THE HILBERT-SCHMIDT PROPERTY FOR EMBEDDING MAPS BETWEEN SOBOLEV SPACES

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1. Introduction. Let $H_0^m(\Omega)$ denote the so-called Sobolev space consisting of functions defined on a region Ω in n-dimensional Euclidean space, which together with their generalized derivatives of all orders $\leqslant m$ belong to $\mathfrak{L}_2(\Omega)$, and which vanish in a certain sense on the boundary $\partial\Omega$. (Precise definitions are given in the next section.) For each pair m, k of non-negative integers the inclusion $H_0^{m+k}(\Omega) \subset H_0^m(\Omega)$ defines a natural "embedding" map. For the case of a bounded region Ω it is well known that these maps are completely continuous, and even, for sufficiently large k, of Hilbert–Schmidt type. We have discussed complete continuity in the case of unbounded regions in an earlier paper; here we consider conditions on Ω which imply the Hilbert–Schmidt property for embeddings. An application is given to the spectral theory of self-adjoint uniformly elliptic differential operators; that is, we show that the resolvent operator corresponding to such a differential operator is of Hilbert–Schmidt type provided that the order of the differential operator is sufficiently large.

2. Embedding theorems. Let Ω be a region in n-dimensional Euclidean space E_n ($n \geqslant 2$). Consider the function

$$\tau(y) = \text{dist}\,(y, \partial\Omega), \qquad y \in \Omega;$$

$\partial\Omega$ denotes the boundary of Ω. If the region Ω is unbounded but satisfies the condition

$$\lim_{y\to\infty,\, y\in\Omega} \tau(y) = 0,$$

then Ω is said to be *quasi-bounded.*

Consider next the well-known Sobolev spaces $H_0^m(\Omega)$, with norms given by

$$||u||_m^2 = \sum_{|\alpha|\leqslant m} \int_\Omega |D^\alpha u(x)|^2\, dx;$$

by definition $H_0^m(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $||\ ||_m$. It is obvious that each space $H_0^{m+1}(\Omega)$ is continuously embedded in the preceding space $H_0^m(\Omega)$; the following stronger result is of importance in partial differential equations.

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THEOREM 1 (Rellich, Sobolev). *If Ω is bounded, then the embedding maps $H_0^{m+1}(\Omega) \to H_0^m(\Omega)$ $(m = 0, 1, 2, \ldots)$ are completely continuous.*

This result was sharpened considerably by K. Maurin **(4)**.

THEOREM 2 (Maurin). *If Ω is bounded and if $t > \frac{1}{2}n$, then the embedding maps $H_0^{m+t}(\Omega) \to H_0^m(\Omega)$ $(m = 0, 1, 2, \ldots)$ are of Hilbert–Schmidt type.*

A mapping $T : X \to Y$ between two Hilbert spaces is said to be of *Hilbert–Schmidt type* if $\sum_i \|Te_i\|_Y^2 < \infty$ for any orthonormal sequence $\{e_i\}$ in X.

In **(3)** we obtained the following generalization of Theorem 1. (Condition I, which we need not describe here, is a sort of regularity condition.)

THEOREM 3. *Let Ω be a quasi-bounded open set in E_n, satisfying the "condition I" **(3)**. Then the embedding maps $H_0^{m+1}(\Omega) \to H_0^m(\Omega)$ are completely continuous. Conversely, if Ω is not bounded or quasi-bounded, the embedding maps are not completely continuous.*

This theorem leads easily to a result of A. M. Molcanov **(5)** on discreteness of the spectrum of the Laplacian operator on Ω, with zero boundary conditions; cf. **(3)**. Theorems 1 and 2 are also valid for the Sobolev spaces denoted by $H^m(\Omega)$, but Theorem 3 fails in general for such spaces. (In particular, if Ω is contained in a cylinder of finite cross-section and if Ω has infinite n-dimensional volume, it can easily be shown that the mapping $H^1(\Omega) \to \mathfrak{L}_2(\Omega)$ is not completely continuous.) For this reason we shall not consider the spaces $H^m(\Omega)$ further.

Our main interest in the present paper is to generalize Maurin's theorem to quasi-bounded regions.

THEOREM 4. *Let Ω be a quasi-bounded region in E_n. Suppose that for some non-negative integer ν we have*

$$\int_\Omega \tau(y)^{2\nu+2}\, dy < \infty. \tag{1}$$

Then the embedding maps $H_0^{m+t}(\Omega) \to H_0^m(\Omega)$ $(m = 0, 1, 2, \ldots)$ are of Hilbert–Schmidt type provided

$$t > \tfrac{1}{2}n + \nu + 1.$$

The proof of Theorem 4 is based on the following lemma.

LEMMA. *Let Ω be an open set in E_n. Then provided $m > \frac{1}{2}n + k$, we have $H_0^m(\Omega) \subset C^k(\bar{\Omega})$, with continuous embedding.*

Proof (suggested by the referee). For $u \in H_0^m(\Omega)$, define

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in E_n - \Omega. \end{cases}$$

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Then $\tilde{u} \in H^m(E_n)$ and $||\tilde{u}||_{H^m(E_n)} = ||u||_{H_0{}^m(\Omega)}$. But by the Sobolev embedding theorem (e.g. **2**, Lemma 5), $H^m(E_n) \subset C^k(E_n)$ continuously, provided $m > \frac{1}{2}n + k$. Therefore $u' = \tilde{u}|_{\bar{\Omega}} \in C^k(\bar{\Omega})$ and

$$||u'||_{C^k(\bar{\Omega})} = ||\tilde{u}||_{C^k(E_n)} \leqslant k||\tilde{u}||_{H^m(E_n)} = k||u||_{H_0{}^m(\Omega)},$$

completing the proof.

Proof of Theorem 4. We write out the proof for the case $m = 0$, leaving to the reader the straightforward alterations needed to obtain the general case.

Let $y \in \Omega$ be given, and consider the linear functional T_y on $H_0{}^t(\Omega)$ defined by $T_y(u) = u(y)$. By the lemma we have, for $u \in H_0{}^t(\Omega)$,

$$\sup_{y \in \Omega} |D^\alpha u(y)| \leqslant \text{const.}\, ||u||_t \qquad \text{if } |\alpha| \leqslant \nu + 1. \tag{2}$$

In particular, T_y is a continuous linear functional on $H_0{}^t(\Omega)$, so that there exists $g_y \in H_0{}^t(\Omega)$ with $||g_y||_t = ||T_y||$ and

$$T_y u = u(y) = (u, g_y)_t \qquad \text{for } u \in H_0{}^t(\Omega).$$

Now let $\{e_i\}$ be an orthonormal sequence in $H_0{}^t(\Omega)$. Then

$$||T_y||^2 = ||g_y||_t{}^2 \geqslant \textstyle\sum_i |(e_i, g_y)_t|^2 = \sum_i |e_i(y)|^2.$$

To show that the embedding $H_0{}^t(\Omega) \to \mathfrak{L}_2(\Omega)$ is Hilbert–Schmidt, it suffices to show that

$$\textstyle\sum_i ||e_i||_0{}^2 = \sum_i \int_\Omega |e_i(y)|^2\, dy < \infty;$$

and for this the following inequality is sufficient:

$$\int_\Omega ||T_y||^2\, dy < \infty. \tag{3}$$

Now let $u \in C_0{}^\infty(\Omega)$, let $y \in \Omega$, and let y_0 be a point of $\partial\Omega$ such that

$$\tau(y) = \text{dist}\,(y, \partial\Omega) = |y - y_0|.$$

Expanding $u(y)$ about y_0 by Taylor's formula with remainder, we have

$$u(y) = \sum_{|\alpha| = \nu+1} \frac{1}{\alpha!} D^\alpha u(y_\alpha)(y - y_0)^\alpha,$$

where $|y_\alpha - y_0| < |y - y_0|$. Using inequality (2), we therefore obtain

$$|u(y)| \leqslant c||u||_t \cdot \tau(y)^{\nu+1}, \tag{4}$$

c being a constant independent of u.

Now we can easily show that (4) holds also for any $u \in H_0{}^t(\Omega)$. For example, let $\epsilon > 0$ and let $u_1 \in C_0{}^\infty(\Omega)$ satisfy $||u - u_1||_t < \epsilon$. By (2) we have

$$|u(y) - u_1(y)| \leqslant K\epsilon,$$

so that applying (4) for $u_1(y)$, we obtain

$$\begin{aligned}|u(y)| &\leqslant c\|u_1\|_t \cdot \tau(y)^{\nu+1} + K\epsilon \\ &\leqslant c(\|u\|_t + \epsilon)\tau(y)^{\nu+1} + K\epsilon \\ &= c\|u\|_t\,\tau(y)^{\nu+1} + c_1\,\epsilon,\end{aligned}$$

c_1 being independent of u. Since ϵ is arbitrary, (4) now follows for any $u \in H_0{}^t(\Omega)$.

The proof of (3) is now immediate:

$$\|T_y\| = \sup_{\|u\|_t \leqslant 1} |u(y)| \leqslant c\tau(y)^{\nu+1};$$

the hypothesis (1) then yields (3) and the theorem is proved. Note that the argument given also works for the case $\nu = -1$, which includes the case of bounded Ω. In fact in this case our proof is the same as Maurin's.

THEOREM 5. *Let Ω be a region in E_n, and suppose that for some non-negative integer ν we have*

$$\int_\Omega \tau(y)^{2\nu+2}\,dy = +\infty. \tag{5}$$

Then the embedding maps $H_0{}^{m+t}(\Omega) \to H_0{}^m(\Omega)$ are not of Hilbert–Schmidt type ($m = 0, 1, 2, \ldots$) if $t \leqslant [\frac{1}{2}n] + \nu + 1$.

Proof. We again treat only the case $m = 0$. There is no loss of generality in assuming that Ω is quasi-bounded, for in the contrary (unbounded) case the embeddings are not even completely continuous, by Theorem 3. Moreover, we need consider only the case $t = [\frac{1}{2}n] + \nu + 1$.

Following the first part of the proof of the preceding theorem, but now taking $\{e_i\}$ to be a *complete* orthonormal sequence in $H_0{}^t(\Omega)$, we see that

$$\sum_i \|e_i\|_0{}^2 = \int_\Omega \|T_y\|^2\,dy,$$

so that we must show that

$$\int_\Omega \|T_y\|^2\,dy = +\infty. \tag{6}$$

Since for quasi-bounded regions the norm $\|\ \|_m$ in $H_0{}^m(\Omega)$ is equivalent to the norm $|\ |_m$ given by

$$|u|_m{}^2 = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^2\,dx$$

(3), (6) is equivalent to

$$\int_\Omega |T_y|^2\,dy = +\infty, \tag{6'}$$

where

$$|T_y| = \sup_{|u|_t \leqslant 1} |u(y)|.$$

(The new norm $|\ |_m$ is introduced here only for convenience; it could be dispensed with at the cost of some complication in the ensuing calculations.)

Let B_1 denote the unit solid ball in E_n, centred at the origin. Choose a function $u \in C_0^\infty(B_1)$ satisfying $|u(0)| = \sigma \neq 0$ and $|u|_t = 1$. For a given point $y \in \Omega$, let $\rho = \tau(y)$ and consider the function

$$\tilde{u}(z) = \rho^{t-n/2}\, u[\rho^{-1}(z-y)].$$

Then $\tilde{u} \in C_0^\infty(\Omega)$ and $|\tilde{u}|_t = 1$, whereas

$$|\tilde{u}(y)| = \sigma\rho^{t-n/2} = \begin{cases} \sigma\rho^{\nu+1} & \text{if } n \text{ is even,} \\ \sigma\rho^{\nu+\frac{1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$|T_y|^2 \geqslant \begin{cases} \sigma^2\rho^{2\nu+2} & (n \text{ even}), \\ \sigma^2\rho^{2\nu+1} & (n \text{ odd}). \end{cases}$$

Since $\rho = \tau(y) < 1$ except on a bounded subset of Ω, the relation (6′) is a consequence of (5). The proof is complete.

Note that the proof shows that when n is odd, the hypothesis (5) can be weakened to

$$(5') \qquad \int_\Omega \tau(y)^{2\nu+1}\, dy = +\infty.$$

In spite of this, Theorem 5 is not a complete converse to Theorem 4. Assuming Ω to be a quasi-bounded domain, let us define $\beta = \beta(\Omega)$ as the smallest integer (if any exist) such that $\int_\Omega \tau(y)^\beta\, dy < \infty$. Likewise, let $\gamma = \gamma(\Omega)$ be the smallest integer making all the embedding maps $H_0^{m+\gamma}(\Omega) \to H_0^m(\Omega)$ Hilbert–Schmidt. Define $\lambda = [\frac{1}{2}(n + \beta + 1)]$. Theorems 4 and 5 show that if $\beta < +\infty$, then either $\gamma = \lambda$ or $\gamma = \lambda + 1$; if n is odd and β even, then $\gamma = \lambda$. We conjecture that in general $\gamma = \lambda + 1$ when n is even and $\gamma = \lambda$ when n is odd; this agrees with the case of a bounded region, corresponding to the case $\beta = 0$.

Theorem 5 has the following obvious consequence, corresponding to the case $\beta = +\infty$.

COROLLARY. *If $\int_\Omega \tau(y)^\beta\, dy = +\infty$ for every positive integer β, then none of the embedding maps $H_0^{m+t}(\Omega) \to H_0^m(\Omega)$ is of Hilbert–Schmidt type.*

3. Application. We can use Theorem 4 to discuss the nature of the resolvent of a uniformly elliptic operator L (with null boundary conditions) acting in $\mathfrak{L}_2(\Omega)$, Ω being a quasi-bounded region. For instance, consider the differential operator

$$L = \sum_{|\alpha| \leqslant 2\mu} a_\alpha(x) D^\alpha.$$

Let the coefficients $a_\alpha(x)$ be real bounded functions on Ω and, without specifying details, let us assume a sufficient degree of smoothness for the a_α. For simplicity, assume also that L is formally self-adjoint. Finally, and this is crucial, suppose that L is uniformly elliptic on Ω, i.e.

$$(-1)^\mu \sum_{|\alpha|=2\mu} a_\alpha(x)\xi^\alpha \geqslant \text{const. } |\xi|^{2\mu}$$

for every $x \in \Omega$, $\xi \in E_n$.

These conditions are sufficient to allow us to apply standard arguments for the construction of an operator $A_\lambda : \mathfrak{L}_2(\Omega) \to H_0^\mu(\Omega)$ with the properties that for sufficiently large λ, A_λ is bounded, one to one, and A_λ^{-1} is an extension of the differential operator $L + \lambda$ in the sense that $A_\lambda^{-1}f = (L + \lambda)f$ for smooth functions $f \in H_0^\mu(\Omega)$; cf. **(1**, p. 198**)** for this construction; also see **(2**, Theorem 16(c)**)** for the proof of the basic Gårding inequality for unbounded regions.

If J_μ denotes the embedding map $H_0^\mu(\Omega) \to \mathfrak{L}_2(\Omega)$, we define the *resolvent* operator $R_\lambda = J_\mu A_\lambda$; this leads to the operator $\tilde{L} = R_\lambda^{-1} - \lambda I$, which is a natural self-adjoint operator to associate with the differential operator L. By Theorem 3, R_λ is completely continuous, and the spectrum of $\tilde{L}$ is therefore discrete. Using Theorem 4 and the simple observation that the product of a Hilbert–Schmidt operator and a bounded operator is again Hilbert–Schmidt, we obtain the following result.

THEOREM 6. *Let Ω satisfy the hypotheses of Theorem 4 and let the operator L satisfy the above conditions. Then for large positive λ the resolvent operator R_λ is of Hilbert–Schmidt type provided that $\mu > \frac{1}{2}n + \nu + 1$.*

If the inequality $\mu > \frac{1}{2}n + \nu + 1$ does not hold, we may consider instead the operator L^k where $k\mu > \frac{1}{2}n + \nu + 1$. Thus the resolvent $R_\lambda^{(k)}$ of L^k will be of Hilbert–Schmidt type for large λ. Since, however, in general $(L^k)^\sim \neq (\tilde{L})^k$, we do not obtain any immediate information about spectral properties of $\tilde{L}$. One might expect the eigenvalues of $(L^k)^\sim$ and those of $(\tilde{L})^k$ to have the same asymptotic growth, but this would probably be difficult to verify in the case of an unbounded region Ω.

REFERENCES

1. L. Bers, F. John, and M. Schechter, *Partial differential equations* (New York, 1964).

2. Felix E. Browder, *On the spectral theory of elliptic differential operators*. I, Math. Ann., *142* (1961), 22–130.

3. Colin Clark, *An embedding theorem for function spaces*, to appear in Pacific J. Math.

4. Krzysztof Maurin, *Abbildungen vom Hilbert-Schmidtschen Typus und ihre Anwendungen*, Math. Scand., *9* (1961), 359–371.

5. A. M. Molcanov, *On conditions for the discreteness of the spectrum of a self-adjoint differential equation of second order* (Russian), Trudi Mosk. Mat. Obshchestva, *2* (1953), 169–200.

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