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MULTIPLIERS ON WEIGHTED FUNCTION SPACES OVER LOCALLY COMPACT VILENKIN GROUPS

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In this note, we consider multipliers on weighted function spaces over totally disconnected locally compact Abelian groups (Vilenkin groups). First we present an (H^1_α, L^q_α) multiplier result. Then we give an (H^p_α, H^p_α) multiplier result under a similar condition of Lu–Yang type. In Section 3, we obtain a result about the boundedness of multipliers on weighted Besov spaces.

1. INTRODUCTION

One of the central problems in Fourier analysis is the study of multipliers and singular integral operators on weighted spaces over $\mathbb{R}^n$. In this paper we shall study the theory with the underlying group $\mathbb{R}^n$ replaced by certain totally disconnected groups G that are the locally compact analogues of Vilenkin groups. As early as the 1970's, pioneering work on multipliers and the boundedness of convolution operators on L^p spaces was done over local fields by Taibleson ([**15**]), and over certain locally compact Abelian groups by Gaudry and Inglis ([**3, 4**]). Later Onneweer and Quek ([**9, 10, 11, 12**]), Kitada ([**5**]) as well as Lu and Yang ([**7**]) extended their results to weighted spaces L^p_ω and H^p_ω. In this paper we extend Taibleson's result in a different direction.

In the last decade, besides Hardy spaces, many other function or distribution spaces have been found to admit a decomposition, in the sense that every member of the space is a linear combination of basic functions of a particularly elementary form. Such decompositions simplify the analysis of the spaces and the operators acting on them. In [**2**], Frazier and Jawerth obtained an atomic and a molecular decomposition for certain distribution spaces (Besov spaces and Triebel–Lizorkin spaces) on $\mathbb{R}^n$. With the use of this powerful result, many authors have studied the boundedness on such spaces of various types of operators. These papers were the motivation for the present paper in which we

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study such subjects in the context of locally compact Vilenkin groups G, instead of $\mathbb{R}^n$. In particular, we extend the atomic decomposition theorems of Frazier and Jawerth to the weighted Besov spaces on G, and apply them to prove the boundedness of multipliers on such spaces.

We now summarise the contents of this paper. In the remainder of this section, we describe the groups G and introduce the necessary definitions and notations. In Section 2, we give the boundedness of multipliers on weighted Hardy spaces $H^p_\alpha(G)$ where $0 \leqslant p < 1$. In Section 3, we first give an atomic decomposition of the weighted Besov spaces on G which is an extension of the atomic decomposition theorems of Frazier and Jawerth to G, and we prove the boundedness of some multipliers on $B^{\alpha,p}_p(\omega)$. In particular, a pseudo-differential operator with kernel in S^m_ρ is bounded on these weighted Besov spaces.

Throughout this paper, we shall denote by G a locally compact Abelian group containing a strictly decreasing sequence of compact open subgroups in the sense of Edwards and Gaudry ([**1**, Section 4.1]). This means that there exists a sequence $\{G_n\}_{n=-\infty}^{\infty}$ such that

(i) $\bigcup\limits_{n=-\infty}^{\infty} G_n = G$ and $\bigcap\limits_{n=-\infty}^{\infty} G_n = \{0\}$;

(ii) $\sup\{\mathrm{order}(G_n/G_{n+1}) : n \in \mathbb{Z}\} < \infty$.

Such a group is totally disconnected. It is a locally compact analogue of the groups described by Vilenkin in [**17**]. Several examples of such groups are given in [**1**, Section 4.1]. Additional examples are the p-adic numbers and, more general, the additive group of a local field (see [**15**]).

Let Γ denote the dual group of G, and for each $n \in \mathbb{Z}$, let

$$\Gamma_n = \{\gamma \in \Gamma : (\gamma, x) = 1 \,\forall\, x \in G_n\}.$$

Then $\{\Gamma_n\}_{n=-\infty}^{\infty}$ is a strictly increasing sequence of open compact subgroups of Γ and

(i′) $\bigcup\limits_{n=-\infty}^{\infty} \Gamma_n = \Gamma$ and $\bigcap\limits_{n=-\infty}^{\infty} \Gamma_n = \{1\}$;

(ii′) $\mathrm{order}(\Gamma_{n+1}/\Gamma_n) = \mathrm{order}(G_n/G_{n+1})$.

We choose Haar measures μ on G and γ on Γ so that $\mu(G_0) = \gamma(\Gamma_0) = 1$. Then $\mu(G_n)^{-1} = \gamma(\Gamma_n) := m_n$ for each $n \in \mathbb{Z}$. If we define the function $d : G \times G \to \mathbb{R}$ by $d(x,y) = 0$ when $x = y$ and $d(x,y) = m_n^{-1}$ when $x - y \in G_n \setminus G_{n+1}$, then d defines a metric on $G \times G$ and the topology on G induced by this metric is the same as the original topology on G. For $x \in G$, we set $|x| = d(x, 0)$. The symbols $\hat{}$ and $\check{}$ denote the Fourier transform and inverse Fourier transform respectively. We have $(\chi_{G_n})\hat{}(\gamma) = m_n^{-1}\chi_{\Gamma_n}(\gamma)$ and $(\chi_{\Gamma_n})\check{}(x) = m_n \chi_{G_n}(x) := \Delta_n(x)$ for each $n \in \mathbb{Z}$, where χ_A denotes the characteristic function of a set A. For the definition of the spaces of test functions and distributions on G and Γ, see [**8**]. These spaces will be denoted by $\mathcal{S}(G), \mathcal{S}'(G), \mathcal{S}(\Gamma)$ and $\mathcal{S}'(\Gamma)$.

The Lebesgue space on G with respect to the weighted measure

$$v_\alpha(x) d\mu(x) = |x|^\alpha d\mu(x) = d\mu_\alpha(x),$$

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where $\alpha \in \mathbb{R}$ and $0 < p < \infty$, will be denoted by L^p_α.

2. Multipliers on Weighted Hardy Spaces

We give the definition of the weighted Hardy spaces $H^p_\alpha(G)$, where $0 < p \leqslant 1$ and $\alpha > -1$.

Definition 1: Suppose that $\alpha > -1$ and $0 < p \leqslant 1$. A function $f \in S'(G)$ belongs to $H^p_\alpha(G)$ if

$$f^*(x) = \sup_{n\in\mathbf{Z}} |f * \Delta_n(x)| \in L^p_\alpha(G)$$

. Moreover, $\|f\|_{H^p_\alpha(G)} = \|f^*\|_{L^p_\alpha(G)}$.

A function a on G is called a $(p,\infty)_\alpha$ atom if

(i) $\operatorname{supp}(a) \subset I := x + G_n$ for some $x \in G$ and $n \in \mathbb{Z}$;

(ii) $\|a\|_\infty \leqslant (\mu_\alpha(I))^{-1/p}$;

(iii) $\displaystyle\int_G a(x)d\mu(x) = 0$.

Lemma 2. (See [9].) *Suppose that $-1 < \alpha \leqslant 0$ and $0 < p \leqslant 1$. A function $f \in S'(G)$ belongs to $H^p_\alpha(G)$ if and only if f can be represented as $f = \sum_i \lambda_i a_i$ with convergence in S', where each a_i is a $(p,\infty)_\alpha$ atom and $\sum_i |\lambda_i|^p < \infty$. Moreover, $\|f\|_{H^p_\alpha} \simeq \inf\left(\sum_i |\lambda_i|^p\right)^{1/p}$, where the infimum is taken over all such expression.*

Theorem 3. *Suppose that $1 \leqslant q \leqslant 2$. Assume that $h \in L^\infty(\Gamma \setminus \{0\})$ and there exist C, $\varepsilon > 0$ such that for all $j \in \mathbb{Z}$, $n > j$,*

$$\sup_{y\in G_n} \left\{ \int_{G_j\setminus G_{j+1}} \left|\check{h}(x-y) - \check{h}(x)\right|^2 d\mu(x) \right\}^{1/2} \leqslant C m_n^{-\varepsilon} m_j^{\varepsilon+1/q-1/2},$$

$$\left\{ \int_{G_j\setminus G_{j+1}} \left|\check{h}(x)\right|^2 d\mu(x) \right\}^{1/2} \leqslant C m_j^{1/q-1/2}.$$

Then h is a bounded multiplier from $H^1(G)$ to $L^q(G)$.

Proof: It suffices to prove that the $L^q(G)$ norm is bounded for $(1,\infty)$ atoms (with a bound that is independent of the atoms). Let a be a $(1,\infty)$ atom with support in G_n for some $n \in \mathbb{Z}$. We set

$$Ta(x) = (h\widehat{a})\check{}(x) = K * a(x).$$

Then

$$\begin{aligned}\|a * K\|_q &\leqslant \left(\int_{G\setminus G_n} |a * K(x)|^q d\mu(x)\right)^{1/q} + \mu(G_n)^{1/q-1/2}\|a * K\|_2 \\ &:= I_1 + I_2\end{aligned}$$

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and

$$\begin{aligned} I_1 &= \Big(\int_{G\setminus G_n} \big|K * a(x)\big|^q d\mu(x)\Big)^{1/q} \\ &\leqslant \Big(\int_{G\setminus G_n}\Big(\int_{G_n} \big|K(x-y)-K(x)\big|\big|a(y)\big|d\mu(y)\Big)^q d\mu(x)\Big)^{1/q} \\ &\leqslant \int_{G_n} \big|a(y)\big|\Big(\int_{G\setminus G_n} \big|K(x-y)-K(x)\big|^q d\mu(x)\Big)^{1/q} d\mu(y) \leqslant C. \end{aligned}$$

For the estimate of $\|a * K\|_2$, we decompose h as $\sum_{j\in\mathbf{Z}} h_j$ with $h_j(\xi) = h(\xi)\chi_{\Gamma_{j+1}\setminus\Gamma_j}(\xi)$.

$$\begin{aligned} \|a * K\|_2^2 &= \sum_{j\in\mathbf{Z}} \int \big|h_j(\xi)\widehat{a}(\xi)\big|^2 d\gamma(\xi) \\ &\leqslant \sum_{j\in\mathbf{Z}} \|h_j\|_{q'}^2 \Big(\int_{\Gamma_{j+1}\setminus\Gamma_j} \big|\widehat{a}(\xi)\big|^{(2q)/(2-q)} d\gamma(\xi)\Big)^{(2-q)/q} \\ &\leqslant \sum_{j\in\mathbf{Z}} \|K_j\|_q^2 \Big(\int_{\Gamma_{j+1}\setminus\Gamma_j} \big|\widehat{a}(\xi)\big|^{(2q)/(2-q)} d\gamma(\xi)\Big)^{(2-q)/q} \\ &\leqslant C\sum_{j\in\mathbf{Z}} \Big(\int_{\Gamma_{j+1}\setminus\Gamma_j} \big|\widehat{a}(\xi)\big|^{(2q)/(2-q)} d\gamma(\xi)\Big)^{(2-q)/q}. \end{aligned}$$

It remains to show that

$$J := \sum_{j\in\mathbf{Z}} \Big(\int_{\Gamma_{j+1}\setminus\Gamma_j} \big|\widehat{a}(\xi)\big|^{(2q)/(2-q)} d\gamma(\xi)\Big)^{(2-q)/q} \leqslant C\big(\mu(G_n)\big)^{-(2-q)/q}.$$

Starting with the case where $q = 1$, we get

$$J = \sum_{j\in\mathbf{Z}} \int_{\Gamma_{j+1}\setminus\Gamma_j} \big|\widehat{a}(\xi)\big|^2 d\gamma(\xi) = \int_\Gamma \big|\widehat{a}(\xi)\big|^2 = \|a\|_2 \leqslant C\big(\mu(G_n)\big)^{-1}.$$

In the case where $q = 2$, we want to estimate

$$\begin{aligned} J &= \sum_{j\in\mathbf{Z}} \sup_{\xi\in\Gamma_{j+1}\setminus\Gamma_j} \big|\widehat{a}(\xi)\big|^2 \\ &= \Big(\sum_{j\geqslant n} + \sum_{j<n}\Big) \sup_{\xi\in\Gamma_{j+1}\setminus\Gamma_j} \big|\widehat{a}(\xi)\big|^2 := J_1 + J_2. \end{aligned}$$

For J_2 we see that

$$\widehat{a}(\xi) = \int_{G_n} a(x)\overline{(\xi, x)}d\mu(x) = 0.$$

For J_1, we write $\nabla_n(\xi) = m_n^{-1}\chi_{\Gamma_n}(\xi)$, and observe that

$$\widehat{a}(\xi) = \widehat{a} * \nabla_n(\xi) = \int_{\Gamma_{j+1}\setminus\Gamma_j} \widehat{a}(\eta)\nabla_n(\xi - \eta)d\gamma(\eta).$$

The last equality holds because if $j \geqslant n$, and $\xi \in \Gamma_{j+1} \setminus \Gamma_j$, then $\eta \in \Gamma_{j+1} \setminus \Gamma_j$. So

$$\begin{aligned}
J_1 &= \sum_{j \geqslant n} \sup_{\xi \in \Gamma_{j+1} \setminus \Gamma_j} \left|\widehat{a}(\xi)\right|^2 \\
&\leqslant \sum_{j \geqslant n} \sup_{\xi \in \Gamma_{j+1} \setminus \Gamma_j} \left| \int_{\Gamma_{j+1} \setminus \Gamma_j} \widehat{a}(\eta) \nabla_n(\xi - \eta) d\gamma(\eta) \right|^2 \\
&\leqslant \sum_{j \geqslant n} \int_{\Gamma_{j+1} \setminus \Gamma_j} \left|\widehat{a}(\eta)\right|^2 d\gamma(\eta) \sup_{\xi \in \Gamma_{j+1} \setminus \Gamma_j} \int_{\Gamma_{j+1} \setminus \Gamma_j} \left|\nabla_n(\xi - \eta)\right|^2 d\gamma(\eta) \\
&\leqslant \sum_{j \geqslant n} \int_{\Gamma_{j+1} \setminus \Gamma_j} \left|\widehat{a}(\eta)\right|^2 d\gamma(\eta) \int_{\Gamma_n} \left|\nabla_n(\eta)\right|^2 d\gamma(\eta) \\
&= (m_n)^{-1} \int_{\Gamma} \left|\widehat{a}(\eta)\right|^2 d\gamma(\eta) \leqslant C.
\end{aligned}$$

Thus $J \leqslant C$ in the case $q = 2$. We shall now interpolate to get the general case. Let $p = 2/(1-\theta)$ where $0 \leqslant \theta \leqslant 1$, $c_j = \|\widehat{a}\chi_{\Gamma_{j+1} \setminus \Gamma_j}\|_\infty$ and $d_j = \|\widehat{a}\chi_{\Gamma_{j+1} \setminus \Gamma_j}\|_2$. We know that

$$\|\widehat{a}\chi_{\Gamma_{j+1} \setminus \Gamma_j}\|_p \leqslant c_j^\theta d_j^{1-\theta}.$$

The $q = 1$ estimate shows that $\|d_j\|_{l^2} \leqslant C\mu(G_n)^{-1/2}$ and the $q = 2$ estimate shows that $\|c_j\|_{l^2} \leqslant C$. This implies that

$$\left\| \|\widehat{a}\chi_{\Gamma_{j+1} \setminus \Gamma_j}\|_p \right\|_{l^2} \leqslant \|c_j^\theta d_j^{1-\theta}\|_{l^2} \leqslant \|c_j\|_{l^2}^\theta \|d_j\|_{l^2}^{1-\theta} \leqslant C\mu(G_n)^{-1/p}.$$

So when we put $p = (2q)/(2-q)$, we get $J \leqslant C\mu(G_n)^{-(2-q/q)}$, that is,

$$\|a * K\|_2 \leqslant C\mu(G_n)^{1/2 - 1/q}.$$

The proof is completed. ∎

Theorem 4. *Suppose that $0 < p \leqslant 1$ and $1 < r < 2$, and that a function $h \in L^\infty(\Gamma)$ satisfies*

$$\sum_{k=n}^{\infty} \left\| (h^k)\check{}\, \chi_{J_i} \right\|_r^p \leqslant C m_n^{p(1-1/r-1/p)} m_i$$

where $i < n$, $J_i = G_i \setminus G_{i+1}$, $h^k = h\chi_{\Gamma_{k+1} \setminus \Gamma_k}$, and C is independent of n, i. Then h is a bounded multiplier on $H_\alpha^p(G)$ when $-(1/r') < \alpha \leqslant 0$.

Proof: It is enough to prove that for every $(p, \infty)_\alpha$ atom a, on writing $f = (h\widehat{a})\check{}$, we have

$$\int_G f^*(x)^p d\mu_\alpha(x) \leqslant C$$

with C independent of a. Suppose $\operatorname{supp}(a) \subset x_0 + G_n := I_n$, for some $x_0 \in G$ and $n \in \mathbb{Z}$. Write

$$\int f^*(x)^p d\mu_\alpha(x) = \left(\int_{I_n} + \int_{G \setminus I_n} \right) f^*(x)^p d\mu_\alpha(x). \tag{1}$$

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We first see that

$$\begin{aligned}\int_{I_n}(f^*(x))^p d\mu_\alpha(x) &\leqslant \Big(\int_{I_n}(f^*(x))^2 d\mu(x)\Big)^{p/2}\Big(\int_{I_n} v_{(2\alpha/2-p)} d\mu(x)\Big)^{1-p/2}\\ &\leqslant C\|a\|_2^p \mu(I_n)^{1-p/2}\inf\{v_\alpha(x) : x\in I_n\setminus\{0\}\}\\ &\leqslant C. \qquad (2)\end{aligned}$$

For $x \notin I_n$, we set $\psi(\gamma) = \overline{\gamma(x_0)}h(\gamma)$ and $b(x) = a(x+x_0)$. Then as shown in the proof of [**5**, Theorem 2] or [**7**, Theorem 2.5],

$$f^*(x) \leqslant \sum_{j=n}^{\infty}\big|(\psi^j)^\check{} * b(x)\big| \leqslant \sum_{j=n}^{\infty}\sum_{i=-\infty}^{n-1}\big|(\psi^j)^\check{}\chi_{L_i} * b(x)\big|$$

where $L_i = I_i \setminus I_{i+1}, I_i = x_0 + G_i$.

But when $i < n, \operatorname{supp}\big((\psi^j)^\check{}\chi_{L_i} * b\big) \subset L_i$, so if $x \in L_i$ with $i < n$, we have

$$f^*(x) \leqslant \sum_{j=n}^{\infty}\big|(\psi^j)^\check{}\chi_{L_i} * b(x)\big|.$$

Therefore,

$$\begin{aligned}\int_{G\setminus I_n}\big|f^*(x)\big|^p d\mu_\alpha(x) &= \sum_{i=-\infty}^{n-1}\int_{I_i\setminus I_{i+1}}\big|f^*(x)\big|^p d\mu_\alpha(x)\\ &\leqslant \sum_{i=-\infty}^{n-1}\sum_{j=n}^{\infty}\int_{J_i}\big|(\psi^j)^\check{}\chi_{L_i} * b(x)\big|^p d\mu_\alpha(x)\\ &\leqslant C\sum_{i=-\infty}^{n-1}\sum_{j=n}^{\infty}\|a\|_1^p\big\|(\psi^j)^\check{}\chi_{L_i}\big\|_r^p m_i^{p/r-1}\inf\{v_\alpha(x) : x\in I_i\setminus\{0\}\}\\ &\leqslant C\sum_{i=-\infty}^{n-1}\sum_{j=n}^{\infty} m_n^{1-p}m_i^{p/r-1}\big\|(h^j)^\check{}\chi_{J_i}\big\|_r^P\\ &\leqslant C\sum_{i=-\infty}^{n-1} m_n^{1-p}m_i^{p/r-1}m_n^{p-1-p/r}m_i \leqslant C. \qquad (3)\end{aligned}$$

Substituting (2) and (3) into (1) completes the proof. ∎

REMARK 5. This result is an improvement on the weak multiplier result which Lu and Yang obtained in [**7**] under similar conditions.

When $p = 1$, we have the following result.

THEOREM 6. *Suppose $h \in L^\infty(\Gamma)$ and*

$$\sup_N\Big(\sum_{j=N+1}^{\infty}\int_{(G_N)^c}\big|(h^j)^\check{}(x)\big|d\mu(x)\Big) < \infty. \qquad (4)$$

Then h is a bounded multiplier on $H^1_\alpha(G)$ with $-1 < \alpha \leqslant 0$.

PROOF: Let T denote the multiplier operator with multiplier h. For T to be bounded on $H^1_\alpha(G)$, it is sufficient to show that there exists a constant B such that

$$\int_G \Big| (T(a))^*(x) \Big| d\mu_\alpha(x) \leqslant B < \infty$$

for all $(1,\infty)_\alpha$ atoms a. Write $f = (h\widehat{a})^\check{}$.

We assume that $\operatorname{supp}(a) \subset x_0 + G_N := I_N$, for some $x_0 \in G, N \in \mathbb{Z}$. Then we have

$$\begin{aligned} \int_G \big|f^*(x)\big| d\mu_\alpha(x) &= \int_{I_N} \big|f^*(x)\big| d\mu_\alpha(x) + \int_{(I_N)^c} \big|f^*(x)\big| d\mu_\alpha(x) \\ &:= (A) + (B). \end{aligned} \tag{5}$$

For the integral (A), we use the usual L^2 argument as in the proof of Theorem 4:

$$\begin{aligned} (A) &\leqslant \|f^*\|_2 \Big(\int_{I_N} d\mu_\alpha(x) \Big)^{1/2} \\ &\leqslant \|a\|_2 \mu(I_N)^{1/2} \inf\big\{ v_\alpha(x) : x \in I_N \setminus \{0\} \big\} \\ &\leqslant C \|a\|_\infty \mu(I_N) \inf\big\{ v_\alpha(x) : x \in I_N \setminus \{0\} \big\} \leqslant C. \end{aligned}$$

For the integral (B), we set $\psi(\gamma) = \overline{\gamma(x_0)} h(\gamma)$ and $b(x) = a(x + x_0)$. As shown in the proof of the above theorem,

$$f^*(x) \leqslant \sum_{j=N}^{\infty} \big| (\psi^j)^\check{} * b(x) \big|.$$

So

$$\begin{aligned} (B) &\leqslant \sum_{j=N}^{\infty} \int_{(I_N)^c} \big| (\psi^j)^\check{} * b(x) \big| d\mu_\alpha(x) \\ &\leqslant \sum_{j=N}^{\infty} \int_G \big| (\psi^j)^\check{}(t) \big| \int_{(I_N)^c} b(x-t) \mid d\mu_\alpha(x) d\mu(t) \\ &= \sum_{j=N}^{\infty} \int_{(I_N)^c} \big| (\psi^j)^\check{}(t) \big| \int_{(I_N)^c} \big| a(x + x_0 - t) \big| d\mu_\alpha(x) d\mu(t) \\ &\leqslant C \sum_{j=N}^{\infty} \int_{(I_N)^c} \big| (\psi^j)^\check{}(t) \big| d\mu(t) = C \sum_{j=N}^{\infty} \int_{(G_N)^c} \big| (h^j)^\check{}(t) \big| d\mu(t). \end{aligned}$$

The first equality comes from the fact that $x_0 \in I_N, x \notin I_N$, so if $t \in I_N$, then $x + x_0 - t \notin I_N$, and hence $a(x + x_0 - t) = 0$. Combining these estimates and the condition (4), we get the desired result. $\square$

3. Multipliers on Weighted Besov Spaces

3.1 Homogeneous Besov spaces. For $\alpha > 0$ and $0 < p$, $q \leqslant \infty$, there exist a large number of equivalent characterisations of the Besov or generalised Lipschitz spaces $\dot{B}_p^{\alpha,q}$ on $\mathbb{R}^n$. For these results see the book by Triebel [**16**], and for the atomic decomposition of Besov spaces on $\mathbb{R}^n$, see [**2**]. In [**8**] and [**13**] Ombe, Onneweer and Su introduced the Besov spaces on the topological groups G and presented several equivalent (quasi-)norms for these spaces. In this section we define the weighted Besov spaces $\dot{B}_p^{\alpha,q}(\omega)$ on G and give a decomposition theorem.

Definition 7: Suppose that $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$ and let ω be a nonnegative doubling weight function. The weighted homogeneous Besov space $\dot{B}_p^{\alpha,q}(\omega)$ is the collection of all $f \in \mathcal{S}'$ such that

$$\|f\|_{\dot{B}_p^{\alpha,q}(\omega)} = \left(\sum_{n=-\infty}^{\infty} \left[m_n^{\alpha}\|f * \varphi_n\|_{p,\omega}\right]^q\right)^{1/q} < \infty$$

with the usual interpretation when $q = \infty$, where $\varphi_n = \Delta_n - \Delta_{n-1}$.

The corresponding sequence space $\dot{b}_p^{\alpha,q}(\omega)$ is the collection of all $s = \{s_I\}$ such that

$$\|s\|_{\dot{b}_p^{\alpha,q}(\omega)} = \left\{\sum_{v=-\infty}^{\infty}\left(\left\|\sum_{|I|=m_v^{-1}} |I|^{-\alpha}|s_I|\widetilde{\chi}_I\right\|_{p,\omega}\right)^q\right\}^{1/q}$$

where $\widetilde{\chi}_I$ is the $L^2(\omega)$-normalised characteristic function of I.

When $\omega \equiv 1$, the basic properties of the $\dot{B}_p^{\alpha,q}$ spaces on G are described in [**8**].

Definition 8: A set of functions $\{a_I\}$ is called a family of smooth ω atoms for $\dot{B}_p^{\alpha,q}(\omega)$ if

(i) each $a_I \in \mathcal{S}$ and $\operatorname{supp}(a_I) \subset I$;

(ii) $\displaystyle\int_G a_I(x)d\mu(x) = 0$;

(iii) for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\left|a_I(x) - a_I(y)\right| \leqslant C_\delta \omega(I)^{-1/2}\left(\frac{|x-y|}{|I|}\right)^{\delta}.$$

Definition 9: Suppose that $\omega \in A_\infty$, $r = r_\omega = \inf\{s : \omega \in A_s\}$, $\alpha \geqslant 0$, and $J = \max(1, 1/p, 1/q)$. A set of functions $\{m_I\}$ is called a family of smooth ω molecules for $\dot{B}_p^{\alpha,q}(\omega)$ if there exist $\delta > \alpha$ and $M > rJ$ such that

(a) $\displaystyle\int_G m_I(x)d\mu(x) = 0$;

(b) $\left|m_I(x)\right| \leqslant \omega(I)^{-1/2}\left\{1 + \frac{|x-x_I|}{|I|}\right\}^{-\max(M,M-\alpha)}$;

$\left|m_I(x) - m_I(y)\right| \leqslant \omega(I)^{-1/2}\left(\frac{|x-y|}{|I|}\right)^{\delta} \sup_{|z|\leqslant|x-y|}\left(1 + \frac{|x-z-x_I|}{|I|}\right)^{-M}$

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where x_I is an element of I.

For each $n \in \mathbb{Z}$, $l \in \mathbb{Z}_+$, we choose elements $x_{n,l} \in G$, so that

$$\bigcup_l (x_{n,l} + G_n) = G$$

and

$$(x_{n,l} + G_n) \cap (x_{n,k} + G_n) = \phi \text{ if } k \neq l.$$

See [**13**]. We put

$$\varphi_I(x) = \omega(I)^{1/2}\varphi_n(x - x_{n,l})$$

and

$$\psi_I(x) = \omega(I)^{-1/2}|I|\varphi_n(x - x_{n,l}) \text{ for } I = x_{n,l} + G_n.$$

It is easily seen that $\{\psi_I\}$ is a family of smooth ω atoms, hence a family of smooth ω molecules.

THEOREM 10. *Suppose that $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$. Then each $f \in \dot{B}_p^{\alpha,q}(\omega)$ can be decomposed as follows:*

$$f = \sum_I \langle f, \varphi_I \rangle \psi_I$$

with convergence in S', where $\|s\|_{\dot{b}_p^{\alpha,q}(\omega)} \leqslant C\|f\|_{\dot{B}_p^{\alpha,q}(\omega)}$. Here $s = \{s_I\} = \{\langle f, \varphi_I \rangle\}$, and C is independent of f.

PROOF:

$$\begin{aligned}
f(x) &= \sum_{n \in \mathbf{Z}} f * \varphi_n(x) \text{ in } S' \\
&= \sum_{n \in \mathbf{Z}} f * \varphi_n * \varphi_n(x) \\
&= \sum_{n \in \mathbf{Z}} \int_G (f * \varphi_n)(y)\varphi_n(x - y)d\mu(y) \\
&= \sum_{n \in \mathbf{Z}} \sum_{l=0}^{\infty} (f * \varphi_n)(x_{n,l})\varphi_n(x - x_{n,l})m_n^{-1} \\
&= \sum_{n \in \mathbf{Z}} \sum_{l=0}^{\infty} (f * \varphi_n)(x_{n,l})\omega(I)^{1/2}\psi_I(x) = \sum_I \langle f, \varphi_I \rangle \psi_I(x)
\end{aligned}$$

where $I = x_{n,l} + G_n$. Moreover

$$\begin{aligned}
\|s\|_{\dot{b}_p^{\alpha,q}(\omega)} &= \left\{ \sum_{n \in \mathbf{Z}} \left(\left\| \sum_{l=0}^{\infty} m_n^{\alpha} (f * \varphi_n)(x_{n,l})\chi_I \right\|_{L^p(\omega)} \right)^q \right\}^{1/q} \\
&= \left\{ \sum_{n \in \mathbf{Z}} m_n^{q\alpha} \left(\sum_{l=0}^{\infty} \int_{I_{n,l}} \left| f * \varphi_n(x_{n,l}) \right|^p \omega(x) d\mu(x) \right)^{q/p} \right\}^{1/q}
\end{aligned}$$

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$$= \left\{ \sum_{n \in \mathbb{Z}} m_n^{q\alpha} \|f * \varphi_n\|_{L^p(\omega)}^q \right\}^{1/q} = \|f\|_{\dot{B}_p^{\alpha,q}(\omega)}. \tag{6}$$ $\square$

The next theorem is the converse of the decomposition in Theorem 10.

THEOREM 11. *Suppose that $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$. If $s = \{s_I\} \in \dot{b}_p^{\alpha,q}(\omega)$ and $\{m_I\}$ is a family of smooth molecules for $\dot{B}_p^{\alpha,q}(\omega)$, and if $f = \sum_{u \in \mathbb{Z}} \sum_{|I| = m_u^{-1}} s_I m_I$, then*

$$\|f\|_{\dot{B}_p^{\alpha,q}(\omega)} \leqslant C \|s\|_{\dot{b}_p^{\alpha,q}(\omega)}$$

where C is independent of s.

To prove the theorem, we need the following lemma.

LEMMA 12. (See [6].)

(a) *Let $a > 1, x_0 \in G$ and $u \geqslant \nu$ with $u, \nu \in \mathbb{Z}$. Suppose that $h : G \to \mathbb{C}$ satisfies $\int_G h(x) d\mu(x) = 0$ and $|h(x)| \leqslant \left(1 + m_u|x - x_0|\right)^{-a}$. Then*

$$|\varphi_\nu * h(x)| \leqslant C \left(\frac{m_\nu}{m_u}\right)^a \left(1 + m_\nu|x - x_0|\right)^{-a}$$

where C is independent of u, ν, x_0 and x.

(b) *Let $a, b > 1$, $\delta > 0$, $x_0 \in G$ and $u \leqslant \nu$ with $u, \nu \in \mathbb{Z}$. Suppose that $h : G \to \mathbb{C}$ satisfies*

$$|h(x)| \leqslant \left(1 + m_u|x - x_0|\right)^{-a},$$

and

$$|h(x) - h(y)| \leqslant \left(m_u|x - y|\right)^\delta \sup_{|z| \leqslant |x-y|} \left(1 + m_u|x - z - x_0|\right)^{-b}.$$

Then

$$|\varphi_\nu * h(x)| \leqslant C \left(\frac{m_u}{m_\nu}\right)^\delta \left(1 + m_u|x - x_0|\right)^{-b}$$

with C independent of u, ν, x_0 and x.

PROOF:

$$\varphi_\nu * \left(\sum_{u \in \mathbb{Z}} \sum_{|I| = m_u^{-1}} s_I m_I \right) = \varphi_\nu * \left(\sum_{u=-\infty}^{\nu} + \sum_{u=\nu+1}^{\infty} \right) \sum_{|I| = m_u^{-1}} s_I m_I.$$

$$\begin{aligned} \|f\|_{\dot{B}_p^{\alpha,q}(\omega)}^q \leqslant \sum_{\nu \in \mathbb{Z}} m_\nu^{p\alpha} \Bigg(& \left\| \sum_{u=-\infty}^{\nu} \sum_{|I| = m_u^{-1}} \varphi_\nu * s_I m_I \right\|_{L^p(\omega)}^q \\ & + \left\| \sum_{u=\nu+1}^{\infty} \sum_{|I| = m_u^{-1}} \varphi_\nu * s_I m_I \right\|_{L^p(\omega)}^q \Bigg) := I_1 + I_2. \end{aligned}$$

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For I_1, applying Lemma 12 (b) and [**6**, Lemma 3], we obtain

$$I_1 \leqslant \sum_{\nu\in\mathbf{Z}} m_\nu^{q\alpha} \Big\| \sum_{u=-\infty}^{\nu} \sum_{|I|=m_u^{-1}} \omega(I)^{-1/2}|s_I| \Big(\frac{m_u}{m_\nu}\Big)^{\delta'} \big(1+m_u|x-x_I|\big)^{-M} \Big\|_{L^p(\omega)}^q$$

$$\leqslant C \sum_{\nu\in\mathbf{Z}} m_\nu^{q\alpha} \Big\| \sum_{u=-\infty}^{\nu} \Big(\frac{m_u}{m_\nu}\Big)^{\delta'} \Big[M_\omega \Big(\sum_{|I|=m_u^{-1}} |s_I|\widetilde{\chi}_I \Big)^A (x) \Big]^{1/A} \Big\|_{L^p(\omega)}^q,$$

where $0 < A \leqslant 1$, and $\delta' = \delta$ if $\alpha \geqslant 0$, while $\delta' = 0$ if $\alpha < 0$.

If $0 < p \leqslant 1$, and $A \in (0,1]$ is chosen such that $p/A > 1$, then

$$I_1 \leqslant C \sum_{\nu\in\mathbf{Z}} m_\nu^{q\alpha} \Big\{ \sum_{u=-\infty}^{\nu} \Big(\frac{m_u}{m_\nu}\Big)^{p\delta'} \Big\| \Big[M_\omega \sum_{|I|=m_u^{-1}} |s_I|\widetilde{\chi}_I(x) \Big]^{1/A} \Big\|_{L^p(\omega)}^p \Big\}^{q/p}$$

$$\leqslant C \sum_{\nu\in\mathbf{Z}} m_\nu^{q\alpha} \Big\{ \sum_{u=-\infty}^{\nu} \Big(\frac{m_u}{m_\nu}\Big)^{p\delta'} \Big\| \sum_{|I|=m_u^{-1}} |s_I|\widetilde{\chi}_I(x) \Big\|_{L^p(\omega)}^p \Big\}^{q/p}.$$

(i) When $q/p \leqslant 1$

$$\begin{aligned}
I_1 &\leqslant C \sum_{\nu\in\mathbf{Z}} m_\nu^{q\alpha} \sum_{u=-\infty}^{\nu} \Big(\frac{m_u}{m_\nu}\Big)^{q\delta'} \Big\| \sum_{|I|=m_u^{-1}} |s_I|\widetilde{\chi}_I(x) \Big\|_{L^p(\omega)}^q \\
&= \sum_{u\in\mathbf{Z}} m_u^{q\delta'} \sum_{\nu=u}^{\infty} m_\nu^{(\alpha-\delta')q} \Big\| \sum_{|I|=m_u^{-1}} |s_I|\widetilde{\chi}_I(x) \Big\|_{L^p(\omega)}^q \\
&\leqslant \sum_{u\in\mathbf{Z}} m_u^{q\delta'} m_u^{(\alpha-\delta')q} \Big\| \sum_{|I|=m_u^{-1}} |s_I|\widetilde{\chi}_I(x) \Big\|_{L^p(\omega)}^q \\
&= \sum_{u\in\mathbf{Z}} \Big(\Big\| \sum_{|I|=m_u^{-1}} |I|^{-\alpha}|s_I|\widetilde{\chi}_I(x) \Big\|_{L^p(\omega)} \Big)^q \\
&= \|s\|_{\dot{b}_p^{\alpha,q}(\omega)}.
\end{aligned}$$

(ii) When $q/p > 1$, let $s = q/p$

$$I_1 = C \sum_{\nu\in\mathbf{Z}} m_\nu^{\alpha q/2} \Big\{ \sum_{u=-\infty}^{\nu} \Big(\frac{m_u}{m_\nu}\Big)^{\delta' p/2} \Big(\frac{m_u}{m_\nu}\Big)^{-\alpha p/2} \Big(\frac{m_u}{m_\nu}\Big)^{\delta' p/2} m_u^{\alpha p/2} \Big\| \sum_{|I|=m_u^{-1}} |s_I|\widetilde{\chi}_I \Big\|_{L^p(\omega)}^p \Big\}^{q/p}$$

$$\leqslant C \sum_{\nu\in\mathbf{Z}} m_\nu^{\alpha q/2} \Big\{ \sum_{u=-\infty}^{\nu} \Big(\frac{m_u}{m_\nu}\Big)^{(\delta'-\alpha)ps'/2} \Big\}^{s/s'} \Big\{ \sum_{u=-\infty}^{\nu} \Big(\frac{m_u}{m_\nu}\Big)^{\delta' q/2} m_u^{\alpha q/2} \Big\| \sum_{|I|=m_u^{-1}} |s_I|\widetilde{\chi}_I \Big\|_{L^p(\omega)}^q \Big\}$$

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$$\leqslant C\sum_{\nu\in\mathbf{Z}} m_\nu^{\alpha q/2}\sum_{u=-\infty}^{\nu}\left(\frac{m_u}{m_\nu}\right)^{q\delta'/2} m_u^{-q\alpha/2}\Big\|\sum_{|I|=m_u^{-1}}|s_I|\widetilde{\chi}_I\Big\|_{L^p(\omega)}^q$$
$$= C\sum_{u\in\mathbf{Z}}\sum_{\nu=u}^{\infty}\left(\frac{m_u}{m_\nu}\right)^{(\delta'-\alpha)q/2}\Big\|\sum_{|I|=m_u^{-1}}|I|^{-\alpha}|s_I|\widetilde{\chi}_I\Big\|_{L^p(\omega)}^q$$
$$\leqslant C\sum_{u\in\mathbf{Z}}\Big\|\sum_{|I|=m_u^{-1}}|I|^{-\alpha}|s_I|\widetilde{\chi}_I\Big\|_{L^p(\omega)}^q = C\|s\|^q_{\dot b_p^{\alpha,q}(\omega)}.$$

If $p > 1$,

$$I_1 \leqslant \sum_{\nu\in\mathbf{Z}} m_\nu^{\alpha q}\left(\sum_{u=-\infty}^{\nu}\left(\frac{m_u}{m_\nu}\right)^{\delta'}\Big\|\Big[M_\omega\Big(\sum_{|I|=m_u^{-1}}|s_I|\widetilde{\chi}_I\Big)A(x)\Big]^{1/A}\Big\|_{L^p(\omega)}\right)^q$$
$$\leqslant \sum_{\nu\in\mathbf{Z}} m_\nu^{\alpha q}\left(\sum_{u=-\infty}^{\nu}\left(\frac{m_u}{m_\nu}\right)^{\delta'}\Big\|\sum_{|I|=m_u^{-1}}|s_I|\widetilde{\chi}_I\Big\|_{L^p(\omega)}\right)^q.$$

(i′) When $0 < q \leqslant 1$

$$I_1 \leqslant \sum_{\nu\in\mathbf{Z}} m_\nu^{\alpha q}\sum_{u=-\infty}^{\nu}\left(\frac{m_u}{m_\nu}\right)^{\delta' q}\Big\|\sum_{|I|=m_u^{-1}}|s_I|\widetilde{\chi}_I\Big\|_{L^p(\omega)}^q$$
$$= \sum_{u\in\mathbf{Z}}\sum_{\nu=u}^{\infty}\left(\frac{m_u}{m_\nu}\right)^{(\delta'-\alpha)q}\Big\|\sum_{|I|=m_u^{-1}}|I|^{-\alpha}|s_I|\widetilde{\chi}_I\Big\|_{L^p(\omega)}^q$$
$$\leqslant C\sum_{u\in\mathbf{Z}}\left(\Big\|\sum_{|I|=m_u^{-1}}|I|^{-\alpha}|s_I|\widetilde{\chi}_I\Big\|_{L^p(\omega)}\right)^q = \|s\|^q_{\dot b_p^{\alpha,q}(\omega)}.$$

(ii′) When $q > 1$, the proof is essentially the same as the case (ii), so we omit the details.

For the estimate of I_2, we use Lemma 12 (a) and obtain

$$I_2 = \sum_{\nu\in\mathbf{Z}} m_\nu^{q\alpha}\Big\|\sum_{u=\nu+1}^{\infty}\sum_{|I|=m_u^{-1}}\varphi_\nu * s_I m_I\Big\|_{L^p(\omega)}^q$$
$$\leqslant \sum_{\nu\in\mathbf{Z}} m_\nu^{q\alpha}\Big\|\sum_{u=\nu+1}^{\infty}\sum_{|I|=m_u^{-1}}|s_I|\omega(I)^{-1/2}\left(\frac{m_\nu}{m_u}\right)^{a}(1+m_\nu|x-x_I|)^{-a}\Big\|_{L^p(\omega)}^q$$
$$\leqslant C\sum_{\nu\in\mathbf{Z}}\Big\|\sum_{u=\nu+1}^{\infty}\left(\frac{m_u}{m_\nu}\right)^{r/A}\left(\frac{m_\nu}{m_u}\right)^{a}\Big[M_\omega\Big(\sum_{|I|=m_u^{-1}}|s_I|\widetilde{\chi}_I\Big)^A(x)\Big]^{1/A}\Big\|_{L^p(\omega)}^q$$
$$= C\sum_{\nu\in\mathbf{Z}} m_\nu^{q\alpha}\Big\|\sum_{u=\nu+1}^{\infty}\left(\frac{m_\nu}{m_u}\right)^{a-r/A}\Big[M_\omega\Big(\sum_{|I|=m_u^{-1}}|s_I|\widetilde{\chi}_I\Big)^A(x)\Big]^{1/A}\Big\|_{L^p(\omega)}^q, \tag{7}$$

where we have taken $a = \max\{M, M - \alpha\}$, M is the constant in the definition of a molecule, and we choose $A \in (0,1)$, such that $a > r/A$. Then we can get $I_2 \leqslant C\|s\|^q_{\dot{b}^{\alpha,q}_p}$ in a similar way to that used in the estimate of I_1. ∎

3.2 Multiplier Theorem. We are now in a position to prove a boundedness result for certain multipliers on weighted Besov spaces, by applying Theorems 10 and 11.

Theorem 13. *Suppose that $\omega \in A_\infty, \alpha \in \mathbb{R}, 0 < p < \infty$ and $0 < q \leqslant \infty$. Also suppose that $h \in L^\infty(\Gamma)$ satisfies the following properties.*

For some $r \in [1,\infty)$, there exist $C > 0$ and $\varepsilon > 1$ such that for all $n \in \mathbb{Z}, l < n$

$$\Big(\int_{\Gamma_{n+1}\backslash\Gamma_n} |h(\xi)|^r d\gamma(\xi)\Big)^{1/r} \leqslant C m_n^{1/r},$$

and

$$\sup_{z\in\Gamma_l}\Big(\int_{\Gamma_{n+1}\backslash\Gamma_n} |h(\xi) - h(\xi - z)|^r d\gamma(\xi)\Big)^{1/r} \leqslant C m_n^{1/r-\varepsilon} m_l^\varepsilon.$$

Then h is a bounded multiplier on $\dot{B}^{\alpha,q}_p(\omega)$.

Proof: We write

$$Tf(x) = (h\widehat{f})^\vee(x) = \int_\Gamma h(\xi)\widehat{f}(\xi)(\xi,x)d\gamma(\xi).$$

It is sufficient to show that up to a constant, $\{T\psi_I\}$ is a family of smooth molecules for $\dot{B}^{\alpha,q}_p(\omega)$. We shall prove that there exist $\delta > \alpha$ and $M > rJ$ such that

(i) $\quad |T\psi_I(x)| \leqslant C\omega(I)^{-1/2}\Big(1 + \dfrac{|x - x_I|}{|I|}\Big)^{-M},$

(ii) $\quad |T\psi_I(x) - T\psi_I(y)| \leqslant C\omega(I)^{-1/2}\Big(\dfrac{|x-y|}{|I|}\Big)^\delta \sup\limits_{|z|\leqslant|x-y|}\Big(1 + \dfrac{|x - z - x_I|}{|I|}\Big)^{-M}$

where C is independent of I.

For $I = x_I + G_n$, we have

$$\begin{aligned} T\psi_I(x) &= \int_\Gamma (\xi,x)h(\xi)\widehat{\psi}_I(\xi)d\gamma(\xi) \\ &= |I|\omega(I)^{-1/2}\int_{\Gamma_n\backslash\Gamma_{n-1}} (\xi, x - x_I)h(\xi)d\gamma(\xi). \end{aligned}$$

If $x \in I$, then

$$|T\psi_I(x)| \leqslant |I|\omega(I)^{-1/2}\int_{\Gamma_n\backslash\Gamma_{n-1}} |h(\xi)|d\gamma(\xi) \leqslant C\omega(I)^{-1/2}.$$

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If $x \notin I$, we take $z \in \Gamma_l \setminus \Gamma_{l-1}, \forall l < n$,

$$\begin{aligned}
\left|(z, x - x_I) - 1\right|\left|T\psi_I(x)\right| &= |I|\omega(I)^{-1/2}\left|\int_{\Gamma_n \setminus \Gamma_{n-1}} \left[(z + \xi, x - x_I) - (\xi, x - x_I)\right] h(\xi) d\gamma(\xi)\right| \\
&= |I|\omega(I)^{-1/2}\left|\int_{\Gamma_n \setminus \Gamma_{n-1}} \left[h(\xi - z) - h(\xi)\right](\xi, x - x_I) d\gamma(\xi)\right| \\
&\leqslant |I|\omega(I)^{-1/2} \int_{\Gamma_n \setminus \Gamma_{n-1}} \left|h(\xi - z) - h(z)\right| d\gamma(\xi) \\
&\leqslant C|I|^{\varepsilon}\omega(I)^{-1/2} m_l^{\varepsilon} = C|I|^{\varepsilon}\omega(I)^{-1/2}|z|^{\varepsilon}.
\end{aligned}$$

Hence

$$\left|T\psi_I(x)\right| \frac{\left|(z, x - x_I) - 1\right|}{|z|^{\varepsilon}} \leqslant C|I|^{\varepsilon}\omega(I)^{-1/2}.$$

Integrating both sides of this inequality over Γ_{n-1}, we have by [**14**, Lemma (II.3)] that

$$|x - x_I|^{\varepsilon - 1}\left|T\psi_I(x)\right| \leqslant C|I|^{\varepsilon - 1}\omega(I)^{-1/2}.$$

So for all $x \in G$ we have

$$\left|T\psi_I(x)\right| \leqslant C\omega(I)^{-1/2}\left(1 + \frac{|x - x_I|}{|I|}\right)^{-\varepsilon + 1}.$$

If $|x - y| \geqslant |I|$, (ii) follows from (i), so it is sufficient to show (ii) for $|x - y| < |I|$.

$$\begin{aligned}
T\psi_I(x) - T\psi_I(y) &= |I|\omega(I)^{-1/2} \int_{\Gamma_n \setminus \Gamma_{n-1}} \left[(\xi, x - x_I) - (\xi, y - x_I)\right] h(\xi) d\gamma(\xi) \\
&= |I|\omega(I)^{-1/2}\Big[\int_{\Gamma_n \setminus \Gamma_{n-1}} (\xi, x - x_I) h(\xi) d\gamma(\xi) \\
&\qquad - \int_{\Gamma_n \setminus \Gamma_{n-1}} (\xi, y - x)(\xi, x - x_I) h(\xi) d\gamma(\xi)\Big] \\
&= 0.
\end{aligned}$$

The last equality comes from the fact that $\xi \in \Gamma_n \setminus \Gamma_{n-1}$, $|x - y| < |I|$, so $x - y \in G_n$ and $(\xi, x - y) = 1$. ∎

REMARK 14. In [**18**] the author proved that a pseudo-differential operator with a kernel in S_ρ^m (defined in [**13**]) satisfies the condition given in Theorem 13, so it is bounded on weighted Besov spaces.

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