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OSCILLATION OF SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH ALTERNATING COEFFICIENTS

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A new result is obtained for the oscillation of second order linear ordinary differential equations with alternating coefficients. This oscillation result extends a recent oscillation criterion due to Kamenev [*Mat. Zametki* 23 (1978), 249-251].

Consider the second order linear ordinary differential equation

$$x''(t) + a(t)x(t) = 0 \ , \quad t \geq t_0 \ , \tag{E}$$

where a *is a continuous real-valued function on the interval* $[t_0, \infty)$ *without any restriction on its sign.* A solution of (E) is said to be *oscillatory* if the set of its zeros is unbounded above, and otherwise it is said to be *nonoscillatory*. Equation (E) is called oscillatory if all its solutions are oscillatory.

For the oscillation of the equation (E) the following integral criterion due to Wintner [*4*] is well-known. *If*

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (t-s)a(s)ds = \infty \ ,$$

then (E) *is oscillatory.* Hartman [*1*] has shown that the limit cannot be replaced by the upper limit in the above condition. Recently, Kamenev [*2*]

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has proved a new integral criterion for the oscillation of the differential equation (E), based on the use of the nth primitive of the coefficient a, which has the result of Wintner as a particular case. More precisely, Kamenev has established that (E) *is oscillatory if, for some integer* $n \geq 3$,

$$(C_0) \qquad \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} a(s)ds = \infty .$$

Kamenev's criterion has been extended in various directions by Yeh [5], [6] and Philos [3]. The purpose of this paper is to extend the result of Kamenev by multiplying the coefficient a in condition (C_0) by a function which belongs to an appropriate function class depending on the integer n. More precisely, our main result is the following theorem.

THEOREM. *Let n be an integer with $n \geq 3$ and ρ be a positive continuously differentiable function on the interval $[t_0, \infty)$ such that*

$$(H) \qquad \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} \frac{(t-s)^{n-3}}{\rho(s)} [(n-1)\rho(s)-(t-s)\rho'(s)]^2 ds < \infty .$$

Equation (E) *is oscillatory if*

$$(C) \qquad \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \rho(s) a(s)ds = \infty .$$

Proof. Let x be a nonoscillatory solution of the differential equation (E) and let $T > \max\{t_0, 0\}$ be such that $x(t) \neq 0$ for all $t \geq T$. As in [2], we set $w = x'/x$ on $[T, \infty)$ and we obtain

$$a(t) = -w'(t) - w^2(t) \quad \text{for every} \quad t \geq T .$$

Then for $t \geq T$ we have

$$\int_T^t (t-s)^{n-1}\rho(s)a(s)ds$$

$$= - \int_T^t (t-s)^{n-1}\rho(s)w'(s)ds - \int_T^t (t-s)^{n-1}\rho(s)w^2(s)ds$$

$$= (t-T)^{n-1}\rho(T)w(T) + \tfrac{1}{4}\int_T^t \frac{(t-s)^{n-3}}{\rho(s)}[(n-1)\rho(s)-(t-s)\rho'(s)]^2 ds$$

$$- \int_T^t \left\{(t-s)^{(n-1)/2}\sqrt{\rho(s)}w(s) + \frac{(t-s)^{(n-3)/2}[(n-1)\rho(s)-(t-s)\rho'(s)]}{2\sqrt{\rho(s)}}\right\}^2 ds$$

$$\le (t-T)^{n-1}\rho(T)w(T) + \tfrac{1}{4}\int_{t_0}^t \frac{(t-s)^{n-3}}{\rho(s)}[(n-1)\rho(s)-(t-s)\rho'(s)]^2 ds .$$

On the other hand, for every $t \ge T$, we get

$$\int_{t_0}^t (t-s)^{n-1}\rho(s)a(s)ds - \int_T^t (t-s)^{n-1}\rho(s)a(s)ds$$

$$= \int_{t_0}^T (t-s)^{n-1}\rho(s)a(s)ds$$

$$\le \int_{t_0}^T (t-s)^{n-1}\rho(s)|a(s)|ds \le (t-t_0)^{n-1}\int_{t_0}^T \rho(s)|a(s)|ds .$$

Thus

$$\frac{1}{t^{n-1}}\int_{t_0}^t (t-s)^{n-1}\rho(s)a(s)ds$$

$$\le \left(1-\frac{T}{t}\right)^{n-1}\rho(T)w(T) + \left(1-\frac{t_0}{t}\right)^{n-1}\int_{t_0}^T \rho(s)|a(s)|ds$$

$$+ \frac{1}{t^{n-1}}\int_{t_0}^t \frac{(t-s)^{n-3}}{\rho(s)}[(n-1)\rho(s)-(t-s)\rho'(s)]^2 ds$$

for all $t \ge T$. This gives

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}}\int_{t_0}^t (t-s)^{n-1}\rho(s)a(s)ds \le \rho(T)w(T) + \int_{t_0}^T \rho(s)|a(s)|ds$$

$$+ \limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^t \frac{(t-s)^{n-3}}{\rho(s)}[(n-1)\rho(s)-(t-s)\rho'(s)]^2 ds ,$$

which contradicts conditions (H) and (C).

REMARK 1. By setting $\rho(t) = 1$, $t \ge t_0$, our theorem leads to

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Kamenev's criterion.

Now we remark that ρ satisfies (H) for any integer $n \geq 3$ if

$$\text{(H')}\qquad \begin{cases} \liminf_{t\to\infty} \rho'(t) > -\infty \, , \\ \limsup_{t\to\infty} \frac{1}{t^2}\int_{t_0}^{t} \rho(s)ds < \infty \quad \text{and} \quad \int^{\infty} \frac{[\rho'(t)]^2}{\rho(t)}dt < \infty \, . \end{cases}$$

A special case where ρ is subject to (H') is that $\rho(t) = t^{\alpha}$, $t \geq t_0 > 0$ for $\alpha \in [0, 1)$. So we have the following corollary.

COROLLARY 1. *Let* $t_0 > 0$. *Equation* (E) *is oscillatory if there exists a* $\alpha \in [0, 1)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}}\int_{t_0}^{t} (t-s)^{n-1}s^{\alpha}a(s)ds = \infty \quad \textit{for some integer} \quad n \geq 3 \, .$$

Also, when $\rho(t) = \log t$ for $t \geq t_0 > 1$, condition (H') is satisfied by itself. Hence we obtain Corollary 2 below.

COROLLARY 2. *Let* $t_0 > 1$. *Equation* (E) *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}}\int_{t_0}^{t} (t-s)^{n-1} \log s \cdot a(s)ds = \infty \quad \textit{for some integer} \quad n \geq 3 \, .$$

Now let us consider the more general differential equation with a damped term

$$\text{(E*)}\qquad x''(t) + q(t)x'(t) + a(t)x(t) = 0 \, , \quad t \geq t_0 \, ,$$

where q *is a continuous real-valued function on the interval* $[t_0, \infty)$ *without any restriction on its sign.*

Let n be an integer with $n \geq 3$ and ρ be a positive continuously differentiable function on $[t_0, \infty)$. We consider a nonoscillatory solution x of (E*) with $x(t) \neq 0$, $t \geq T$, for some $T > \max\{t_0, 0\}$ and we set $w = x'/x$ on $[T, \infty)$. Then $a = -w' - qw - w^2$ on $[T, \infty)$ and hence, for every $t \geq T$, we obtain

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$$\int_T^t (t-s)^{n-1}\rho(s)a(s)ds$$

$$= -\int_T^t (t-s)^{n-1}\rho(s)w'(s)ds - \int_T^t (t-s)^{n-1}\rho(s)q(s)w(s)ds$$

$$- \int_T^t (t-s)^{n-1}\rho(s)w^2(s)ds$$

$$= (t-T)^{n-1}\rho(T)w(T) + \tfrac{1}{4}\int_T^t \frac{(t-s)^{n-3}}{\rho(s)}\{(n-1)\rho(s)+(t-s)[\rho(s)q(s)-\rho'(s)]\}^2 ds$$

$$- \int_T^t \left\{(t-s)^{n-1}\sqrt{\rho(s)}w(s) + \frac{(t-s)^{(n-3)/2}\{(n-1)\rho(s)+(t-s)[\rho(s)q(s)-\rho'(s)]\}}{2\sqrt{\rho(s)}}\right\}^2 ds$$

$$\le (t-T)^{n-1}\rho(T)w(T) + \tfrac{1}{4}\int_{t_0}^t \frac{(t-s)^{n-3}}{\rho(s)}\{(n-1)\rho(s)+(t-s)[\rho(s)q(s)-\rho'(s)]\}^2 ds .$$

Thus we derive the following generalization of our theorem.

Let n *be an integer with* $n \ge 3$ *and* ρ *be a positive continuously differentiable function on the interval* $[t_0, \infty)$. *Suppose that*

$$(\mathrm{H}_1) \quad \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^t \frac{(t-s)^{n-3}}{\rho(s)}\{(n-1)\rho(s)+(t-s)[\rho(s)q(s)-\rho'(s)]\}^2 ds < \infty .$$

Equation (E*) *is oscillatory if* (C) *holds.*

We remark that (H_1) holds for any integer $n \ge 3$ if the function ρ satisfies

$$(\mathrm{H}_1') \quad \begin{cases} \liminf_{t\to\infty} [\rho'(t)-\rho(t)q(t)] > -\infty , \\ \limsup_{t\to\infty} \frac{1}{t^2}\int_{t_0}^t \rho(s)ds < \infty \quad \text{and} \quad \int^\infty \frac{[\rho'(t)-\rho(t)q(t)]^2}{\rho(t)} dt < \infty . \end{cases}$$

Also we note that for $\rho(t) = 1$, $t \ge t_0$, condition (H_1) becomes

$$(\mathrm{H}_1'') \qquad \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^t (t-s)^{n-3}[n-1+(t-s)q(s)]^2 ds < \infty .$$

So we obtain the follcwing result due to Yeh [6].

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Equation (E*) *is oscillatory if, for some integer* $n \geq 3$, (C_0) *and* (H_1'') *hold.*

REMARK 2. It is easy to see that (*cf.* [3], [5] and [6]) the results obtained for the differential equation (E) or for the more general equation (E*) hold also for the case of the not necessarily linear differential equation

$$x''(t) + a(t)f[x(t)] = 0 \ , \quad t \geq t_0 \ ,$$

or for the case of the equation

$$x''(t) + q(t)x'(t) + a(t)f[x(t)] = 0 \ , \quad t \geq t_0 \ ,$$

respectively, where f is a continuous function on the real line R which is differentiable on $R - \{0\}$ and such that, for some constant $k > 0$,

$$yf(y) > 0 \quad \text{and} \quad f'(y) \geq k \quad \text{for} \quad y \neq 0 \ .$$

REMARK 3. With the use of some well-known transformations the results of this paper can be extended for more general differential equations involving the term $(rx')'$ in place of the second derivative of the unknown function x , where r is a positive continuous function on the interval $[t_0, \infty)$.

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