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A HOMOMORPHISM THEOREM FOR MULTIPLIERS

by NAKHLÉ HABIB ASMAR

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1. Notation and introduction

Throughout the paper, the symbols G_1 and G_2 will denote two locally compact abelian groups with character groups X_1 and X_2, respectively. Haar measures on G_j are denoted by μ_j; the ones on X_j are denoted by θ_j ($j=1,2$). The measures μ_j and θ_j are normalized so that the Plancherel Theorem holds (see [7, p. 226, Theorem 31.18]).

If G is a locally compact abelian group with character group X, and if f is a complex-valued function on G, then f is said to be measurable means that f is measurable with respect to Haar measure on G. The class of measurable functions on G, with integrable pth power, is denoted by $\mathscr{L}_p(G)$ $1 \leqq p < \infty$; the class of essentially bounded measurable functions by $\mathscr{L}_\infty(G)$; the class of continuous functions with compact support by $\mathscr{C}_{00}(G)$.

If A is a subset of G, the complement of A in G is denoted by A' or $G \backslash A$. The symbol 1_A will denote the indicator function of the set A. All other notation used in this paper without explanation is as in [6] and [7]. A bounded measurable function m on X is called an $\mathscr{L}_p(G)$-multiplier, $1 \leqq p < \infty$, if for every f in $\mathscr{L}_p(G) \cap \mathscr{L}_2(G)$ there is a g in $\mathscr{L}_p(G)$ such that $\hat{g} = m\hat{f}$, and $\|g\|_p \leqq N_p(m)\|f\|_p$, where $N_p(m)$ is the norm of the unique extension of the bounded linear operator $f \to g$ to all of $\mathscr{L}_p(G)$. We shall denote this extension by T_m. The set of all multipliers on $\mathscr{L}_p(G)$ will be denoted by $M_p(G)$.

Suppose that τ is a continuous nonzero homomorphism from X_2 into X_1. A well-known theorem for multipliers asserts that if m is continuous on X_1, then $m \circ \tau$ is in $M_p(G_2)$ and $N_p(m \circ \tau) \leqq N_p(m)$. (See [5, Theorem B.2.1, p. 187]). We will refer to this fact as the homomorphism theorem for continuous multipliers.

Many interesting multipliers are not continuous; e.g. the sgn function on $\mathbb{R}$ which is an $\mathscr{L}_p$ ($\mathbb{R}$)—multiplier for $1 < p < \infty$. Our goal, in this essay, is to give a new proof of the homomorphism theorem for continuous multipliers based on the so-called transference methods, then derive a more general version that applies to multipliers like the sgn function.

2. The homomorphism theorem

We continue with the notation of Section 1: m is a bounded continuous function on X_1, and τ is a continuous nonzero homomorphism from X_2 into X_1.

2.1. *An approximate unit in* $\mathscr{L}_1(X_1)$.

By interchanging the group G and its character group X in Theorem 33.12 p. 298 of [7], we see that $\mathscr{L}_1(X_1)$ contains a net of functions $(\hat{u}_\iota)_{\iota\in I}$ such that, for all ι in I, we have:

$$\hat{u}_\iota \geqq 0; \tag{1}$$

$$\int_{X_1} \hat{u}_\iota \, d\theta_1 = 1; \tag{2}$$

$$u_\iota \geqq 0; \quad \text{and} \quad u_\iota \in \mathscr{C}_\infty(G_1). \tag{3}$$

From (1) and (2), it follows that

$$\lim_\iota \hat{u}_\iota * m = m \tag{4}$$

uniformly on compact subsets of X_1. (Use (1), (2), and (28.52) of [7]).

Clearly, from (4), we have

$$\lim_\iota \hat{u}_\iota * m \circ \tau = m \circ \tau \tag{5}$$

uniformly on compact subsets of X_2.

Theorem 2.2. *Suppose that m is a bounded and continuous function on* X_1 *which is also in* $M_p(G_1)$, $1 \leqq p < \infty$. *Let* τ *be a continuous homomorphism from* X_2 *into* X_1. *Then* $m \circ \tau$ *is an* $\mathscr{L}_p(G_2)$*-multiplier with* $N_p(m \circ \tau) \leqq N_p(m)$.

The proof of Theorem 2.2 combines a transference results and well-known properties of translation-invariant operators. We shall start with the transference set-up. Suppose that k is in $\mathscr{L}_1(G_1)$ with compact support. Let T_k denote the operator $f \mapsto f * k$, and let $N_p(k)$ denote its norm as an operator from $\mathscr{L}_p(G_1)$ into $\mathscr{L}_p(G_1)$. Let ϕ denote a continuous nonzero homomorphism from G_1 into G_2. If f is in $\mathscr{L}_p(G_2)$, using [**6**, Lemma 20.6, p. 287], one can easily show that the function $(t, x) \mapsto f(x - \phi(t))$ is measurable with respect to the product measure on $G_1 \times G_2$.

Let $T_k^{\#}$ denote the operator, defined in $\mathscr{L}_p(G_2)$ by

$$T_k^{\#} f(x) = \int_{G_1} f(x - \phi(t)) k(t) \, d\mu_1(t). \tag{6}$$

Applying Theorem 2.4 of [**4**], we see that the inequality

$$\|T_k^{\#} f\|_p \leqq N_p(k) \|f\|_p \tag{7}$$

holds for all f in $\mathscr{L}_p(G_2)$ with $1 \leqq p < \infty$. (While in [**4**] it is required that G_2 be

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σ-compact, one can check that the proof of [4, Theorem 2.4], still holds when G_2 is not σ-compact and the operator is of the particular form (6). See also Theorem 2.3 of [3]).

Lemma 2.3. *Let m and u_ι be as in* (2.1). *Let h be in $\mathscr{L}_2(G_1)\cap\mathscr{C}_\infty(G_1)$ such that $\hat{h}$ is in $\mathscr{L}_1(X_1)$. Set*

$$k_\iota=((\hat{u}_\iota*m)\hat{h})^\vee.$$

Then k_ι is in $\mathscr{L}_1(G_1)$ with support contained in $\operatorname{supp}u_\iota+\operatorname{supp}h$. *In particular,* $\operatorname{supp}k_\iota$ *is compact.*

Proof. We have

$$\begin{aligned}((m*\hat{u}_\iota)\hat{h})^\vee(x)&=\int_{X_1} m*\hat{u}_\iota(\gamma)\hat{h}(\gamma)\gamma(x)\,d\theta_1(\gamma)\\
&=\int_{X_1}\int_{X_1} m(\eta)\hat{u}_\iota(\gamma-\eta)\,d\theta_1(\eta)\hat{h}(\gamma)\gamma(x)\,d\theta_1(\gamma)\\
&=\int_{X_1} m(\eta)\int_{X_1}\hat{u}_\iota(\gamma-\eta)\hat{h}(\gamma)\gamma(x)\,d\theta_1(\gamma)\,d\theta_1(\eta)\\
&=\int_{X_1} m(\eta)h*(\eta u_\iota)(x)\,d\theta_1(\eta).\end{aligned}$$

Note that $\operatorname{supp}\eta u_\iota\subseteq\operatorname{supp}u_\iota$. Thus,

$$\operatorname{supp}(h*(\eta u_\iota))\subseteq\operatorname{supp}h+\operatorname{supp}u_\iota,$$

from which the lemma follows. □

Lemma 2.4. *Suppose that m is in $M_p(X_1)$ (m need not be continuous). With the notation of Lemma* 2.3, *we have*

(a) $N_p(u_\iota*m)\leqq N_p(m)$,

(b) $\|k_\iota*f\|_p\leqq N_p(m)\|h\|_1\|f\|_p$

for all $\iota\in I$, and all f in $\mathscr{L}_p(G_1)$, $1\leqq p<\infty$.

Proof. Part (a) is a well-known property of multipliers. For its proof see [5, B.1.2. (iii), p. 185].

For (b), it is enough to consider f in $\mathscr{L}_p(G)$ with compactly supported $\hat{f}$. We have

$$\|f*((\hat{u}_\iota*m)\hat{h})^\vee\|_p=\|h*(\hat{f}(\hat{u}_\iota*m))^\vee\|_p$$

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$$\leqq \|h\|_1 \|\hat{f}(\hat{u}_\iota * m))^\vee\|_p$$

$$\leqq N_p(m)\|h\|_1\|f\|_p \quad \text{(from (a)).} \qquad \square$$

2.5.

We now go back to our set-up of (2.1). We have a continuous homomorphism τ from X_2 into X_1. To use the transference results, we introduce the adjoint homomorphism ϕ of τ; thus ϕ is the continuous homomorphism from G_1 into G_2 satisfying the identity

$$\chi \circ \phi(s) = \tau(\chi)(s)$$

for all χ in X_2, and all s in G_1.

For every $\iota \in I$ and $f \in \mathscr{L}_p(G_2)$, $1 \leqq p < \infty$, we let

$$T^{\#}_{k_\iota} f(x) = \int_{G_1} f(x - \phi(t))\, k_\iota(t)\, d\mu_1(t)$$

where $k_\iota = ((m * \hat{u}_\iota)\hat{h})^\vee$, h is an arbitrary but fixed element in $\mathscr{L}_2(G_1) \cap \mathscr{C}_\infty(G_1)$ such that $\|h\|_1 \leqq 1$, and $\hat{h}$ is an $\mathscr{L}_1(X_1)$.

Using (2.2.7) and (2.4.b) we see that

$$\|T^{\#}_{k_\iota} f\|_p \leqq N_p(m)\|f\|_p \tag{8}$$

for all f in $\mathscr{L}_p(G_2)$.

Lemma 2.6. *Notation is as in* (2.5). *Let f be in $\mathscr{L}_p(G_2) \cap \mathscr{L}_1(G_2)$, $1 \leqq p < \infty$. We have*

(*a*) $$(T^{\#}_{k_\iota} f)\hat{}(\chi) = \hat{f}(\chi)\hat{u}_\iota * m(\tau(\chi))\hat{h}(\tau(\chi))$$

for all χ in X_2 and all $\iota \in I$;

(b) $$\lim_\iota (T^{\#}_{k_\iota} f)\hat{}(\chi) = \hat{f}(\chi) m(\tau(\chi))\hat{h}(\tau(\chi))$$

uniformly on compact subsets of X_2.

Proof. We have

$$(T^{\#} f)\hat{}(\chi) = \int_{G_2} \bar{\chi}(x) \int_{G_1} f(x - \phi(t)) k_\iota(t)\, d\mu_1(t)\, d\mu_2(x)$$

$$= \int_{G_1}\int_{G_2} \bar{\chi}(x) f(x - \phi(t))\, d\mu_2(x) k_\iota(t)\, d\mu_1(t)$$

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$$= \int_{G_1} \int_{G_2} \bar{\chi}(x+\phi(t)) f(x)\, d\mu_2(x) k_\iota(t)\, d\mu_1(t)$$

$$= \hat{f}(\chi) \int_{G_1} \bar{\chi}(\phi(t)) k_\iota(t)\, d\mu_1(t)$$

$$= \hat{f}(\chi) \int_{G_1} \overline{\tau(\chi)}(t) k_\iota(t)\, d\mu_1(t)$$

$$= \hat{f}(\chi) \hat{k}_\iota(\tau(\chi))$$

$$= \hat{f}(\chi) u_\iota * m(\tau(\chi)) \hat{h}(\tau(\chi)).$$

Part (b) is an immediate consequence of (a) and (2.1.5). □

2.7 *Proof of Theorem 2.2.* Let $1 \leqq p < \infty$, and let $q = (p/p-1)$ if $1 < p < \infty$, and $q = \infty$ if $p = 1$. It is enough to show that

$$\left| \int_{G_2} (\hat{f}(m \circ \tau))^{\vee}(x) \bar{g}(x)\, d\mu_2(x) \right| \leqq N_p(m) \|f\|_p \|g\|_q \tag{9}$$

for all $f \in \mathscr{L}_p(G_2) \cap \mathscr{L}_1(G_2)$, $g \in \mathscr{L}_q(G_2) \cap \mathscr{L}_1(G_2)$, and $\hat{f}$ and $\hat{g}$ are in $\mathscr{C}_\infty(X_2)$. (See [5, 1.2.2. (iii), p. 7]). We have from (2.6b)

$$\lim_\iota \bar{\hat{g}}(\chi) \hat{f}(\chi) \hat{u}_\iota * m(\tau(\chi)) \hat{h}(\tau(\chi)) = \bar{\hat{g}}(\chi) \hat{f}(\chi) m(\tau(\chi)) \hat{h}(\tau(\chi)) \tag{10}$$

uniformly on X_2. Also note that the inequality

$$|\hat{f}(\chi) \bar{\hat{g}}(\chi) (\hat{u}_\iota * m)(\tau(\chi)) \hat{h}(\tau(\chi))| \leqq \|\hat{g}\|_\infty \|m\|_\infty \|\hat{h}\|_\infty \|\hat{f}\|_\infty \tag{11}$$

holds for all χ in X_2 and all $\iota \in I$. From (10), (11), (2.5.8), and Parseval's identity ([7, 31.19, p. 226]), we infer that

$$\left| \int_{X_2} \bar{\hat{g}} \hat{f} m \circ \tau \hat{h} \circ \tau\, d\theta_2 \right| = \left| \lim_\iota \int_{X_2} \bar{\hat{g}} \hat{f} (\hat{u}_\iota * m) \circ \tau \hat{h} \circ \tau\, d\theta_2 \right|$$

$$= \lim_\iota \left| \int_{G_2} \bar{g} T^{\#}_{k_\iota} f\, d\mu_2 \right|$$

$$\leq N_p(m)\|f\|_p\|g\|_q. \tag{12}$$

We now show that (12) implies (9). Using Parseval's identity, rewrite the left side of (9) as

$$\left|\int_{X_2} \hat{f}(\chi) m \circ \tau(\chi) \bar{\hat{g}}(\chi)\, d\theta_2(\chi)\right|.$$

Denote by K the support of $\hat{f}\hat{g}$. Given $\varepsilon > 0$, let h in $\mathscr{L}_2(G_1) \cap \mathscr{C}_\infty(G_1)$ be such that $\|h\|_1 = 1$, $\hat{h} \in \mathscr{L}_1(X_1)$, and

$$|\hat{h}(\chi) - 1| < \varepsilon$$

for all χ in $\tau(K)$. (To find h, use [7, Theorem 33.11 p. 298]). We have

$$\left|\int_{X_2} \hat{f}(\chi) m \circ \tau(\chi) \bar{\hat{g}}(\chi)\, d\theta_2(\chi) - \int_{X_2} \hat{f}(\chi) \hat{h}(\tau(\chi)) m \circ \tau(\chi) \bar{\hat{g}}(\chi)\, d\theta_2(\chi)\right| < \varepsilon \|f\|_\infty \|g\|_\infty \|m\|_\infty \theta_2(K).$$

Clearly, this together with (12) implies (9). □

2.8. *Remark.* The assumption 2.1.5 can be replaced by the requirement that $(\hat{u}_\iota * m) \circ \tau$ converges to $m \circ \tau$ in the weak-star topology of $\mathscr{L}_\infty(X_2)$. For in this case, to establish 2.7.1, we would start with the equality

$$\left|\int_{X_2} \hat{f}\bar{\hat{g}} m \circ \tau \hat{h} \circ \tau\, d\theta_2\right| = \left|\lim_{\iota} \int_{X_2} \hat{f}\bar{\hat{g}}(u_\iota * m) \circ \tau \hat{h} \circ \tau\, d\theta_2\right|,$$

and then continue the proof 2.7 from 2.7.12 until the end without a hitch.

Our next version of the homomorphism theorem applies to normalized multipliers.

Definition 2.9. A bounded function m on X is said to be *normalized* if there is an approximate identity $(k_n)_{n=1}^{\infty}$ in $\mathscr{L}_1(X_1)$ such that $\lim_{n\to\infty} k_n * m(\chi)$ exists for all χ in X. We denote this limit by m^*.

Theorem 2.10. *Let m be a normalized function in $M_p(X_1)$, $1 \leq p < \infty$, and let τ be a continuous homomorphism of X_2 into X_1. Then the function $m^* \circ \tau$ is in $M_p(X_2)$, $1 \leq p < \infty$, with $N_p(m^* \circ \tau) \leq N_p(m)$.*

Theorem 2.10 is an immediate consequence of Theorem 2.2 and the following lemma whose proof can be reconstructed from [**5**, pp. 190–191, B.2.2, (i)–(iv)].

Lemma 2.11. *Let X be a locally compact abelian group with character group G. Let $(m_n)_{n=1}^{\infty}$ be a sequence of continuous functions in $M_p(X)$, $1 \leqq p < \infty$, such that:*

(*a*) $$\sup_n \|m_n\|_\infty < \infty;$$

(*b*) $$\lim_{n\to\infty} m_n(\chi) = m(\chi)$$

for all χ in X; and

(*c*) $$\sup_n N_p(m_n) = c_p < \infty.$$

Then m is in $M_p(X)$ with $N_p(m) \leqq c_p$.

To prove Theorem 2.10 note that the functions $k_n * m \circ \tau$ have the following properties:

$k_n * m \circ \tau$ are continuous, and

$$\|k_n * m \circ \tau\|_\infty \leqq \|k_n * m\|_\infty$$
$$\leqq \|k_n\|_1 \|m\|_\infty = \|m\|_\infty;$$
$$\lim_{n\to\infty} k_n * m \circ \tau = m^* \circ \tau$$

pointwise everywhere on X_2; and

$$N_p(k_n * m \circ \tau) \leqq N_p(k_n * m) \text{ (by Th. 2.2)}$$
$$\leqq N_p(m) \text{ (by (2.4)(a)).}$$

Now apply Lemma 2.11 to the sequence $(k_n * m \circ \tau)_{n=1}^{\infty}$ in $M_p(X_2)$.

A version of Theorem 2.10 appears in [**3**, Theorem 2.7]. Its proof, while quite different from ours, also uses the transference methods.

3. Applications

An interesting application of Theorem 2.10 to multiple Fourier series is obtained by taking: $X_1 = \mathbb{T}$ (the unit circle parametrized by the interval $[-\pi, \pi[$); $X_2 = \mathbb{Z}^n$ where n is a positive integer; and $m = 1_{]a,b[}$ where $-\pi \leqq a < b < \pi$. The homomorphism τ is given by

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$$\tau(m_1, m_2, \ldots, m_n) = \sum_{j=1}^{n} \alpha_j m_j (\text{mod } 2\pi) \text{ where}$$

$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a subset of $\mathbb{R}$ which is linearly independent over $\mathbb{Q}$. The case $n=1$ is presented in [**8**], Section 1.

We now derive a generalization of M. Riesz's theorem on conjugate functions by using the original version on $\mathbb{R}$. This approach to the abstract of M. Riesz's theorem is due to [**2**] for compact abelian groups, and to [**1**] for arbitrary locally compact abelian groups.

We take $X_1 = \mathbb{R}$, $G_1 = \mathbb{R}$; and we write X and G for X_2 and G_2, and μ and θ for μ_2 and θ_2. We suppose that X contains a measurable subset P such that $P+P=P$; $P \cap (-P) = \{0\}$; $P \cup (-P) = X$. Such a set is called an order on X. With P we associate the function sgn_P defined on X by

$$\text{sgn}_P(\chi) = \begin{cases} 1 & \text{if } \chi \in P \backslash \{0\}; \\ 0 & \text{if } \chi = 0; \\ -1 & \text{if } \chi \in (-P) \backslash \{0\}. \end{cases}$$

An abstract version of M. Riesz's theorem for conjugate function can be stated as follows.

Theorem 3.1. *Notation is as above. Let f be in $\mathscr{L}_p(G) \cap \mathscr{L}_2(G)$, $1 < p < \infty$. We have*

(*i*) $$\|(-i\,\text{sgn}_P \hat{f})^{\vee}\|_p \leqq A_p \|f\|_p$$

where the constant A_p is the same as the constant appearing in M. Riesz's theorem on $\mathbb{R}$ (or $\mathbb{T}$).

Proof. It is enough to consider f in $\mathscr{L}_p(G)$ such that $\hat{f}$ is in $\mathscr{C}_\infty(X)$.

Let K be the support of $\hat{f}$. Apply Theorem (5.14) of [**1**] to obtain a homomorphism τ from X into $\mathbb{R}$ such that the equality

$$\text{sgn}_P(\chi) = \text{sgn}(\tau(\chi))$$

holds for θ-almost all χ in X. We clearly have

$$(-i\,\text{sgn}_P \hat{f})^{\vee} = (-i\,\text{sgn} \circ \tau \hat{f})^{\vee} \tag{13}$$

θ-almost everywhere on X.

The function $s \mapsto -i\,\text{sgn}(s)$ is normalized on $\mathbb{R}$. M. Riesz's theorem on $\mathbb{R}$ asserts that the function $s \mapsto -i \operatorname{sgn}(s)$ is an $\mathscr{L}_p(\mathbb{R})$—multiplier with norm A_p. The inequality (i) follows now from (13) and Theorem 2.10. □

The following results is due to [**8**, Section 3], for the case $X = \mathbb{R}^n$; to [**4**, (3.16)] for the case G σ-compact; and to [**9**, (4.6)(b)], for the general case under more hypothesis than we require below.

Theorem 3.2. *Let m be a normalized function on X which is an $\mathscr{L}_p(G)$-multiplier with norm $N_p(m)$, $1 \leqq p < \infty$. Suppose further that $m^* = m$. Let Y be a closed subgroup of X. Suppose that f, the restriction of m to Y, is measurable with respect to the Haar measure on Y. Then f is an $\mathscr{L}_p(G/A(G, Y))$-multiplier, where $A(G, Y) = \{g \in G: g(\chi) = 1$ for all χ in $Y\}$. Moreover, we have $N_p(f) \leqq N_p(m)$.*

Proof. Let τ be the identity homomorphism from Y into X. Apply Theorem 2.10. □

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Department of Mathematics and Computer Science
California State University, Long Beach
Long, Beach, California 90840
USA