ON FUNDAMENTAL CONSTRUCTIONS AND ADJOINT FUNCTORS

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A fundamental construction of a category $\mathcal{C}$ ((2), Appendice) is a triple (S, p, k), where S is a functor from $\mathcal{C}$ to itself and where $p:S^2 \to S$ and $k:1_{\mathcal{C}} \to S$ are natural transformations such that

$$p(p*S) = p(S*p)$$
$$p(k*S) = 1_S = p(S*k)$$

Given two fundamental constructions (S, p, k) and (S', p', k') of $\mathcal{C}$, a morphism from the first to the second is a natural transformation $m:S \to S'$ such that

$$k' = mk$$
$$mp = p'(m*m)$$

The fundamental constructions of $\mathcal{C}$ with their morphisms, multiplied as natural transformations, form a category.

Given two categories $\mathcal{C}$ and $\mathcal{C}'$, by an adjoint morphism from $\mathcal{C}'$ to $\mathcal{C}$, we mean a quadruple (T, U, t, u), where T is a functor from $\mathcal{C}'$ to $\mathcal{C}$ with left adjoint U defined by the natural transformations $t:1_{\mathcal{C}} \to TU$ and $u:UT \to 1_{\mathcal{C}'}$. This means that the following conditions are satisfied:

$$(T*u)(t*T) = 1_T$$
$$(u*U)(U*t) = 1_U$$

Every adjoint morphism $(T, U, t, u):\mathcal{C}' \to \mathcal{C}$ defines a fundamental

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construction (S, p, k) of $\mathcal{C}$ (3), where

$$S = TU, \quad p = T*u*U \quad \text{and} \quad k = t$$

and conversely, it was shown by Kleisli [4] and Eilenberg-Moore [1] that every fundamental construction of $\mathcal{C}$ is defined in this way by an adjoint morphism to $\mathcal{C}$. Furthermore, the adjoint morphism constructed in [1] was shown to be, let us say, the "coarsest" possible. In this paper, we will broaden this characterization of the Eilenberg-Moore adjoint morphism in such a way that morphisms of fundamental constructions will be involved, and similarly, we will characterize the Kleisli adjoint morphism as the "finest" one possible, here again, the characterization involving morphisms of fundamental constructions.

For two objects A and B of a category $\mathcal{C}$, $M_{\mathcal{C}}(A, B)$ will denote the set of morphisms of $\mathcal{C}$ from A to B. All functors are of course assumed to be covariant. A functor $F: \mathcal{C} \to \mathcal{C}'$ is "faithful" if for any two morphisms $f, g \in M_{\mathcal{C}}(A, B)$, $F(f) = F(g)$ implies $f = g$, while F is "full" if for any morphism $f' \in M_{\mathcal{C}'}(F(A), F(B))$, there exists $f \in M_{\mathcal{C}}(A, B)$ such that $f' = F(f)$.

THEOREM 1. Given a fundamental construction (S,p, k) of $\mathcal{C}$, there exists an adjoint morphism

$$(\overline{T}, \overline{U}, k, \overline{u}): \overline{\mathcal{C}} \to \mathcal{C}$$

defining (S, p, k), where $\overline{\mathcal{C}}$ has the same objects as $\mathcal{C}$ and $\overline{U}(A) = A$ for each object A, such that if

$$(T, U, k', u): \mathcal{C}' \to \mathcal{C}$$

is an adjoint morphism defining the fundamental construction (S', p', k') and if $m: (S, p, k) \to (S', p', k')$, then there exists a unique pair $(V, \overline{m})$, where V is a functor from $\overline{\mathcal{C}}$ to $\mathcal{C}'$ and $\overline{m}$ is a natural transformation from $\overline{T}$ to TV, such that $U = V\overline{U}$ and $m = \overline{m}*\overline{U}$. Furthermore,

$$V*\overline{u} = (u*V)\,(U*\overline{m})$$

and if each m_A is a monomorphism, then V is faithful, while if each m_A has a right inverse, then V is full. Finally, if $(k*S')m = S*k'$, then V has a right adjoint.

Proof. The constructions of $\bar{\mathcal{C}}$ and $(\bar{T}, \bar{U}, k, \bar{u})$ are those of Kleisli [4]. The objects of $\bar{\mathcal{C}}$ are those of $\mathcal{C}$. For any two objects A and B, $M_{\bar{\mathcal{C}}}(A, B) = M_{\mathcal{C}}(A, S(B))$. If $f \in M_{\bar{\mathcal{C}}}(A, B)$ and $g \in M_{\bar{\mathcal{C}}}(B, C)$ their product in $\bar{\mathcal{C}}$ is given by

$$g \cdot f = p_C S(g) f \in M_{\bar{\mathcal{C}}}(A, C)$$

The functors $\bar{U}$ and $\bar{T}$ are defined as follows: if $f \in M_{\mathcal{C}}(A, B)$, then $\bar{U}(A) = A$ and $\bar{U}(f) = k_B f$, while if $f \in M_{\bar{\mathcal{C}}}(A, B)$, then $\bar{T}(A) = S(A)$ and $\bar{T}(f) = p_B S(f)$. Finally, for each object A, $\bar{u}_A = 1_{S(A)} \in M_{\bar{\mathcal{C}}}(S(A), A)$. We refer the reader to [4] for the proof of the statement that $\bar{\mathcal{C}}$ is a category and $(\bar{T}, \bar{U}, k, \bar{u})$ is an adjoint morphism defining (S, p, k).

Now, let (T, U, k', u) and m be as above. Assume first of all that there is a pair $(V, \bar{m})$ with the required properties. Then, for each object A of $\bar{\mathcal{C}}$, $V(A) = V\bar{U}(A) = U(A)$, and if $f \in M_{\bar{\mathcal{C}}}(A, B)$, then

$$\begin{aligned}
V(f) &= V(f \bar{u}_{\bar{U}(A)} \bar{U}(k_A)) = V(\bar{u}_{\bar{U}(B)} \bar{U}\bar{T}(f) \bar{U}(k_A)) \\
&= V(\bar{u}_{\bar{U}(B)}) U(\bar{T}(f) k_A) = V(\bar{u}_{\bar{U}(B)}) U(p_B S(f) k_A) \\
&= V(\bar{u}_{\bar{U}(B)}) U(f) = V(\bar{u}_{\bar{U}(B)}) u_{U\bar{T}\bar{U}(B)} U(k'_{\bar{T}\bar{U}(B)}) U(f) \\
&= u_{V\bar{U}(B)} UTV(\bar{u}_{\bar{U}(B)}) U(k'_{\bar{T}\bar{U}(B)} f) \\
&= u_{U(B)} U(TV(\bar{u}_{\bar{U}(B)}) \bar{m}_{\bar{U}\bar{T}\bar{U}(B)} k_{\bar{T}\bar{U}(B)} f) \\
&= u_{U(B)} U(\bar{m}_{\bar{U}(B)} \bar{T}(\bar{u}_{\bar{U}(B)}) k_{\bar{T}\bar{U}(B)} f) \\
&= u_{U(B)} U(m_B p_B k_{S(B)} f) = u_{U(B)} U(m_B f)
\end{aligned}$$

and of course, $\bar{m}_A = \bar{m}_{\bar{U}(A)} = m_A$. Thus, there is no choice; if $(V, \bar{m})$ exists, it is unique.

Now we show that if we define, for each $f \in M_{\bar{\mathcal{C}}}(A, B)$, $V(A) = U(A)$, $V(f) = u_{U(B)} U(m_B f)$ and $\bar{m}_A = m_A$, then V is a functor and $\overline{m}$ is a natural transformation, with the required properties.

If $f \in M_{\overline{\mathcal{C}}}(A, B)$ and $g \in M_{\overline{\mathcal{C}}}(B, C)$, then

$$V(g \cdot f) = u_{U(C)} U(m_C p_C S(g) f)$$

$$= u_{U(C)} U(p'_C S'(m_C) m_{S(C)} S(g) f)$$

$$= u_{U(C)} U(p'_C S'(m_C) S'(g) m_B f)$$

$$= u_{U(C)} UT(u_{U(C)}) UTU(m_C g) U(m_B f)$$

$$= u_{U(C)} u_{UTU(C)} UTU(m_C g) U(m_B f)$$

$$= u_{U(C)} U(m_C g) u_{U(B)} U(m_B f) = V(g) V(f)$$

while

$$V(1_A) = V(k_A) = u_{U(A)} U(m_A k_A) = u_{U(A)} U(k'_A) = 1_{U(A)} = 1_{V(A)} .$$

Thus, V is a functor. If $f \in M_{\mathcal{C}}(A, B)$, $V\overline{U}(A) = V(A) = U(A)$ and $V\overline{U}(f) = V(k_B f) = u_{U(B)} U(m_B k_B f) = u_{U(B)} U(k'_B) U(f) = U(f)$ so that $V\overline{U} = U$. If $f \in M_{\overline{\mathcal{C}}}(A, B)$, then

$$TV(f)\overline{m}_A = T(u_{U(B)} U(m_B f)) m_A = p'_B S'(m_B f) m_A$$

$$= p'_B m_{S'(B)} S(m_B f) = m_B p_B S(f) = m_B \overline{T}(f)$$

so that $\overline{m}$ is a natural transformation from $\overline{T}$ to TV. Obviously, for each object A of $\overline{\mathcal{C}}$,

$$V(\overline{u}_A) = V(1_{S(A)}) = u_{U(A)} U(m_A) = u_{V(A)} U(\overline{m}_A)$$

so that $V * \overline{u} = (u * V)(U * \overline{m})$.

Now, assume that for each A, m_A is a monomorphism. If $f \in M_{\overline{\mathcal{C}}}(A, B)$,

$$TV(f) k'_A = TV(f) \overline{m}_A k_A = \overline{m}_B \overline{T}(f) k_A = m_B p_B S(f) k_A = m_B f$$

so that if $g \in M_{\bar{\mathcal{C}}}(A, B)$ is such that $V(f) = V(g)$, then $m_B f = m_B g$ and therefore $f = g$. Thus V is faithful.

Then, assume that each m_B has a right inverse n_B. If $f' \in M_{\mathcal{C}'}(V(A), V(B))$, then $n_B T(f')k'_A \in M_{\bar{\mathcal{C}}}(A, B)$ and

$$V(n_B T(f')k'_A) = u_{U(B)} U(m_B n_B T(f')k'_A) = u_{U(B)} UT(f')U(k'_A)$$

$$= f' u_{U(A)} U(k'_A) = f'$$

Thus, in this case, V is full.

Finally, assume that $(k * S')m = S * k'$. We set $W = \bar{U}T$. For each object A of $\mathcal{C}$,

$$k_{S'(A)} k'_A : A \to SS'(A)$$

so that it is a morphism v_A in $\bar{\mathcal{C}}$ from A to $S'(A) = \bar{U}S'(A) = \bar{U}TU(A) = WV(A)$. Let us show that $v = \{v_A\}$ is a natural transformation from $1_{\bar{\mathcal{C}}}$ to WV. If $f \in M_{\bar{\mathcal{C}}}(A, B)$, then

$$WV(f) \cdot v_A = p_{S'(B)} S(\bar{U}T(u_{U(B)} U(m_B f)) k_{S'(A)} k'_A$$

$$= p_{S'(B)} S\bar{U}(p'_B S'(m_B f)) k_{S'(A)} m_A k_A$$

$$= p_{S'(B)} S(k_{S'(B)} p'_B S'(m_B f)) S(m_A) k_{S(A)} k_A$$

$$= S(p'_B S'(m_B f) m_A) S(k_A) k_A$$

$$= S(p'_B S'(m_B) m_{S(B)} S(f) k_A) k_A$$

$$= S(m_B p_B k_{S(B)} f) k_A = S(m_B f) k_A$$

$$= k_{S'(B)} m_B f = S(k'_B) f$$

$$= p_{S'(B)} S(k_{S'(B)} k'_B) f = v_B \cdot f$$

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Now, $u: UT = V\bar{U}T = VW \to 1_{\mathcal{C}'}$. For each object A of $\bar{\mathcal{C}}$,

$$u_{V(A)}V(v_A) = u_{U(A)}u_{US'(A)}U(m_{S'(A)}S(k'_A)k_A)$$

$$= u_{U(A)}u_{US'(A)}U(S'(k'_A)m_A k_A)$$

$$= u_{U(A)}u_{US'(A)}UTU(k'_A)U(k'_A)$$

$$= u_{U(A)}U(k'_A)u_{U(A)}U(k'_A) = 1_{U(A)} = 1_{V(A)}$$

so that $(u*V)(V*v) = 1_V$. Then, for each object A' of $\mathcal{C}'$,

$$W(u_{A'})\cdot v_{W(A')} = \bar{U}T(u_{A'})\cdot v_{W(A')}$$

$$= (k_{T(A')}T(u_{A'}))\cdot v_{W(A)}$$

$$= p_{T(A')}S(k_{T(A')}T(u_{A'}))S(k'_{T(A')})k_{W(A')}$$

$$= k_{W(A')}$$

and of course, $k_{T(A')}$ is the identity of $W(A')$ in $\bar{\mathcal{C}}$ so that we have shown that $(W*u)(v*W) = 1_W$. Thus, (W, V, v, u) is an adjoint morphism.

As an immediate corollary of this theorem, one can obtain a result of F.E.J. Linton (Notices A.M.S., February 1966, 631-1), i.e. there is an equivalence between the category of fundamental constructions of $\mathcal{C}$ and their morphisms and the category of adjoint morphisms

$$(T, U, t, u): \mathcal{C}' \to \mathcal{C}$$

where $\mathcal{C}'$ has the same objects as $\mathcal{C}$ and $U(A) = A$ for all A, and their "morphisms".

PROPOSITION. For a fundamental construction (S, p, k) of $\mathcal{C}$, the following statements are equivalent:

(i) p is a natural equivalence,

(ii) $k*S$ is a natural equivalence,

(iii) $\mathcal{S}*k$ is a natural equivalence,

(iv) $k*S = \mathcal{S}*k$,

(v) $p*S = \mathcal{S}*p$.

Proof. That (i), (ii) and (iii) are equivalent and that they imply (iv) and (v) is obvious. If one assumes (iv), then

$$(\mathcal{S}*k)p = (p*S)(S^2*k) = (p*S)(\mathcal{S}*k*S) = 1_{S^2}$$

so that p is a natural equivalence, while if one assumes (v), then

$$(\mathcal{S}*k)p = (p*S)(S^2*k) = (\mathcal{S}*p)(S^2*k) = 1_{S^2}$$

so that p is a natural equivalence.

A fundamental construction satisfying the conditions of this proposition will be called an idempotent construction.

As a special case of theorem 1, we have that if the adjoint morphism

$$(T, U, k, u) : \mathcal{C}' \to \mathcal{C}$$

defines the fundamental construction (S, p, k), then there exists a unique functor $V: \bar{\mathcal{C}} \to \mathcal{C}'$ such that $U = V\bar{U}$ and $\bar{T} = TV$,[1) and if (S, p, k) is an idempotent construction, then V has a right adjoint.

By a regular construction of $\mathcal{C}$, we mean a couple (S, k), where S is a functor from $\mathcal{C}$ to itself and $k:1_{\mathcal{C}} \to S$ is a natural transformation, such that if $f \in M_{\mathcal{C}}(A, S(B))$, then there exists a unique $g \in M_{\mathcal{C}}(S(A), S(B))$ such that $f = gk_A$. It was shown in [5] (§ 3, Proposition 1), that if (S, k) is a regular construction of $\mathcal{C}$, then there exists a unique natural transformation $p:S^2 \to S$ such that (S, p, k) is a fundamental construction, and that p is then a natural equivalence. Regular constructions are thus essentially idempotent constructions and there are of course

1) This is apparently known. See the review of [4] by P. J. Huber, Math. Reviews, February, 1966.

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very many examples of regular constructions.

THEOREM 2. Given a fundamental construction (S, p, k) of $\mathcal{C}$, there exists an adjoint morphism

$$(\underline{T}, \underline{U}, k, \underline{u}): \underline{\mathcal{C}} \to \mathcal{C}$$

defining (S, p, k), where $\underline{T}$ is faithful, such that if

$$(T, U, k', u): \mathcal{C}' \to \mathcal{C}$$

is an adjoint morphism defining the fundamental construction (S', p', k') and if $m:(S, p, k) \to (S', p', k')$, then there exists a unique functor $Z: \mathcal{C}' \to \underline{\mathcal{C}}$ such that $T = \underline{T}Z$ and $\underline{T}*\underline{u}*Z = (T*u)(m*T)$. Furthermore, there exists a unique natural transformation $\underline{m}:\underline{U} \to ZU$ such that $\underline{T}*\underline{m} = m$ and if (S, p, k) is a regular construction, then Z has a left adjoint.

Proof. The constructions of $\underline{\mathcal{C}}$ and $(\underline{T}, \underline{U}, k, \underline{u})$ are those of Eilenberg and Moore [1]. The objects of $\underline{\mathcal{C}}$ are the couples (A, φ), where A is an object of $\mathcal{C}$ and $\varphi \in M_{\mathcal{C}}(S(A), A)$ is such that $\varphi k_A = 1_A$ and $\varphi S(\varphi) = \varphi p_A$. Given two such objects (A, φ) and (B, ψ), the morphisms from the first to the second in $\underline{\mathcal{C}}$ are the morphisms $f \in M_{\mathcal{C}}(A, B)$ such that $\psi S(f) = f\varphi$. Morphisms in $\underline{\mathcal{C}}$ are multiplied as in $\mathcal{C}$. The functors $\underline{U}$ and $\underline{T}$ are defined as follows: if $f \in M_{\mathcal{C}}(A, B)$, then $\underline{U}(A) = (S(A), p_A)$ and $\underline{U}(f) = S(f)$, while if $f \in M_{\underline{\mathcal{C}}}((A, \varphi), (B, \psi))$, then $\underline{T}(A, \varphi) = A$ and $\underline{T}(f) = f$. For any object (A, φ) of $\underline{\mathcal{C}}$, $\underline{u}_{(A, \varphi)} = \varphi$. We refer the reader to [1] for the proof of the statement that $(\underline{T}, \underline{U}, k, \underline{u})$ is an adjoint morphism defining (S, p, k).

Now, let (T, U, k', u) and m be as above. For each object A' of $\mathcal{C}'$, $Z(A') = (T(A'), T(u_{A'})m_{T(A')})$ is an object of $\underline{\mathcal{C}}$ since

$$T(u_{A'})m_{T(A')}k_{T(A')} = T(u_{A'})k'_{T(A')} = 1_{T(A')}$$

and

$$T(u_{A'})m_{T(A')}S(T(u_{A'})m_{T(A')}) = T(u_{A'})S'T(u_{A'})m_{S'T(A')}S(m_{T(A')})$$

$$= T(u_{A'}UT(u_{A'}))(m*m)_{T(A')}$$

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$$= T(u_{A'}u_{UT(A')})(m*m)_{T(A')}$$

$$= T(u_{A'})p'_{T(A')}(m*m)_{T(A')} = T(u_{A'})m_{T(A')}p_{T(A')}$$

If $f':A' \to B'$ in $\mathcal{C}'$, then

$$T(u_{B'})m_{T(B')}ST(f') = T(u_{B'})S'T(f')m_{T(A')}$$

$$= T(u_{B'}UT(f'))m_{T(A')} = T(f')T(u_{A'})m_{T(A')}$$

so that

$$T(f'):(T(A'), T(u_{A'})m_{T(A')}) \to (T(B'), T(u_{B'})m_{T(B')})$$

in $\underline{\mathcal{C}}$ and therefore, we may set $Z(f') = T(f')$. It is obvious that we have defined a functor Z from $\mathcal{C}'$ to $\underline{\mathcal{C}}$ and that $T = \underline{T}Z$. Also, for any object A' of $\mathcal{C}'$,

$$\underline{T}(\underline{u}_{Z(A')}) = \underline{T}(\underline{u}_{(T(A'), T(u_{A'})m_{T(A')})})$$

$$= \underline{T}(T(u_{A'})m_{T(A')}) = T(u_{A'})m_{T(A')}$$

so that $\underline{T}*\underline{u}*Z = (T*u)(m*T)$.

Now, assume that Z' is a functor from $\mathcal{C}'$ to $\underline{\mathcal{C}}$ such that $T = \underline{T}Z'$ and

$$\underline{T}*\underline{u}*Z' = (T*u)(m*T)$$

If A' is any object of $\mathcal{C}'$ and if we set $Z'(A') = (A, \varphi)$, then

$$A = \underline{T}(A, \varphi) = \underline{T}Z'(A') = T(A')$$

and

$$\varphi = \underline{T}(\varphi) = \underline{T}(\underline{u}_{(A, \varphi)}) = \underline{T}(\underline{u}_{Z'(A)}) = T(u_{A'})m_{T(A')}$$

so that we see that $Z'(A') = Z(A')$. Thus, since $\underline{T}Z' = T = \underline{T}Z$ and since $\underline{T}$ is faithful, $Z' = Z$.

For each object A of $\mathcal{C}$, m_A is a morphism in $\underline{\mathcal{C}}$ from

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$\underline{U}(A) = (S(A), p_A)$ to $ZU(A) = (S'(A), p'_A m_{S(A)})$ since

$$p'_A m_{S(A)} S(m_A) = p'_A (m^* m)_A = m_A p_A$$

so that, $\underline{T}$ being faithful, if we set $\underline{m}_A = m_A$ for each A, $\underline{m}$ is the only natural transformation from $\underline{U}$ to ZU such that $\underline{T}^* \underline{m} = m$.

Finally, let us assume that (S, p, k) is a regular construction. If (A, φ) is an object of $\underline{\mathcal{C}}$, then $k_A \varphi k_A = k_A$ so that $k_A \varphi = 1_{S(A)}$, i.e. k_A is an isomorphism and φ is its inverse. If (A, φ) and (B, ψ) are objects of $\underline{\mathcal{C}}$ and if $f \in M_{\mathcal{C}}(A, B)$, then

$$f\varphi = \psi k_B f \varphi = \psi S(f) k_A \varphi = \psi S(f)$$

so that $f:(A, \varphi) \to (B, \psi)$ in $\underline{\mathcal{C}}$. Thus we see that in this case, $\underline{\mathcal{C}}$ is essentially the full subcategory of $\mathcal{C}$ whose objects are the injectives of the regular injective structure of $\mathcal{C}$ underlying (S, p, k) ([5], § 3). Now, if we set $Y = U\underline{T}$, then

$$u: UT = U\underline{T}Z = YZ \to 1_{\underline{\mathcal{C}}}$$

For each object (A, φ) of $\underline{\mathcal{C}}$, we have

$$ZY(A, \varphi) = ZU\underline{T}(A, \varphi) = ZU(A) = (TU(A), T(u_{U(A)}) m_{TU(A)})$$

so that we may set

$$y_{(A, \varphi)} = k'_A : (A, \varphi) \to ZY(A, \varphi)$$

and y is obviously a natural transformation from $1_{\underline{\mathcal{C}}}$ to ZY. For each object A' of $\mathcal{C}'$,

$$Z(u_{A'}) y_{Z(A')} = T(u_{A'}) k'_{T(A')} = 1_{T(A')} = 1_{Z(A')}$$

so that $(Z^* u)(y^* Z) = 1_Z$. For each object (A, φ) of $\underline{\mathcal{C}}$,

$$u_{Y(A, \varphi)} Y(y_{(A, \varphi)}) = u_{U(A)} U(k'_A) = 1_{U(A)} = 1_{Y(A, \varphi)}$$

so that $(u*Y)(Y*y) = 1_Y$. Thus, (Z, Y, y, u) is an adjoint morphism from $\mathcal{C}'$ to $\mathcal{C}$.

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