CORRECTION TO "ARITHMETIC LINEAR TRANSFORMATIONS"

S. A. AMITSUR

The elementary proof of the prime number theorem and its generalizations (**2**) to primes in arithmetic progressions and to generalized primes in the sense of Beurling (**3**) were based on the following (**2**, Theorem 10) lemma:

(L) *Let $g(n)$ be a non-negative real function, $M > 0$ and*

$$\sum_{n \leqq x} g(n) = Mx \log^m x + o(x \log^m x), \qquad m > 1,$$

and let $h(x)$ be a real- (complex)-valued function satisfying:
(h1) $h(x) = O(1)$;
(h2) $\sum_{\nu \leqq x} \nu^{-1} h(\nu) = O(1)$;
(h3) $h(tx) - h(x) = o(1)$ *as* $(t, x) \to (1, \infty)$;
then $h(x) = o(1)$ *if the following holds:*

$$|h(x)| \log^{m+1} x \leqq M^{-1}(m+1) \sum_{n \leqq x} n^{-1} g(n) |h(n^{-1}x)| + o(\log^{m+1} x).$$

The proof of this result is based on the fact that this function $h(x)$ has the following property:

(P) *For every* $\Delta > 0$, *there exists* x_Δ, $T > 1$, *such that every interval* $[x, Tx]$, $x \geqq x_\Delta$, *contains a point* u_0 *for which* $|h(u_0)| < \Delta$.

It was pointed out correctly by Ahern (**1**) that the result (L) is false in this generality for an arbitrary *complex*-valued function $h(x)$. The reason for this failure is that the proof of (P) depends on the fact that $h(x)$ is real. Thus, the proofs of the prime number theorem and the density of the generalized primes (**3**), where (P) was applied for real $h(x)$, remain valid as they stand. But the proofs of the main theorems (**2**, Theorems A and B) are affected; the other results, like (**2**, Theorems 11 and 13), will remain valid for real functions.

By passing from complex functions to its real and imaginary parts, it is easy to verify (P) for complex-valued functions $h(x)$ whose values lie in a cone with zero as centre. This is not sufficient to correct the results of (**2**). Theorems A and B of (**2**) are based on Theorem 12 of (**2**), which remains valid for real characters f and we are unable to provide a proof for arbitrary complex characters f; but a corrected proof is given here for Theorem 12 for complex characters of the type appearing in Theorems A and B, and thus the main results of (**2**) are valid.

Received November 21, 1966 and in revised form May 10, 1968.

https://doi.org/10.4153/CJM-1969-001-0 Published online by Cambridge University Press

2. We adopt, henceforth, the notation of (**2**) and we reprove its Theorem 12 in the following form.

THEOREM. *Let f be a character on the multiplicative semigroup W such that $f(\omega)$ is an nth root of unity for all $\omega \in W$ and such that:*

$$(1)\quad S_{f^r}1 = \sum_{N\omega \leqq x} f^r(\omega) = a_r x + O(x \log^{-\gamma} x), \qquad r = 0, 1, 2, \dots, n-1,$$

then

$$(2)\qquad S_{f^r\Lambda}1 = \sum_{N\omega \leqq x} f^r(\omega)\Lambda(\omega) = \begin{cases} x + o(x) & \textit{in Case† I} \quad \textit{for } \gamma > 2, \\ o(x) & \textit{in Case II} \quad \textit{for } \gamma > 2, \\ -x + o(x) & \textit{in Case III} \quad \textit{for } \gamma > 3. \end{cases}$$

Proof. Let ζ be a fixed primitive nth root of unity, and let $e_k(\omega)$ (for $k = 0, 1, \dots, n-1$) be the functions defined on W by:

$$e_k(\omega) = 1 \quad \text{if} \quad f(\omega) = \zeta^k \quad \text{and} \quad e_k(\omega) = 0 \quad \text{otherwise.}$$

These functions are real-valued and satisfy

$$(3)\qquad f^r = \sum_{k=0}^{n-1} \zeta^{rk} e_k, \qquad e_k = \frac{1}{n} \sum_{r=0}^{n-1} \zeta^{-rk} f^r.$$

Let $h_r(x) = x^{-1}(S_{f^r\Lambda}1) - \sigma_r = x^{-1}[\sum f^r(\omega)\Lambda(\omega)] - \sigma_r$, where $\sigma_r = 1, 0, -1$, respectively, for the three different cases of our theorem. First we prove that

$$(4)\qquad I_{f^r\Lambda_2} h_r(x) = o(\log^2 x).$$

This fact is true for all characters dealt with in (2); therefore, we shall prove it for f and we shall drop the subscript r and it will hold for all f^r.

$$I_{f\Lambda_2} h(x) = I_{f\Lambda_2}[x^{-1}S_{f\Lambda}1 - \sigma] = x^{-1}S_{f\Lambda_2}S_{f\Lambda}1 - \sigma I_{f\Lambda_2}1 =$$
$$x^{-1}S_{f\Lambda}(S_{f\Lambda_2}1) - \sigma I_{f\Lambda_2} = I_{f\Lambda}(x^{-1}S_{f\Lambda_2}1) - \sigma_r I_{f\Lambda_2}1$$

and, as in the proof of (7.1) of (**2**, p. 98), we obtain, in view of (**2**, (6.5b) and Theorem 8, p. 94):

$$\begin{aligned} x^{-1}S_{f\Lambda_2}1 &= I_{f\mu}(x^{-1}S_{fL^2}1) \\ &= I_{f\mu}[a \log^2 x - 2a \log x + 2a + O(\log^{1-\delta} x)] \\ &= 2 \log x + O(1) + O(\log^{2-\delta} x) \quad \text{if } a = a_r \neq 0 \\ &= O(\log^{2-\delta} x) \quad \text{if } a = 0 \text{ in Case II or Case III.} \end{aligned}$$

Thus, in Case I: $\sigma = 1$, $I_{f\Lambda}(2 \log x) = \log^2 x + o(\log^2 x)$, $I_{f\Lambda_2}1 = \log^2 x + o(\log^2 x)$, and $I_{f\Lambda} o(\log x) = o(\log^2 x)$ from which our assertion (4) follows.

†The different cases are listed in (**2**, p. 94).

https://doi.org/10.4153/CJM-1969-001-0 Published online by Cambridge University Press

The proof of Case II is immediate (and in fact, this has already been settled in the proof of (**2**, Theorem 11)). In Case III: $I_{f\Lambda}O(\log^{2-\delta}x) = O(\log^{3-\delta}x)$ and $I_{f\Lambda_2}1 = b\log x + O(\log^{3-\delta}x)$ for some b and (2.4) follows by applying (**2**, Theorem 9).

Next, as in (**2**, p. 102), we have that $I_f h = c + o(\log^{-\delta}x)$ for some c and also $(I_{f\mu}\log x\, I_f)h = [\log x + I_{f\Lambda}]h = o(\log x)$. Applying $\log x - I_{f\Lambda}$ on both sides of the last equation, we obtain

$$(\log x - I_{f\Lambda})(\log x + I_{f\Lambda})h = (\log x + I_{f\Lambda})o(\log x) = o(\log^2 x)$$

but $(\log x - I_{f\Lambda})(\log x + I_{f\Lambda})h = (\log^2 x + \log x I_{f\Lambda} - I_{f\Lambda}\log x - I_{f\Lambda^2})h.$

Noting that $(\log x)I_{f\Lambda} - I_{f\Lambda}\log x = I_{f\Lambda L}$ and $I_{f\Lambda L} + I_{f\Lambda^2} = I_{f\Lambda_2}$, and substituting these in the last relation, we obtain, in view of (2.4), the following fundamental two relations which hold for all f^r:

$$(5)\qquad (\log^2 x + 2I_{f^r\Lambda L})h_r = I_{f^r\Lambda_2}h_r + o(\log^2 x) = o(\log^2 x),$$

$$(6)\qquad (\log^2 x + 2I_{f^r\Lambda^2})h_r = -I_{f^r\Lambda_2}h_r + o(\log^2 x) = o(\log^2 x).$$

Let

$$H_k = \frac{1}{n}\sum_{r=0}^{n-1}\zeta^{-rk}h_r;$$

then

$$h_r = \sum_{k=0}^{n-1}\zeta^{rk}H_k.$$

Multiply (2.5) by ζ^{-rk} and sum over all $0 \leqq r \leqq n-1$ and obtain (in view of (3) and the last relation):

$$\begin{aligned} o(\log^2 x) &= H_k(x)\log^2 x + \frac{2}{n}\sum_{r=0}^{n-1}\zeta^{-rk}I_{f^r\Lambda L}h_r \\ &= H_k\log^2 x + \frac{2}{n}\sum_{r=0}^{n-1}\zeta^{-rk}\sum_{h=0}^{n-1}\zeta^{rh}I_{e_h\Lambda L}\sum_{j=0}^{n-1}\zeta^{rj}H_j \\ &= H_k\log^2 x + \frac{2}{n}\sum_{r=0}^{n-1}\sum_{h=0}^{n-1}\sum_{j=0}^{n-1}\zeta^{-rk+rh+rj}I_{e_h\Lambda L}H_j. \end{aligned}$$

The sum over all r is zero unless $h + j - k \equiv 0 \pmod n$, where the sum is equal to n; we then obtain

$$-H_k(x)\log^2 x = 2\sum_{k\equiv h+j(n)} I_{e_h\Lambda L}H_j + o(\log^2 x),$$

and it follows that for the real part of H_j we have (noting that $[e_h\Lambda L](\omega) \geqq 0$) that

$$|\operatorname{Real} H_k(x)|\log^2 x \leqq 2\sum_{k\equiv h+j(n)} I_{e_h\Lambda L}|\operatorname{Real} H_j| + o(\log^2 x).$$

Similarly, we obtain from (6)

$$|\operatorname{Real} H_k(x)|\log^2 x \leqq 2\sum_{k\equiv h+j(n)} I_{e_h\Lambda^2}|\operatorname{Real} H_j| + o(\log^2 x).$$

https://doi.org/10.4153/CJM-1969-001-0 Published online by Cambridge University Press

Adding these two and noting that $e_h\Lambda L + e_h\Lambda^2 = e_h\Lambda_2$, we have that

$$(7) \qquad |\operatorname{Real} H_k(x)| \log^2 x \leqq \sum_{h+j\equiv k(n)} I_{e_h\Lambda_2}|\operatorname{Real} H_j| + o(\log^2 x).$$

It follows from (3) and (1) that

$$I_{e_h\Lambda_2}1 = \frac{1}{n}\sum \zeta^{-rh} I_{f^r\Lambda_2}1 = b_h \log^2 x + o(\log^2 x)$$

for some b_h, and since $\sum_h e_h\Lambda_2 = \Lambda_2$ and $I_{\Lambda_2}1 = \log^2 x + o(\log^2 x)$, it follows that $\sum b_h = 1$. Furthermore, since $e_h\Lambda_2$ is non-negative, we must have that $b_h \geqq 0$.

Finally, in (**2**, p. 102) we have shown that h_r satisfies (h1)–(h3) of the quoted result (L); hence, Real H_k also satisfy these conditions.

Let $\limsup|\operatorname{Real} H_j| = A_j < \infty$ and let $A_k = \operatorname{Max} A_j$. Hence,

$$\begin{aligned}\log^2 x|\operatorname{Real} H_k| &\leqq \sum_{h+j\equiv k(n)} (A_j + \epsilon) I_{e_h\Lambda_2}1 + o(\log^2 x) \\ &= \sum_{h+j\equiv k(n)} (A_j + \epsilon) b_h \log^2 x + o(\log^2 x)\end{aligned}$$

and by dividing by $\log^2 x$ and as $x \to \infty$ we obtain $A_k \leqq \sum'(A_j + \epsilon)b_h$, where the sum ranges over all $h + j \equiv k(n)$ for which $b_h \neq 0$. Thus, for these we obtain $A_k \leqq \operatorname{Max} A_j$. Thus, $\operatorname{Max} A_j$ is obtained for some index (say j_0) for which $b_{j_0} \neq 0$. Then (7) now yields:

$$|\operatorname{Real} H_k| \log^2 x \leqq \sum_{j\neq j_0} I_{e_h\Lambda_2}|\operatorname{Real} h_j| + I_{e_{h_0}\Lambda_2}|\operatorname{Real} H_{j_0}| + o(\log^2 x),$$

where $h + j \equiv k(n)$ and $h_0 + j_0 \equiv k(n)$. Divide by $\log^2 x$ and, as $x \to \infty$, the left-hand side tends to A_k and the first term of the right-hand side will be $\leqq \sum_{h\neq h_0} b_h(A_k + \epsilon) = (1 - b_{h_0})(A_k + \epsilon)$.

To compute the second term of the right-hand side we can follow the proof of (L) as given in (**2**, p. 100), in view of the fact that Real H_{j_0} satisfies (h1)–(h3), and obtain that if $A_{h_0} \neq 0$, it is less than $b_{h_0}[A_{h_0} + \epsilon + \eta]$ for some $\eta < 0$. Thus, as $\epsilon \to 0$ we have a contradiction since $b_{h_0} \neq 0$ and $A_{h_0} = A_k$. Hence, $A_{h_0} = 0$, and therefore all $A_j = 0$. Thus, Real $H_k = o(1)$, and similarly one proves that $\operatorname{Im} H_k = o(1)$, and consequently $H_k(x) = o(1)$. The remainder follows since $h_r(x) = \sum \zeta^{rk} H_k = o(1)$ for all r.

5. We use this opportunity to point out some misprints in (**2**; **3**).

In (**2**):

p. 93: in the two displays after Theorem 6, $O(\log^{\delta-1} x)$ should read $O(\log^{\nu-\delta} x)$;

p. 97: line 9, $O(\sum_{j=0}^{n+p} |R; (f^{-1})|)$ should read $O(\sum_{j=0}^{n+p} |R_j(f^{-1})|)$;

p. 99: delete the words "a complex" in Theorem 10;

p. 101: an absolute value sign should be added after $\sigma \log x$;

p. 102: line 3, $x + o(w)$ should read $x + o(x)$;

https://doi.org/10.4153/CJM-1969-001-0 Published online by Cambridge University Press

p. 102: line 17 should read as follows:

$$|\log x - I_{|f|\Lambda}|\ |h(x)| \leqq |(\log x + I_{f\Lambda})h(x)| = |I_{f\mu} \log x I_f h(x)| = o(\log x);$$

lines 19 and 20: Λ should be added after $|f|$, i.e., the line should read as follows:

$$(\log^2 x - I_{|f|\Lambda_2})|h(x)| = (\log x + I_{|f|\Lambda})(\log x - I_{|f|\Lambda})|h(x)| \leqq$$
$$(\log x + I_{|f|\Lambda})o(\log x) = o(\log^2 x);$$

p. 103: line 9, $O(\sum_{j=0}^{n+p-1} |R;\ (x; f\Lambda, -F^{-1}F')|)$ should read

$$O(\sum_{j=0}^{n+p-1} |R_j(x; f\Lambda, -F^{-1}F')|);$$

p. 105: line 3,

$$h^{-1}\left[\sum_{x\in\Gamma_1} \chi(H) - \sum_{x\in\Gamma_3} \chi(H)\right]x$$

should read

$$h^{-1}\left[\sum_{\chi\in\Gamma_1} \chi(H) - \sum_{\chi\in\Gamma_3} \chi(H)\right]x;$$

p. 108, line 29 and p. 109, line 3: replace $c + o(1)$ by $O(1)$ in order that the proof of § 10 be valid.

In (**3**):

p. 104: line 24, the definition of X_n should be $X_n = \inf\{x|\ h(t) \leqq n, t \leqq x\}$;

p. 105: line 13, in Theorem 4, read $\sum_{\nu\leqq x}(h(\nu)/\nu) = O(1)$;

In Proposition 1, put "real" instead of "complex";

p. 108, line 7, replace "complex" by "real or roots of unity of a fixed order"; line 10 should have $y_{ni} < x$.

References

1. R. P. Ahern, *Elementary methods in the theory of primes*, Trans. Amer. Math. Soc. *118* (1965), 221–242.

2. S. A. Amitsur, *Arithmetic linear transformations and abstract prime number theorems*, Can. J. Math. *13* (1961), 83–109.

3. ——— *On a lemma in elementary proofs*, Bull. Res. Council Israel Sect. F *10* (1962), 101–108.

4. W. Forman and H. N. Shapiro, *Abstract prime number theorems*, Comm. Pure Appl. Math. *7* (1954), 587–619.

The Hebrew University of Jerusalem,
Jerusalem, Israel

https://doi.org/10.4153/CJM-1969-001-0 Published online by Cambridge University Press