show that either of the two semigroups is nondualisable.

Going back to the case where $p = 3$, we see that separability holds whenever at least three functions in $\{f_{\{1\}}, f_{\{1,2\}}, f_{\{1,3\}}, f_{\{1,2,3\}}\}$ are included, or when just $f_{\{1,2\}}$ and $f_{\{1,3\}}$ are. But rigidity fails for all of these, except when the set of included functions is $\{f_{\{1\}}, f_{\{1,2\}}, f_{\{1,3\}}\}$ or $\{f_{\{1,2\}}, f_{\{1,3\}}, f_{\{1,2,3\}}\}$. Since these yield isomorphic semigroups, we shall only look at the former set. So the smallest nondualisable semigroup given by our theorem has its operation given by Table 1.

Even for the class of 2-idempotent derived semigroups, the dualisability problem is not yet solved. By Lemma 2.5, every 2-idempotent derived semigroup generates the same quasivariety as one which satisfies the separability condition. So within our class, the problem reduces to the dualisability of semigroups which satisfy separability but not rigidity.

The simplest such semigroup is $\langle\{\mathbf{0},\mathbf{1},\mathbf{i}\},\circ\rangle$, where $\mathbf{i}$ denotes the identity function on $\{0,1\}$. This is the one obtained by taking $p = 1$ as mentioned before. As do the other 2-idempotent derived semigroups which fail rigidity, it has an endomorphism which switches $\mathbf{0}$ and $\mathbf{1}$. The presence of such an endomorphism is a serious obstacle to the kind of ghost element argument used in this paper.

It would be interesting to know if this semigroup was dualisable. This semigroup $\langle\{\mathbf{0},\mathbf{1},\mathbf{i}\},\circ\rangle$ satisfies the quasi-identity $(y\cdot z = z \wedge z\cdot y = y) \implies y = z$ which is true in a 2-idempotent derived semigroup if and only if it has only one 2-idempotent function. This shows that it does not generate the same quasivariety as any of the semigroups we have shown to be dualisable.

David Clark has done some unpublished work on the dualisability of $\langle\{\mathbf{0},\mathbf{1},\mathbf{i}\},\circ\rangle$. He has made a good deal of progress in characterising the subalgebras of its powers, but has not succeeded in resolving the question of the dualisability of this semigroup.

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