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INCLUSION THEOREMS FOR *FK*-SPACES

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1. Introduction. The main result of this paper is Theorem 1 (in connection with Corollary 1 (e)), which says that the implication

$$(*) \qquad Y \cap W_E \subset F \Rightarrow Y \cap W_E \subset W_F$$

holds for every separable *FK*-space F, for every *FK*-space E containing φ and for certain (for example, solid) *FK-AB*-spaces Y. At this, φ denotes the space of all finite sequences and W_E is the set of all elements of E being weakly sectionally convergent.

This result was proved by Bennett and Kalton ([**1**] and [**3**]) in the special case that E contains all null sequences and that Y is the space m of all bounded sequences or the space of all sequences almost converging to zero. Also $(*)$ was proved by Snyder [**10**] in case of a semiconservative *FK*-space E containing a *BK-AK*-space K_0 with $\overline{\varphi} = K_0^\beta$ and in case of a sequence space Y with $K_0 \subset Y \subset M(K_0)$ where $M(Y)$ has a "gliding humps property" and $M(X)$ denotes the set of all factor sequences from X into X.

If E and F are the convergence domains of matrices A and B, respectively, then the implication $(*)$ is a result of Leiger and the author [**5**]. This theorem of Mazur-Orlicz-type has as corollary a general consistency theorem, which includes the well known bounded consistency theorem of Mazur and Orlicz.

2. Notation and preliminaries. As usual ω, m, f_0, c, c_0 and φ denote the vector space of all complex (or real) sequences

$$x = (x_k) = (x_k)_{k \in \mathbf{N}},$$

the space of all bounded sequences, of all sequences almost converging to zero, of all convergent sequences, of all null sequences and of all finite sequences, respectively.

For fixed p, $1 \leqq p < \infty$, let

$$l^p := \left\{ x = (x_k) \mid \sum_k |x_k|^p < \infty \right\}, \quad l := l^1$$

and for fixed $\mu = (\mu_k)$, $0 < \mu_k \quad (k \in \mathbf{N})$, let

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$$m_\mu := \left\{x = (x_k) \,|\left(\frac{x_k}{\mu_k}\right) \in m\right\} \quad (\mu\text{-bounded sequences}),$$

$$c_{o\mu} := \left\{x = (x_k) \,|\left(\frac{x_k}{\mu_k}\right) \in c_0\right\} \quad \text{and}$$

$$l_\mu := \{x = (x_k) \,| \quad (\mu_k x_k) \in l\}.$$

If $p = (p_k) \in m$ and $0 < p_k \quad (k \in \mathbf{N})$, then the following notations will be used:

$$l(p) := \{x = (x_k) \,| \quad (|x_k|^{p_k}) \in l\},$$

$$c_0(p) := \{x = (x_k) \,| \quad (|x_k|^{p_k}) \in c_0\} \quad \text{and}$$

$$m(p) := \{x = (x_k) \,| \quad (|x_k|^{p_k}) \in m\}.$$

Furthermore, we consider the sequence spaces

$$bv := \left\{x = (x_k) \,|\, \|x\|_{bv} := |x_1| + \sum_k |x_k - x_{k+1}| < \infty\right\},$$

$$\delta := \{x = (x_k) \,| \overline{\lim_k} |x_k|^{1/k} = 0\}$$

$$= c_0\left(\left(\frac{1}{k}\right)\right) \text{ (entire sequences)},$$

$$\Pi_r := \left\{x = (x_k) \,| \overline{\lim_k} |x_k|^{1/k} \leqq \frac{1}{r}\right\} \quad (r > 0)$$

and

$$|D| := \left\{x = (x_k) \,|\, \sup_n \sum_k |d_{nk} x_k| < \infty\right\}$$

(absolute-D-bounded sequences)

in case of a matrix $D = (d_{nk})$. Let $e := (1, 1, \dots)$ and $e^k := (0, \dots, 0, 1, 0, \dots)$, where "1" is in the k-th position. For fixed $x = (x_k) \in \omega$ and $n \in \mathbf{N}$, the *n-th section of* x is

$$x^{[n]} := \sum_{k=1}^{n} x_k e^k = (x_1, \dots, x_n, 0, \dots).$$

For a sequence space E we put

$$E^\beta := \left\{y \in \omega |\, \forall\, x \in E: \sum_k y_k x_k \text{ exists}\right\} \quad (\beta\text{-dual of } E)$$

and

$$M(E) := \{y \in \omega | \forall x \in E: (x_k y_k) \in E\}$$

(factor sequences from E into E).

Provided that $m \subset M(E)$ then E is *solid*.

A locally convex sequence space is called *K-space* if the coordinate functional $x = (x_k) \to x_j$ is continuous for each $j \in \mathbf{N}$. A K-space E being also a Fréchet space is called *FK-space*; if the topology is normable, then E is called *BK-space*. If (E, F) is a dual pair then $\sigma(E, F)$ and $\tau(E, F)$ denote the weak topology and the Mackey topology, respectively. In case of a sequence space E and $F := E^{\beta}$ we consider the natural bilinear form.

For a fixed K-space E with $\varphi \subset E$ we consider distinguished subsets of E:

$$L_E := \{x \in E| \{x^{[n]}|n \in \mathbf{N}\} \text{ is bounded in } E\}$$

(sectionally bounded sequences),

$$F_E := \left\{x \in E \mid \sum_k x_k f(e^k) \text{ exists for each } f \in E'\right\}$$

(sequences with functionally convergent sections),

$$W_E := \{x \in E \,|x^{[n]} \to x(\sigma(E, E'))\}$$

(sequences with weakly convergent sections),

$$S_E := \{x \in E \,|x^{[n]} \to x \text{ in } E\}$$

(sectionally convergent sequences).

Obviously $\varphi \subset S_E \subset W_E \subset F_E \subset L_E$ holds for every K-space E with $\varphi \subset E$. An *FK-AB*-space (*BK-AB*-space) and an *FK-AK*-space (*BK-AK*-space) is an *FK*-space (*BK*-space) E satisfying $E = L_E$ and $E = S_E$, respectively. It is well known that each special sequence space, introduced above, is an *FK-AB*-space. An *FK*-space E, containing $\varphi \oplus \langle e \rangle$, is called *conull*, if $e \in W_E$.

Let

$$B = (b_{nk}) = (b_{nk})_{n,k \in \mathbf{N}}$$

(with coefficients in $\mathbf{C}$). We put

$$c_B := \left\{x \in \omega \mid Bx := \left(\sum_k b_{nk} x_k\right)_{n \in \mathbf{N}} \text{ exists and } Bx \in c\right\}$$

((convergence) domain of B) and

$$\lim_B x := \lim Bx \quad \text{for each } x \in c_B.$$

Obviously $\varphi \subset c_B$ holds if and only if each column of B is convergent. In this case

$$b_k := \lim_n b_{nk}$$

denotes the limit of the k-th column of B. Two matrices A and B are called *consistent on* M, $M \subset c_A \cap c_B$, if $\lim_A x = \lim_B x$ holds for each $x \in M$. Furthermore we use the following notations:

$$I_B := \left\{x \in c_B \mid \sum_k b_k x_k \text{ exists}\right\}$$

$$\Lambda_B : I_B \to \mathbf{C},\ x \to \Lambda_B(x) := \lim_B x - \sum_k b_k x_k \quad \text{and}$$

$$\Lambda_B^{\perp} := \{x \in I_B \mid \Lambda_B(x) = 0\}.$$

It is well known, that c_B is an FK-space, and we write L_B, F_B, W_B and S_B instead of L_{c_B}, F_{c_B}, W_{c_B} and S_{c_B}, respectively. The inclusions

$$\varphi \subset W_B = \Lambda_B^{\perp} \cap L_B \subset F_B = I_B \cap L_B \subset L_B$$

were proved by Wilansky **[12]** in case of $c \subset c_B$, and they are also true in the general case $\varphi \subset c_B$ (**[13]**).

If E is a vector space and M is a subset of E, then conv M denotes the convex hull of M in E and $\bar{M}^{\tau} = \bar{M}$ denotes the closure of M relative to a topology τ on E.

In the following an *index sequence* is a sequence (k_ν) in $\mathbf{N}$ with $k_\nu < k_{\nu+1}$ $(\nu \in \mathbf{N})$.

3. Main result and corollaries. First of all we formulate the main result of this paper, then we draw some corollaries and discuss some special cases of this result and its corollaries. Furthermore we make some remarks on the bibliography.

The proof of the main result will be given in the next section. To formulate the main result we need a special class $\mathscr{E}^*$ of "factor sequences" motivated by the proof of Theorem 1.

Definition. Let $y = (y_k) \in \omega$, then, by definition, $y \in \mathscr{E}^*$ if

$$(y_k - y_{k+1}) \in c_0 \quad \text{and} \quad y_k \geqq 0 \quad (k \in \mathbf{N})$$

and there exist two index sequences (k_j) and (k_j^*) with the following properties:

(i) $k_j^* < k_j < k_{j+1}^*$ $(j \in \mathbf{N})$,

(ii) $$y_k = \begin{cases} 0 & \text{if } k_{2\mu-1} < k \leqq k_{2\mu}^* \\ 1 & \text{if } k_{2\mu} < k \leqq k_{2\mu+1}^* \end{cases} \quad (\mu \in \mathbf{N}),$$

(iii) $y_k \leqq y_n$ if $k^*_{2\mu} < k \leqq n \leqq k_{2\mu}$ $(\mu \in \mathbf{N})$ and

(iv) $y_n \leqq y_k$ if $k^*_{2\mu+1} < k \leqq n \leqq k_{2\mu+1}$ $(\mu \in \mathbf{N})$.

THEOREM 1. *Let E be an FK-space containing φ, and let Y be an FK-AB-space such that $\mathcal{E}^* \subset M(Y)$. Then the implication*

$$Y \cap W_E \subset c_B \Rightarrow Y \cap W_E \subset W_B$$

holds for every matrix B.

Proof. A proof of this result will be given in the next section.

With the following remark we demonstrate that the condition "$\mathcal{E}^* \subset M(Y)$" is fulfilled by the elements of a large class of *FK-AB*-spaces Y.

Remarks 1. (a) By definition, every solid sequence space Y satisfies the condition $\mathcal{E}^* \subset M(Y)$; for example this is true if Y equals ω, m, c_0, l^p $(1 \leqq p < \infty)$ or m_μ, $c_{0\mu}$, l_μ $(\mu = (\mu_k), 0 < \mu_k\ (k \in \mathbf{N}))$ or $l(p)$, $c_0(p)$, $m(p)$ $(p = (p_k) \in m$ with $p_k > 0\ (k \in \mathbf{N}))$ or δ or Π_r $(r > 0)$ or $|D|$ (in case of a matrix D). All these special sequence spaces are *FK-AB*-spaces.

(b) The *FK-AB*-space $Y := f_0$ is not solid, but $\mathcal{E}^* \subset M(f_0)$ is true because every $y \in \mathcal{E}^*$ satisfies

$$(y_k - y_{k+1}) \in c_0$$

([7], Theorem 5).

An immediate consequence of Theorem 1 and a result of Bennett and Kalton ([2], Theorem 5) is the following corollary which for example says that the β-dual of $Y \cap W_E$ is weakly sequentially complete (if Y and E satisfy the assumptions in Theorem 1).

COROLLARY 1. *Let E be an FK-space containing φ, let Y be an FK-AB-space such that $\mathcal{E}^* \subset M(Y)$, and let $X := Y \cap W_E$. Then each of the following (equivalent) conditions is true*:

(a) $(X^\beta, \sigma(X^\beta, X))$ *is sequentially complete*;

(b) *if (F, τ) is a separable FK-space and*

$$T:(X, \tau(X, X^\beta)) \rightarrow (F, \tau)$$

is a linear mapping with closed graph, then T is continuous.

(c) *Every matrix mapping from $(X, \tau(X, X^\beta))$ into any separable FK-space is continuous.*

(d) *If (F, τ) is any separable FK-space containing $Y \cap W_E$ then τ induces on $Y \cap W_E$ a weaker topology than $\tau(X, X^\beta)$.*

(e) *For each separable FK-space F the implication*

$$Y \cap W_E \subset F \Rightarrow Y \cap W_E \subset W_F$$

holds.

(f) *For any matrix B the implication*

$$Y \cap W_E \subset c_B \Rightarrow Y \cap W_E \subset W_B$$

is true.

We use the next remarks to point out some known special cases of Theorem 1, and to enter into similar results of the bibliography.

Remarks 2. (a) In case of an FK-space E with $c_0 \subset E$ and in case $Y := m$ and $Y := f_0$, respectively, Theorem 1 was proved by Bennett and Kalton ([**1**], Theorem 16, and [**3**], Theorem 9, respectively). The assumption "$c_0 \subset E$" is a decisive factor for the proofs of Bennett and Kalton.

(b) Snyder ([**10**], Theorem 9) proved a general result similar to Theorem 1, but with certain assumptions on Y and E: Let K_0 be a semiconservative BK-AK-space such that φ is dense in K_0^β and $K_0 \subset M(K_0)$, let Y be a sequence space such that $K_0 \subset Y \subset M(K_0)$ is fulfilled and such that $M(Y)$ has a "gliding humps property". Then the implication

$$Y \cap W_E \subset c_B \Rightarrow Y \cap W_E \subset W_B$$

holds for each matrix B and any FK-space E containing K_0.

(c) Leiger and the author ([**5**], Satz 1) proved Theorem 1 in the case $\varphi \subset E := c_A$ and $\mathscr{E} \subset M(Y)$, where A is an infinite matrix and $\mathscr{E}$ is a special class of sequences with $\mathscr{E} \subset \mathscr{E}^*$. We remark that the usual FK-AB-spaces Y satisfy $\mathscr{E}^* \subset M(Y)$ if $\mathscr{E} \subset M(Y)$.

To point out the significance of the result in Remark 2 (c) we formulate this theorem and some of its applications as corollaries of Theorem 1.

COROLLARY 2 (see [**5**], Satz 1). *Let A be a matrix with $\varphi \subset c_A$, and let Y be an FK-AB-space such that $\mathscr{E}^* \subset M(Y)$. Then the implication*

$$Y \cap W_A \subset c_B \Rightarrow Y \cap W_A \subset W_B$$

is true for every matrix B.

Since W_A has codimension 1 in F_A the following consistency theorem is an immediate consequence of Corollary 2.

COROLLARY 3 (see [**5**], Folgerung 2). *Let Y be an FK-AB-space with $\mathscr{E}^* \subset M(Y)$, and let A, B be matrices with $\varphi \subset Y \cap F_A \subset c_B$. If*

$$u \in \{0\} \cup (Y \cap F_A \backslash W_A)$$

is chosen such that

$$Y \cap F_A = (Y \cap W_A) \oplus \langle u \rangle,$$

then the consistency of A and B on $\varphi \oplus \langle u \rangle$ implies the consistency on $Y \cap F_A$.

Remarks 3. If $Y := m$ and A is a coregular (conservative) matrix, then in Corollary 3 we can choose $u := e$ and thus Corollary 3 is the well known bounded consistency theorem of Mazur-Orlicz-Brudno (see **[8]** and **[9]**, Theorem 2, and **[6]**, Satz 1). [In this case $m \cap F_A = m \cap c_A$ holds, and A, B are consistent on c if and only if they are consistent on $\varphi \oplus \langle e \rangle$.)

If A is coregular and $Y := |A| \cap |B|$, Corollary 3 is the consistency theorem of Volkov (**[11]**, Satz 1).

Furthermore Corollary 3 was proved by the author in case of $Y := \omega$ and a coregular matrix A.

Section 2 of **[5]** contains more details about consequences of Corollary 2 and 3 and about the bibliography.

4. Proof of the main result. We prepare the proof of Theorem 1 by three lemmas. The first of them says that every $x \in W_E$, E an *FK*-space, can be approximated by modificated sections of x. In case $x := e \in W_E$ this can be illustrated as follows: e can be approximated by sequences in φ which first equal 1 and then they decrease to 0 in finitely many steps, where the minimal length and the maximal height of the steps can be chosen.

LEMMA 1. *Let (E, τ) be an FK-space containing φ, let !! be a paranorm on E providing τ, and let (η_ν) be an index sequence. If $x \in W_E$, then*

$$x \in \overline{\operatorname{conv}\{x^{[\eta_\nu]} \mid \nu \in \mathbf{N}\}}^{\tau}$$

and there exists a sequence $(x^{(r)})$ of convex combinations

$$(1) \qquad x^{(r)} := \sum_{\nu=s_r}^{t_r} \mu_{r\nu} x^{[\eta_\nu]}$$

$$\left(s_r, t_r \in \mathbf{N} \text{ with } s_r < t_r, \right.$$

$$\left. 0 \leqq \mu_{r\nu} \leqq \frac{1}{r+1}, \ \mu_{rt_r} \neq 0 \quad \text{and} \ \sum_{\nu=s_r}^{t_r} \mu_{r\nu} = 1 \right)$$

of sections of x such that

$$(2) \qquad x^{(r)} \rightarrow x \quad \text{in } (E, \tau).$$

Proof. Because of $x \in W_E$ the sequence $(x^{[n]})$, and hence $(x^{[\eta_\nu]})$, is $\sigma(E, E')$-convergent to x. Thereby

$$x \in \overline{\operatorname{conv}\{x^{[\eta_\nu]} \mid \nu \geqq p\}}^{\tau}$$

holds for every $p \in \mathbf{N}$. Therefore we can choose inductively a sequence $(z^{(n)})$ in

$$\operatorname{conv}\{x^{[\eta_\nu]} \mid \nu \in \mathbf{N}\}$$

such that

$$\|z^{(n)} - x\| < \frac{1}{n} \quad (n \in \mathbf{N})$$

and $z^{(n)}$ has a representation

$$z^{(n)} = \sum_{\nu=v_n}^{w_n} \lambda_{n\nu} x^{[\eta_\nu]}$$

$$\Bigg(v_n, w_n \in \mathbf{N} \text{ with } n \leqq v_n \leqq w_n < v_{n+1},$$

$$0 \leqq \lambda_{n\nu} \leqq 1, \lambda_{nw_n} \neq 0 \text{ and } \sum_{\nu=v_n}^{w_n} \lambda_{n\nu} = 1\Bigg).$$

A straightforward calculation shows that $(x^{(r)})$, defined by

$$x^{(r)} := \frac{1}{r+1} \sum_{n=(1/2)r(r+1)+1}^{(1/2)(r+1)(r+2)} z^{(n)} \quad (r \in \mathbf{N}),$$

satisfies the conditions (1) and (2).

The next lemma is due to Snyder **[10]** and can be used to modify sequences $x \in W_E$ by certain factor sequences y such that $yx \in W_E$.

LEMMA 2. *Let E be an FK-space containing φ and let (γ_j) be an index sequence with $\gamma_1 = 1$. Furthermore let $y = (y_k) \in \omega$ and*

$$y^{(j)} := \sum_{k=\gamma_j}^{\gamma_{j+1}-1} y_k e^k \quad (j \in \mathbf{N})$$

such that

$$\sup_j \|y^{(j)}\|_{bv} < \infty.$$

Then for each $x \in W_E$ the condition

$$yx \in E \quad \text{and} \quad yx = \sum_{j=1}^{\infty} y^{(j)} x \quad \text{in } (E, \sigma(E, E'))$$

implies $yx \in W_E$.

Proof. The proof of this result is quite similar to that of Lemma 3 in **[10]** and will be left to the reader.

LEMMA 3. *Let (E, τ) be a conull FK-space containing φ, and let D be a matrix with $\varphi \oplus \langle e \rangle \subset c_D$, that is, the limits*

$$d := \lim_n \sum_{k=1}^{\infty} d_{nk} \quad \textit{and} \quad d_k := \lim_n d_{nk} \quad (k \in \mathbf{N})$$

exist. If there is a subsequence of $(\sum_{k=1}^{N} d_k)$ *converging to a limit unequal to* d, *then there exists an element* $y = (y_k) \in \mathcal{E}^*$ *such that* $y \in W_E$ *and* $y \notin c_D$.

Proof. First, we choose an $\alpha \in \mathbf{K}$ and an index sequence (η_ν) such that

$$\alpha \neq d \quad \text{and} \quad \alpha = \lim_\nu \sum_{k=1}^{\eta_\nu} d_k.$$

Without loss of generality we may assume that

$$(3) \qquad \left| \sum_{k=\eta_\nu+1}^{\eta_{\nu+\mu}} d_k \right| < 2^{-\nu} \quad (\nu, \mu \in \mathbf{N}).$$

According to Lemma 1 (in case of $x := e$) we choose a sequence $(x^{(r)})$ in $\mathrm{conv}\{e^{[\eta_\nu]} \mid \nu \in \mathbf{N}\}$ such that

$$(4) \qquad x^{(r)} \to e \text{ in } (E, \tau)$$

and

$$(5) \qquad x^{(r)} = \sum_{\nu=s_r}^{t_r} \mu_{r\nu} e^{[\eta_\nu]}$$

$$\Bigg(s_r, t_r \in \mathbf{N} \text{ with } s_r < t_r < s_{r+1},$$

$$0 \leqq \mu_{r\nu} \leqq \frac{1}{r+1}, \mu_{rt_r} \neq 0 \text{ and } \sum_{\nu=s_r}^{t_r} \mu_{r\nu} = 1\Bigg).$$

Now we construct inductively index sequences (k_j), (n_j), (r_j) and (k_j^*).

Let $\alpha_j := 2^{-j}$ $(j \in \mathbf{N})$ and $k_1^* := k_1 := 1$, and let ! ! be a paranorm providing the *FK* topology τ of E. Because of $\varphi \subset c_D$ we may choose a positive integer n_1 such that

$$\sum_{k=1}^{k_1} |d_{n_1 k} - d_k| = |d_{n_1 1} - d_1| < \alpha_1.$$

Furthermore, by (4) and $e \in c_D$ we may choose a positive integer r_1 such that

$$!x^{(r)} - x^{(r+\mu)}! < \alpha_1 \quad (r \geqq r_1 \text{ and } \mu \in \mathbf{N})$$

and

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$$\left|\sum_{k=\nu}^{\nu+\mu} d_{nk}\right| < \alpha_2 \quad (n \leqq n_1,\ \nu \geqq \eta_{s_{r_1}} \text{ and } \mu \in \mathbf{N}).$$

We assume that k^*_{j-1}, k_{j-1}, n_{j-1} and r_{j-1} have been chosen. Then we put

(6) $\quad k^*_j := \eta_{s_{r_{j-1}}} \quad$ and $\quad k_j := \eta_{t_{r_{j-1}}}.$

Especially

(7) $\quad k^*_{j-1} < k_{j-1} < k^*_j < k_j$

is valid (compare (5)). Because of $\varphi \subset c_D$ we may take $n_j > n_{j-1}$ such that

$$(8) \quad \sum_{k=1}^{k_j} |d_{n_jk} - d_k| < \alpha_j.$$

Furthermore, by (4) and $e \in c_D$, we may choose $r_j > r_{j-1}$ so that

(9) $\quad \|x^{(r)} - x^{(r+\mu)}\| < \alpha_{j+1} \quad (r \geqq r_j \text{ and } \mu \in \mathbf{N})$

and

$$(10) \quad \left|\sum_{k=\nu}^{\nu+\mu} d_{nk}\right| < \alpha_{j+1} \quad (n \leqq n_j,\ \nu \geqq \eta_{s_{r_j}} \text{ and } \mu \in \mathbf{N}).$$

Now we define

(11) $\quad y^{(j)} := x^{(r_{2j})} - x^{(r_{2j-1})} \quad (j \in \mathbf{N})$

and

$$(12) \quad y := \sum_{j=1}^{\infty} y^{(j)} \quad \text{(pointwise sum)}.$$

The sequences y and $y^{(j)}$ have the following properties:

(13) $\quad y^{(j)}_k = 0 \quad (k \leqq k^*_{2j} \text{ or } k > k_{2j+1})$

and

(14) $\quad y^{(j)}_k = 1 \quad (k_{2j} < k \leqq k^*_{2j+1})$

because of (6), (11) and (5),

$$(15) \quad y_k = \begin{cases} 0, \text{ if } k_{2\mu-1} < k \leqq k^*_{2\mu} \\ 1, \text{ if } k_{2\mu} < k \leqq k^*_{2\mu+1} \end{cases} \quad (\mu \in \mathbf{N})$$

because of (13), (14) and (12),

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(16) $\|y^{(j)}\|_{bv} \leqq 2 \quad (j \in \mathbf{N}),\ 0 \leqq y_k \leqq 1 \quad (k \in \mathbf{N}),$

$(y_k - y_{k+1}) \in c_0$ and $y \in \mathscr{E}^*$

because of (5), (13), (14) and (15).

Furthermore, by (9), the sequence

$$\left(\sum_{j=1}^{N} y^{(j)}\right)_{N\in\mathbf{N}}$$

is a Cauchy sequence in (E, τ), and therefore $y \in E$ since E is complete and y is the pointwise sum of $\sum_j y^{(j)}$ (see (12)).

Applying Lemma 2 we still have to verify $y \notin c_D$. Obviously, we may assume that $\sum_k d_{nk}y_k$ exists for each $n \in \mathbf{N}$. We prove the divergence of

$$\left(\sum_k d_{n_jk}y_k\right)_{j\in\mathbf{N}}.$$

First of all we establish that

$$\beta := \lim_\nu \sum_{k=1}^{\eta_\nu} d_k y_k$$

exists; this is easy to check by condition (3) because

$$y_k = y_\mu \quad (\eta_\nu < k \leqq \mu \leqq \eta_{\nu+1})$$

holds for every $\nu \in \mathbf{N}$.

For each $j \in \mathbf{N}$ we put

$$A_j := \sum_{k=1}^{k_j} d_{n_jk}y_k, \quad B_j := \sum_{k=k_j+1}^{k^*_{j+1}} d_{n_jk}y_k \quad \text{and}$$

$$C_j := \sum_{k=k^*_{j+1}+1}^{\infty} d_{n_jk}y_k.$$

Let j be an odd, positive integer, that is $j := 2\mu - 1$ for a certain $\mu \in \mathbf{N}$.

Then the following statements are true:

$$B_{2\mu-1} = 0 \quad (\mu > 1)$$

(because of (14)) and

$$|C_{2\mu-1}| \leqq \sum_{\nu=\mu}^{\infty} \left| \sum_{k=k^*_{2\nu}+1}^{k_{2\nu+1}} d_{n_{2\mu-1}k}y_k \right|$$

(see (15))

$$< 2 \sum_{\nu=\mu}^{\infty} \alpha_{2\nu}$$

(because of (6), (10) and (16))

$$\to 0 \quad (\mu \to \infty)$$

and

$$A_{2\mu-1} \to \beta = \lim_{\mu} \sum_{k=1}^{k_{2\mu-1}} d_k y_k$$

because of (put $n := n_{2\mu-1}$)

$$\begin{aligned} &|A_{2\mu-1} - \beta| \\ &\leqq \sum_{k=1}^{k_{2\mu-1}} |d_{nk} - d_k| y_k + \left| \sum_{k=1}^{k_{2\mu-1}} d_k y_k - \beta \right| \\ &\leqq \alpha_{2\mu-1} + \sum_{\nu=2\mu-1}^{\infty} \alpha_{\nu} \end{aligned}$$

(see (8) and (3))

$$\to 0 \quad (\mu \to \infty).$$

Consequently we have demonstrated

$$(17) \quad \sum_{k=1}^{\infty} d_{n_{2\mu-1}k} y_k \to \beta \quad (\mu \to \infty).$$

Now let j be an even, positive integer, that is $j := 2\mu$ for a certain $\mu \in \mathbf{N}$.

The statements

$$C_{2\mu} \to 0 \quad (\mu \in \mathbf{N}) \quad \text{and} \quad A_{2\mu} \to \beta \quad (\mu \to \infty)$$

may be proved similar to the case "$j := 2\mu - 1$".

Furthermore we have

$$B_{2\mu} = \sum_{k=k_{2\mu}+1}^{k^*_{2\mu+1}} d_{n_{2\mu}k}$$

(see (15)) and consequently

$$B_{2\mu} \to d - \alpha \quad (\mu \to \infty),$$

because in (put $n := n_{2\mu}$)

$$\sum_{k=k_{2\mu}+1}^{k^*_{2\mu+1}} d_{nk}$$
$$= \left(\sum_{k=1}^{\infty} d_{nk} - \sum_{k=1}^{k_{2\mu}} d_k\right) - \sum_{k=1}^{k_{2\mu}} (d_{nk} - d_k) - \sum_{k=k^*_{2\mu+1}+1}^{\infty} d_{nk}$$

the first term converges to $d - \alpha$ and the second and third term converge to 0 because of (8) and (10), respectively. Hence we have verified

$$\text{(18)} \quad \sum_{k=1}^{\infty} d_{n_\mu k} y_k \to \beta + d - \alpha \quad (\mu \to \infty).$$

Because of $d - \alpha \neq 0$ the statements (17) and (18) imply $Dy \notin c$, that is $y \notin c_D$. This completes the proof of Lemma 3.

Proof of Theorem 1. Theorem 1 is established if we prove the following implications:

(i) $Y \cap W_E \subset c_B \Rightarrow Y \cap W_E \subset L_B$.

(ii) $Y \cap W_E \subset c_B \Rightarrow Y \cap W_E \subset I_B$ (thus $Y \cap W_E \subset F_B$).

(iii) $Y \cap W_E \subset I_B \Rightarrow Y \cap W_E \subset \Lambda_B^{\perp}$ (thus $Y \cap W_E \subset W_B$).

The first statement is contained in [**5**], Satz 2. We are going to reduce the conditions (ii) and (iii) to Lemma 3.

(ii) Let $Y \cap W_E \subset c_B$, thus $Y \cap W_E \subset L_B$ by (i), but

$$Y \cap W_E \not\subset I_B.$$

Consequently we may choose an $x \in Y \cap W_E$ so that $x \notin I_B$, and we may assume $x_k \neq 0$ for each $k \in \mathbf{N}$ without loss of generality. Therefore we define

$$z = (z_k) := \left(\frac{1}{x_k}\right) \quad \text{and} \quad D = (d_{nk}) := (b_{nk} x_k).$$

Then

$$e \in zW_E = W_{zE} \quad \text{and} \quad \varphi \oplus \langle e \rangle \subset c_D$$

and

$$\left(\sum_{k=1}^{N} d_k\right)_{N \in \mathbf{N}} \in m \backslash c$$

(because of $x \in L_B \backslash I_B$), where d_k denotes the limit of the k-th column of D.

Consequently, by Lemma 3 (for zE) there exists a sequence

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$$y \in \mathscr{E}^* \quad \text{with } y \in zW_E = W_{zE} \text{ and } y \notin c_D,$$

that is

$$yx \in Y \cap W_E$$

(because of $\mathscr{E}^* \subset M(Y)$) and

$$yx \notin c_B;$$

this is a contradiction to $Y \cap W_E \subset c_B$.

(iii) Now we suppose

$$Y \cap W_E \subset I_B \quad \text{but} \quad Y \cap W_E \not\subset \Lambda_B^{\perp}.$$

Then we may choose

$$x \in Y \cap W_E \subset I_B \text{ such that } x \notin \Lambda_B^{\perp}$$

and without loss of generality we can assume $x_k \neq 0 \quad (k \in \mathbf{N})$. Furthermore we define z and D as in part (ii).

Consequently we obtain

$$e \in zW_E = W_{zE} \quad \text{and} \quad \varphi \oplus \langle e \rangle \subset I_D$$

and even

$$\lim_n \sum_k d_{nk} - \lim_n \sum_k d_k \neq 0.$$

If we again apply Lemma 3 in case of zE (instead of E), we obtain a sequence $y \in \mathscr{E}^*$ such that

$$y \in zW_E = W_{zE} \quad \text{and} \quad y \notin c_D,$$

that is

$$yx \in Y \cap W_E \quad \text{and} \quad yx \notin c_B;$$

this is a contradiction to $Y \cap W_E \subset I_B \subset c_B$ and completes the proof of Theorem 1.

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