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On character values in finite groups

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Let u be a nonidentity element of a finite group G and let c be a complex number. Suppose that every nonprincipal irreducible character X of G satisfies either $X(1) - X(u) = c$ or $X(u) = 0$. It is shown that c is an even positive integer and all such groups with $c \leq 8$ are described.

1. Introduction

In [10] the first author completely classified finite groups G containing a nonidentity element u, with respect to which every nonprincipal irreducible character X satisfies $X(1) - X(u) = c$ for some fixed complex number c.

In this paper we investigate finite groups G satisfying the following, more general, condition.

HYPOTHESIS. *There exist* $u \in G$, $u \neq 1$, *and a complex number* c, *such that every nonprincipal irreducible (complex) character of* G *which does not satisfy*

$$X(1) - X(u) = c \tag{1}$$

satisfies

$$Z(u) = 0 . \tag{2}$$

We shall denote by X and Z the sets of irreducible characters of G satisfying (1) and those satisfying (2), but not (1), respectively. The

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groups G mentioned in Theorem 3 of [10] certainly satisfy the hypothesis with an empty Z. The group $SL(2, 5)$ is an example of a group satisfying the hypothesis with a nonempty Z with respect to any element of order 4 and $c = 4$ (see the character table in [4, p. 228]).

In Section 2 we analyze groups satisfying the hypothesis. We show, among other facts, that c is a positive even rational integer (Lemma 2) and if $c = 2$, then $|u| = 2$, $G = \langle u \rangle G'$, and $C_G(u) \cap G' = 1$ (Proposition 12). Our main result is the following Theorem 2, the proof of which is given in Sections 3 and 4.

THEOREM 2. *Let G be a finite group satisfying the hypothesis with respect to $c \leq 8$. Then $O(G)$ is abelian and one of the following holds:*

$$c = 2\,,\quad G/O(G) \cong C_2\,,\qquad |C_G(u)| = |u| = 2\,;$$

$$c = 4\,,\quad G/O(G) \cong \mathrm{PSL}(2, 5)\,,\quad |C_G(u)| = 4\,,\quad |u| = 2\,;$$

$$c = 4\,,\quad G/O(G) \cong \mathrm{SL}(2, 5)\,,\quad |C_G(u)| = |u| = 4\,;$$

$$c = 8\,,\quad G/O(G) \cong \mathrm{PSL}(2, 8)\,,\quad |C_G(u)| = 8\,,\quad |u| = 2\,.$$

In this paper G denotes a finite group and $\mathrm{Irr}\, G$ is the set of irreducible (complex) characters of G. If $x, y \in G$, then $x \sim y$ means that x is conjugate to y in G. The principal character of G will be denoted by 1_G and its values will sometimes be written as $1(x)$. An integer in this paper means a rational integer, and if n is an integer then its 2-part is denoted by $|n|_2$.

2. General results

From now on G denotes a finite group satisfying the hypothesis. The summations $\sum X^i(g)$, $i = 1, 2$, $\sum Z^i(g)$, $i = 1, 2$, $\sum Y^i(g)$, $i = 1, 2$, where $g \in G$, will run over all $X \in \mathcal{X}$, all $Z \in \mathcal{Z}$, and all $Y \in \mathrm{Irr}(G)$, respectively. If $h \in G$, a similar convention will be applied to $\sum X(g)X(h)$, $\sum Y(g)Y(h)$, and $\sum Z(g)Z(h)$. For centralizers in G the subscript G in $C_G(g)$ will be dropped. The principal 2-block of G will be denoted by $B_0(G)$.

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In this section we show that G can be characterized if

(i) $B_0(G) \subseteq X \cup \{1_G\}$ (see [7]), or

(ii) $G \neq G'$ (Proposition 12), or

(iii) $c = 2$ (Proposition 12).

If $G = G'$, then G has a unique maximal normal subgroup H (Corollary 14), with $u \notin H$ (Lemma 11). Clearly then G/H is a nonabelian simple group which satisfies the hypothesis with respect to uH and the same c .

Finally, if Y is a nonprincipal irreducible character of $G = G'$ of minimal degree m , then $1 < m \leq c-1$ and $\overline{G} = G/(\ker Y)$ is a primitive unimodular irreducible group in dimension m with $Z(\overline{G}) \subseteq \overline{G} = \overline{G}'$.

LEMMA 1.

$$1 + \sum X^2(1) = c \sum X(1) = |G| - \sum Z^2(1) .$$

Proof. By the orthogonality relations between irreducible characters and the hypothesis,

$$0 = \sum Y(u)Y(1) = \sum Y^2(1) - c \sum X(1) - \sum Z^2(1)$$

$$= 1 + \sum X^2(1) - c \sum X(1) = |G| - c \sum X(1) - \sum Z^2(1) .$$

LEMMA 2. *c is a positive even integer. Hence $Y(u)$ is an integer for each $Y \in \mathrm{Irr}(G)$ and $u \sim u^{-1}$ in G.*

Proof. By Lemma 1, c is a positive rational number. Since by (1), c is an algebraic integer, it follows that c is a positive integer. As $\sum X(1)$ and $1 + \sum X^2(1)$ are of opposite parity, c is even by Lemma 1. Consequently, in view of the hypothesis, $Y(u)$ is an integer for each $Y \in \mathrm{Irr}(G)$, which implies that $u \sim u^{-1}$ in G .

LEMMA 3. *$\sum X(1)$ and $\sum X(u)$ are odd integers.*

Proof. By the argument of Lemma 2, $\sum X(1)$ is an odd integer. Since c is even, $\sum X(u) \equiv \sum X(1) \pmod 2$.

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LEMMA 4. $|C(u)| = -c\sum X(u)$. *Hence* $\sum X(u)$ *is a negative odd integer,* $c \mid |C(u)|$ *, and*

(3) $$|c|_2 = \left|1 + \sum X^2(1)\right|_2 = |C(u)|_2 .$$

Proof. By the orthogonality relations between irreducible characters and the hypothesis

$$0 = \sum Y(1)Y(u) = \sum Y^2(u) + c\sum X(u) = |C(u)| + c\sum X(u) .$$

The other statements follow by Lemmas 3 and 1.

LEMMA 5. *If either* $B_0(G) \nsubseteq X \cup {}_G$ *or* $u^2 \neq 1$ *, then* $2 \mid |G : C(u)|$.

Proof. Suppose first that $Z \in Z \cap B_0(G)$. Then, using Brauer's criterion for block membership,

$$0 = \frac{|G|Z(u)}{|C(u)|Z(1)} \equiv \frac{|G|.1(u)}{|C(u)|.1(1)} \pmod 2 ;$$

hence $2 \mid |G : C(u)|$.

Next, suppose that $u^2 \neq 1$. By Lemma 2 there exists $g \in G$ such that $u^g = u^{-1}$ and consequently

$$|\langle g, C(u)\rangle| = 2|C(u)| \mid |G| .$$

REMARK. If $B_0(G) \subseteq X \cup 1_G$, then the groups were classified in [1].

LEMMA 6. u *is a 2-element iff* $2 \mid Z(1)$ *for all* Z .

Proof. If $Z(1)$ are all even, then by Lemma 2,

(4) $$Y(1) \equiv Y(u) \pmod 2 \quad \text{for all } Y \in \operatorname{Irr} G .$$

Let u' be the $2'$-part of u . Then, by [5, (6.4)],

$$Y(1) \equiv Y(u) \equiv Y(u') \pmod P$$

for all $Y \in \operatorname{Irr} G$, where P is a prime ideal over 2 in the ring of integers of $Q(\sqrt[|G|]{1})$. It follows by [2, (3c), p. 412] that $u' = 1$; hence u is a 2-element.

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If, on the other hand, u is a 2-element, then both $Y(1)$ and $Y(u)$ are a sum of $Y(1)$ 2-power roots of 1 and both being integers, it follows that (4) holds. In particular, $2 \mid Z(1)$ for all Z.

LEMMA 7. *For* $x \in G$ *let* $d(x) = 0$ *if* $x \sim u$ *in* G *and* $d(x) = 1$ *otherwise. Then for all* x,

$$1 + \sum X(1)X(x) = d(x)c \sum X(x) .$$

Proof. By the hypothesis,

$$(5) \qquad \sum Y(1)Y(x) - \sum Y(u)Y(x) = c \sum X(u) + \sum Z(1)Z(x) .$$

If $x \sim u$ in G, then by the orthogonality relations between irreducible characters,

$$0 = \sum Y(1)Y(x) = 1 + \sum X(1)X(x) ,$$

as required. Otherwise, $\sum Y(u)Y(x) = 0$ and, cancelling $\sum Z(1)Z(x)$ on both sides of (5), the formula follows.

NOTATION. Since c is even, let $c = 2c'$. Denote by r_i the number of irreducible characters of degree i in X. By the hypothesis, $r_i = 0$ for $i < c'$. $\sum_i$ will denote summation over i, $c' \le i \le \infty$. One of the irreducible characters of X of minimal degree will be denoted by X_1.

LEMMA 8. $|C(u)| = 1 + \sum_i (i-c)^2 r_i = (-c) \sum_i (i-c) r_i$.

Proof. By the orthogonality relations between irreducible characters and Lemma 4,

$$|C(u)| = 1 + \sum X^2(u) = -c \sum X(u) .$$

In view of the hypothesis, the lemma follows.

LEMMA 9. $1 + \sum_i i(i-c)r_i = 0$; *hence*

$$1 + \sum_{i=c+1}^{\infty} i(i-c)r_i = \sum_{i=c'}^{c-1} i(c-i)r_i .$$

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Proof. By Lemma 8,

$$1 + \sum_i \left(i^2 - 2ic + c^2 + ic - c^2\right) r_i = 0 .$$

Lemma 9 immediately yields

COROLLARY 10. $X_1(1) \le c - 1$.

Since u does not belong to the kernel of any $Y \in \operatorname{Irr} G$ other than 1_G , we get

LEMMA 11. *If* $u \in H \trianglelefteq G$ *, then* $H = G$.

PROPOSITION 12. *The following statements are equivalent:*

(a) $|u| = 2$, $G = \langle u \rangle O(G)$ *, and* $C(u) = \langle u \rangle$ *;*

(b) $G' \ne G$ *;*

(c) $c = 2$;

(d) $X_1(1) = c'$.

Proof. *(a)* clearly implies *(b)*. Suppose now that $G' \ne G$ and let Y be a nonprincipal linear character of G . Then $Y \in X$ and by Lemma 2, $c = 2$.

Suppose, next, that $c = 2$. By Corollary 10, $X_1(1) = 1 = c'$.

Suppose, finally, that $X_1(1) = c'$. As $X \in X$, it follows that $X_1(u) = -c' = -X_1(1)$. Let

$$\ker^* X_1 = \{g \in G \mid X_1(g) = \pm X_1(1)\} .$$

Then $u \in \ker^* X_1 \triangleleft G$; hence, by Lemma 11, $\ker^* X_1 = G$. As $\left(\ker^* X_1\right) / \left(\ker X_1\right)$ is elementary abelian, it follows that $G' \ne G$; hence $c = 2$ by previous argument. Thus, by the hypothesis, $Y(u) = -1$ for every nonprincipal linear character of G , and consequently $|G/G'| = 2$. follows that $r_1 = 1$ and Lemma 9 implies that $r_i = 0$ for $i > 2$. Hence, by Lemma 8, $|C(u)| = 2$ and *(a)* follows. The proof of Proposition 12 is complete.

LEMMA 13. *If* $H \triangleleft G$ *,* $H \ne G$ *, then* G/H *is not a direct product of*

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its proper subgroups.

Proof. In view of Lemma 11, G/H satisfies the hypothesis with respect to uH. Thus it suffices to prove Lemma 13 for $H = 1$. Suppose that $G = G_1 \times G_2$, $G_1 \neq 1$ or G, and $u = u_1 u_2$, $u_i \in G_i$. By Lemma 11, $u_i \neq 1$ for $i = 1, 2$, and consequently G_i satisfy the hypothesis with respect to u_i and the same c as in G.

Let χ_a and χ_b be characters of X of G_1 and of G_2, respectively, satisfying $\chi_a(u_1) \neq 0$ and $\chi_b(u_2) \neq 0$. By Lemma 9, χ_a and χ_b exist. Now χ_a, χ_b, and $\chi_a \chi_b$ may be regarded as characters of G belonging to X. Thus

$$\chi_a(1)\chi_b(1) - \chi_a(u_1)\chi_b(u_2) = c ,$$

$$\chi_a(1) - \chi_a(u_1) = c ,$$

and

$$\chi_b(1) - \chi_b(u_2) = c .$$

It follows that $\chi_a(1) + \chi_b(1) = c + 1$. Thus G_i have no characters of degree larger than c in their X. By Lemmas 9 and 8, $|C_{G_i}(u_i)| = c = 2$, in contradiction to Proposition 12.

COROLLARY 14. *Let H be a maximal normal subgroup of G. If $K \triangleleft G$, $K \neq G$, then $K \subseteq H$.*

Proof. Suppose that $K \not\subseteq H$. Then $G = HK$ and $G/(H \cap K) \cong (G/H) \times (G/K)$, in contradiction to Lemma 13.

NOTATION. The maximal normal subgroup of G will be denoted by H. The minimal degree of a nonprincipal character in $\operatorname{Irr} G$ will be denoted by m.

LEMMA 15. *Suppose that G also satisfies the hypothesis with respect to $v \in G$ and the same c, but possibly with different X and Z. Then $u \sim v$ in G.*

Proof. Suppose that $u \not\sim v$ in G. Then by the orthogonality

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relations between irreducible characters

$$0 = \sum Y(u)Y(v) = 1 + \left[\sum Y(u)Y(v) - 1(u)1(v)\right] \geq 1 ,$$

as $Y(u) < 0$ implies that $Y(1) < c$. Hence either $Y(v) = 0$ or $Y(v) < 0$.

ASSUMPTIONS. From now on $G = G'$. By Corollary 10 and Proposition 12 we have

(6) $$1 < m \leq c-1 \quad \text{and} \quad X_1(1) > c' .$$

LEMMA 16. *Let* $Y \in \operatorname{Irr} G$, $Y(1) = m$. *Then* $\overline{G} = G/(\ker Y)$ *is a primitive unimodular irreducible group in dimension* m *with* $Z(\overline{G}) \subseteq \overline{G} = \overline{G}'$.

Proof. As $G = G'$, $\overline{G} = \overline{G}'$ is unimodular with $Z(\overline{G}) \subseteq \overline{G}'$. By definition of Y , $\overline{G}$ is irreducible. Finally, primitivity of $\overline{G}$ follows from the minimality of m , since otherwise $\overline{G}$ would have a subgroup L of index r , $1 < r < m$ (see [3], Theorem 4.2B), and $(1_L)^{\overline{G}}$ would contain nonprincipal irreducible components of degree less than m .

3. $c = 4$

In this section we prove the following

THEOREM 1. *Let* G *be a finite group satisfying the hypothesis with respect to* $c = 4$. *Then* $O(G)$ *is abelian and*

$$G/O(G) \cong \mathrm{PSL}(2, 5) \text{ , or } \mathrm{SL}(2, 5) .$$

In the first case $|u| = 2$ *and in the second case* $|u| = 4$.

Proof. As by Proposition 12, $G = G'$, (6) implies that $1 < m \leq c-1$; hence either $m = 2$ or $m = 3$. Let H be the maximal normal subgroup of G (Corollary 14). Then by Lemma 16 applied to G/H , G/H is a simple primitive irreducible group in dimension n , $1 < m \leq n \leq 3$. By Feit's list [6, p. 72], $n = 3$ and

$$G/H \cong A_5 \quad \text{or} \quad \mathrm{PSL}(2, 7) .$$

As the characters of $\mathrm{PSL}(2, 7)$ do not satisfy the hypothesis, $G/H \cong A_5$ and X of G contains at least two characters of degree 3 .

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Let $Y \in \operatorname{Irr} G$ be of degree 3. By Lemma 14, $K = \ker Y \subseteq H$ and $\overline{G} = G/K$ is a unimodular irreducible group in dimension 3 satisfying $Z(\overline{G}) \subseteq \overline{G}'$. As $\overline{G} = \overline{G}'$, it follows by Theorem 4.2B in [3, p. 68] that $\overline{G}$ is primitive. Hence by [6, p. 76] and the fact that $K \trianglelefteq H$ and $G/H \cong A_5$, we get $K = H$.

Thus, using the notation and the statements of Lemmas 9 and 8 with respect to G, it follows that $r_3 = 2$, $r_5 = 1$, and $r_i = 0$ for $i > 5$; hence $|C(u)| = 4$.

Groups with a centralizer of an element of order 4 were characterized by Suzuki [12] and Wong [13]. As $G = G'$ and $G/H \cong A_5$, it follows by [13, Theorems 1 and 2, statements and proofs] that $G/O(G) \cong \mathrm{PSL}(2, 5)$ or $\mathrm{SL}(2, 5)$. If $|u| = 2$, then the first case holds and certainly $O(G)$ is abelian. If $|u| = 4$, then the second case holds and $O(G)$ is abelian by [12, Proposition 5]. The proof of Theorem 1 is complete.

4. $c \le 8$

This section is devoted to the proof of the main theorem, Theorem 2, stated in Section 1. The case $c = 2$ was treated in Proposition 12 and the case $c = 4$ was analyzed in Section 3, Theorem 1.

So suppose that $c = 6$ or 8. Let H be the maximal normal subgroup of G. Then by Proposition 12, Lemma 16, and (6), G/H is a simple primitive irreducible group of dimension n, $1 < m \le n \le 7$, where m is the minimal degree of $Y \in \operatorname{Irr} G$, $Y \ne 1_G$. By [6, pp. 76-77], G/H is isomorphic to one of the groups:

$$A_5,\ \mathrm{PSL}(2, 7),\ A_6,\ \mathrm{PSL}(2, 11),\ O_5'(3),\ A_7,\ \mathrm{PSU}(3, 3),\ \mathrm{PSL}(2, 13),$$
$$\mathrm{PSL}(2, 8),\ A_8,\ \text{and}\ S_p(6, 2).$$

Since G/H satisfies the hypothesis with $c = 6$ or 8, inspection of the character tables of these groups (for $S_p(6, 2)$ see [7]; for the other groups see [11]) yields a single candidate:

$$(7) \qquad c = 8,\quad n = 7,\quad G/H \cong \mathrm{PSL}(2, 8),\quad u^2 \in H,\quad u \notin H.$$

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Let Y be an irreducible character of G of degree m. Then by Lemma 14, $K = \ker Y \subseteq H$ and by Lemma 16, G/K is a perfect primitive irreducible group of dimension m, $1 < m \leq 7$. In view of (7), the table [6, pp. 76-77] yields $m = 7$, $K = H$. Thus X of G has exactly 4 characters of degree 7, and other nonprincipal irreducible characters of G are of degree greater than or equal to 8. Applying the notation and the results of Lemmas 9 and 8 to G, we get

$$r_7 = 4 , \quad r_9 = 3 , \quad r_i = 0 \quad \text{for } i \neq 7, 8, 9 ;$$

hence $|C(u)| = 8 = c$. Also, by Lemmas 11 and 4, if K is a proper normal subgroup of G, then $|C_{G/K}(uK)| = 8$.

Define $Z_0 = O(G)$ and for $i \geq 1$,

$$Z_i/Z_{i-1} = Z\left(G/Z_{i-1}\right) .$$

It is easy to see that Z_i/Z_{i-1} are 2-groups for $i \geq 1$, and $O\left(G/Z_i\right) = 1$ for $i \geq 0$. Let j be the least nonnegative integer for which $Z_j = Z_{j+1}$. Then $\overline{G} = G/Z_j$ satisfies:

$$\overline{G}' = \overline{G} , \quad Z(\overline{G}) = O(\overline{G}) = 1 , \quad |C_{\overline{G}}(\overline{u})| = 8 ,$$

where $\overline{u} = uZ_j$. By Harada [9, Theorem 2], G is of sectional 2-rank less than or equal to 4. Thus by Gorenstein and Harada [8, Corollary C], it follows, in view of (7) and Lemma 13, that $G/Z_j \cong \mathrm{PSL}(2, 8)$.

Suppose that $j > 0$. Then Z_j/Z_{j-1} is an even nontrivial Schur multiplier of $\mathrm{PSL}(2, 8)$, a contradiction. Thus $G/O(G) \cong \mathrm{PSL}(2, 8)$ and as $|C(u)| = 8$, u is an involution and $O(G)$ is abelian.

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