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A RENORMING THEOREM FOR DUAL SPACES

A. C. YORKE

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Abstract

If the second dual of a Banach space E is smooth at each point of a certain norm dense subset, then its first dual admits a long sequence of norm one projections, and these projections have ranges which are suitable for a transfinite induction argument. This leads to the construction of an equivalent locally uniformly rotund norm and a Markuschevich basis for E^*.

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1. Preliminaries

Let E be a real Banach space, E^* its dual, $\hat{E}$ the canonical embedding of E in E^{**}, and $S(E)$ the unit sphere of E. The mapping $D\colon E \to 2^{E^*}$ which associates with each $x \in E$ the $\{f \in E^*\colon f(x) = \|f\|\|x\| \text{ and } \|f\| = \|x\|\}$ is called the *duality mapping*. The duality mappings on E^*, E^{**}, and Y, a subspace of E, will be denoted by D_1, D_2, and D_Y, respectively. The set $D(S(E))$ will be denoted by $D(S)$ and elements of $D(S)$ by f_x. Recall that the Bishop-Phelps Theorem states that $D(S)$ is norm dense in $S(E^*)$.

E is *smooth* [*very smooth*] *at* $x \in S(E)$ if $D(x)$ $[D_2(\hat{x})]$ is a singleton (see Giles (1975) and Yorke (1977) for equivalent definitions). E is *smooth* [*very smooth*] if E is smooth [very smooth] at each $x \in S(E)$. E is *rotund* if $x, y \in S(E)$ with $\|x + y\| = 2$ implies $x = y$. E is *weakly locally uniformly rotund* (*WLUR*) [*locally uniformly rotund* (*LUR*)] *at* $x \in S(E)$ if every sequence (or net) $\{x_n\}$ in $S(E)$ with $\|x_n + x\| \to 2$ has $x_n \to x$ in the $\sigma(E, E^*)$ topology [the norm topology]. E is *WLUR* (*LUR*) if E is WLUR [LUR] at each $x \in S(E)$.

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A biorthogonal set $\{(x_i, f_i): i \in I\} \subset E \times E^*$ is called a *Markuschevich basis* (*M-basis*) for E if $E = \overline{\text{sp}}\{x_i\}$ and $\text{sp}\{f_i\}$ is total over E. The *density character* of a subspace $Y \subset E$, denoted dens Y, is the minimum cardinality of a (relatively) norm dense subset of Y. E is *weakly compactly generated* (*WCG*) if there is a $\sigma(E, E^*)$ compact set $K \subset E$ such that $E = \overline{\text{sp}}\{K\}$. The symbol "$\cong$" will be used to mean "isometrically isomorphic to".

Tacon (1970) showed that if E^{**} is smooth at each point of $S(\hat{E})$, that is, if E is very smooth, then E^* can be equivalently renormed to be rotund. The main step in obtaining this norm involves the construction of a long sequence $\{P_\alpha: \omega \leqslant \alpha \leqslant \mu, \bar{\bar{\mu}} = \text{dens } E^*\}$ of norm one projections on E^*. However, in order to use the transfinite induction method of Troyanski (1971) to improve this result, one must be able to identify each $(P_{\alpha+1} - P_\alpha)E^*$, $\omega \leqslant \alpha < \mu$, with the dual of a very smooth space, or, equivalently, prove that certain quotient spaces of E are also very smooth. The purpose of this paper is to show that if E^{**} is given a stronger type of partial smoothness: if E^{**} is smooth at each $F_f \in D_1(S)$, then the necessary amount of smoothness will pass to the required quotient spaces. Now the standard transfinite induction argument can be used to construct an M-basis and an equivalent LUR norm for E^*. Unfortunately, as with Tacon's rotund norm, this LUR norm need not be a dual norm.

2. The theorem

In what follows the expression "$D(S)$ is smooth" will be used to mean that E^* is smooth at each $f_x \in D(S)$.

PROPOSITION. *Let Y be a subspace of E. If $D(S)$ is smooth, then so is $D_Y(S)$.*

PROOF. If Y^* is not smooth at $f \in D_Y(S)$ there are sequences $\{y_n\}$ and $\{z_n\}$ in $S(Y)$, a $g \in S(Y^*)$, and an $\varepsilon > 0$ such that $f(y_n + z_n) \to 2$ but $|g(y_n - z_n)| \geqslant \varepsilon$ for all n (Yorke (1977; Proposition 3)). But, by the Hahn-Banach Theorem, there are $\tilde{f}$ and $\tilde{g}$ in $S(E^*)$ such that $\tilde{f} = f$ and $\tilde{g} = g$ on Y. Thus $|\tilde{g}(y_n - z_n)| \geqslant \varepsilon$ for all n even though $\tilde{f}(y_n + z_n) \to 2$, hence E^* is not smooth at $\tilde{f}$. However, since $f \in D_Y(S)$ and $\tilde{f} = f$ on Y, $f \in D(S)$, so $D(S)$ can not be smooth either.

LEMMA. *Let $D_1(S)$ be smooth. Then there is a transfinite sequence of projections $\{T_\delta: \delta \in D\}$ defined on E^* such that, for each δ,*

(1) $\|T_\delta\| < \infty$,

(2) $T_\delta T_\gamma = T_\gamma T_\delta = T_\delta$ *if* $\gamma < \delta$,

(3) $\bigcup_{\gamma<\delta} T_{\gamma+1}E^*$ *is norm dense in* $T_\delta E^*$,

(4) *for any* $\varepsilon > 0$ *and* $f \in E^*$ *the* $\{d\colon \|(T_{\delta+1} - T_\delta)f\| > \varepsilon(\|T_\delta\| + \|T_{\delta+1}\|)\}$ *is finite*,

(5) $(T_{\delta+1} - T_\delta)E^*$ *is separable, and*

(6) $\overline{\bigcup_{\delta \in D}(T_{\delta+1} - T_\delta)E^*} = E^*$.

PROOF. Let μ be the first ordinal number of cardinality dens E^*. Then since $D_1(S)$ is smooth and $S(\hat{E}) \subset D_1(S)$, E is very smooth. Thus for every α, $\omega \leqslant \alpha \leqslant \mu$, there is a subspace $E_\alpha \subset E$, with dens $E_\alpha \leqslant \bar{\bar{\alpha}}$, and a projection P_α on E^* such that

(1) $\|P_\alpha\| = 1$,

(2) $P_\alpha E^* = D(\overline{E_\alpha}) \cong E_\alpha^*$,

(3) $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ if $\beta < \alpha$,

(4) $\bigcup_{\beta<\alpha} P_{\beta+1}E^*$ is norm dense in $P_\alpha E^*$,

(5) $P_\mu = I$, and

(6) $(P_{\alpha+1} - P_\alpha)E^* \cong (E_{\alpha+1}/E_\alpha)^*$, $\alpha < \mu$.

(For (1) through (5) use Tacon (1970; Theorem 2); (6) is given in John and Zizler (1975, Lemma 1(8)).) Now proceed by induction on dens E^*. If E^* is separable the result is immediate. Assume that the lemma holds for all cardinal numbers less than dens E^*. Let G_α denote $E_{\alpha+1}/E_\alpha$ for each α, $\omega \leqslant \alpha < \mu$. Since $G_\alpha^* \cong (P_{\alpha+1} - P_\alpha)E_\alpha^*$ and $D_1(S)$ is smooth, the proposition shows that each $D_{G_\alpha^*}(S)$ is smooth. (This means that each $E_{\alpha+1}/E_\alpha$ is very smooth.) Thus, since dens $(P_{\alpha+1} - P_\alpha)E^* < \bar{\bar{\mu}}$, $\omega \leqslant \alpha < \mu$, the inductive hypothesis gives a transfinite sequence of projections $\{S_\beta^\alpha\colon\ \omega \leqslant \beta \leqslant \Gamma_\alpha, \bar{\bar{\Gamma}}_\alpha = \text{dens}(P_{\alpha+1} - P_\alpha)E^*\}$ on each $(P_{\alpha+1} - P_\alpha)E^*$ which satisfies the conditions of the lemma. Let $D = \{(\alpha, \beta)\colon \omega \leqslant \alpha \leqslant \mu, \omega \leqslant \beta \leqslant \Gamma_\alpha\}$ and order this set lexiographically. For each $\delta \in D$ define

$$T_\delta = \begin{cases} S_\alpha^\beta(P_{\alpha+1} - P_\alpha) + P_\alpha & \text{if } \alpha < \mu\,(S_\omega^\omega = 0), \\ I & \text{if } \alpha = \mu. \end{cases}$$

Now as in Troyanski (1971, page 177) one can show that the elements of $\{T_\delta\colon \delta \in D\}$ have the desired properties.

THEOREM. *If* $D_1(S)$ *is smooth, then*

(1) E^* *admits an M-basis, and*

(2) E^* *can be equivalently renormed to be LUR.*

PROOF. (1) Use the standard transfinite induction argument (for example, see John and Zizler (1975, page 294)).

(2) Since $D_1(S)$ is smooth, E is very smooth, hence there is a continuous one-to-one linear operator which maps E^* into $c_0(\Gamma)$ for some set Γ (Tacon

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(1970, Theorem 1)). This, together with properties (3), (4), (5), and (6) of the lemma, shows that E^* satisfies the conditions of Proposition 1 of Troyanski (1971, page 175).

COROLLARY. *If E^* is WLUR, then*
(1) E^* admits an M-basis, and
(2) E^* can be equivalently renormed to be LUR.

PROOF. E^* WLUR implies $D_1(S)$ is smooth (Yorke (1979, Theorem 1)).

Notice that although this WLUR norm for E^* must be a dual norm, the LUR norm, although equivalent, need not be dual.

If the projections of the lemma are $\sigma(E^*, E)$ continuous it is not difficult to show that E must be WCG. On the other hand if E is WCG, as well as very smooth, then the projections will be $\sigma(E^*, E)$ continuous (use John and Zizler (1974, Lemma 4)). However, even though it is likely that if E^{**} is smooth, E is WCG, it appears that the methods used in the proof of the lemma will not give this result without additional assumptions.

References

D. Amir and J. Lindenstrauss (1968), 'The structure of weakly compact sets in Banach spaces,' *Ann. of Math.* (2) **88**, 35–46.

J. Diestel (1975), *Geometry of Banach spaces—selected topics* (Springer-Verlag, Berlin, Heidelberg, New York).

J. R. Giles (1975), 'On smoothness of the Banach space embedding,' *Bull. Austral. Math. Soc.* **13**, 69–74.

K. John and V. Zizler (1974), 'Smoothness and its equivalents in weakly compactly generated Banach spaces', *J. Functional Analysis* **15**, 1–11.

K. John and V. Zizler (1975), 'Markuschevich bases in some dual spaces,' *Proc. Amer. Math. Soc.* **50**, 293–296.

D. G. Tacon (1970), 'The conjugate of a smooth Banach space,' *Bull. Austral. Math. Soc.* **2**, 415–425.

S. L. Troyanski (1971), 'On locally uniformly convex and differentiable norms in certain non-separable Banach spaces,' *Studia Math.* **37**, 173–180.

A. C. Yorke (1977), 'Weakly rotundity in Banach spaces,' *J. Austral. Math. Soc. Ser. A* **24**, 224–233.

A. C. Yorke (1979), 'Differentiability and local rotundity,' *J. Austral. Math. Soc. Ser. A* **28**, 205–213.

School of Mathematical and Physical Sciences
Murdoch University
Murdoch
Western Australia 6150