THE DISTRIBUTION OF THE MAXIMUM OF PARTIAL SUMS OF INDEPENDENT RANDOM VARIABLES

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1. The problem. It has been shown [1] that if $X_1, + X_2, \ldots$ are independent random variables each of density $(2\pi)^{-\frac{1}{2}}e^{-\frac{x^2}{2}}$, and if $s_k = X_1 + X_2 + \ldots + X_k$ then

$$(1.1) \quad \lim_{n\to\infty} \text{Prob}\left\{ \max_{k<nt} |s_k| < a\sqrt{n}\right\} = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \exp\left\{-\frac{(2j+1)^2\pi^2 t}{8a^2}\right\}.$$

This result, which can be generalized to random variables which need not be normally distributed [2], is closely connected with the theory of diffusion and Brownian motion. In fact the above limiting distribution is equal to

$$\int_{-a}^{a} P(x, t)dx, \text{ where } P(x, t) \text{ is the fundamental solution of } P_t = \tfrac{1}{2} P_{xx}$$

which becomes singular at $x = 0$ as $t \to 0$ and which is subject to the condition $P(-a, t) = P(a, t) = 0$, corresponding to absorbing barriers.

The connection between this problem and the theory of diffusion hinges on the fact that the second moments of the X_j are finite. It is therefore of interest to investigate analogous problems for random variables with an infinite second moment. For this purpose we have chosen the case in which the X_j have the same Cauchy density $[\pi(1 + x^2)]^{-1}$. Our principal goal is the computation (§6) of the limiting distribution

$$(1.2) \qquad p(a, t) = \lim_{n\to\infty} \text{Prob}\left\{\max_{k<nt} |s_k| < an\right\},$$

where the s_k retain their preceding significance as the partial sums $X_1 + X_2 + \ldots + X_k$.

The computation will be effected by a series of reductions which culminate in an integral equation of the Hilbert-Schmidt type. It unfortunately does not seem soluble in terms of known functions.

We are, nevertheless, able to compute the mean time of absorbtion for a Cauchy process $x(t)$. Define the random variable $T(a)$ as the greatest lower bound of those t's for which

$$\underset{0\leqslant\nu\leqslant t}{\text{l.u.b.}} \; |x(\nu)| < a.$$

$T(a)$ is thus the time during which $x(t)$ remains in the strip between $-a$ and $+a$ before it leaves that strip for the first time. It turns out that $E\{T(a)\} \equiv a$, and

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$$E\{T^2(a)\} = \frac{4}{\pi}\, a^2 \left\{\frac{1}{2} + \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} \cdots\right\}. \tag{1.3}$$

2. Existence of the distribution $p(a, t)$. Suppose then that the X_j are independent and Cauchy distributed with density $[\pi(1 + x^2)]^{-1}$. Consider the space of all functions $x(t)$, $t \geqslant 0$, $x(0) = 0$, and for $0 < t_1 < t_2 < \ldots < t_n$ assign to the set of $x(t)$ for which

$$a_1 < x(t_1) < b_1, \ldots, a_n < x(t_n) < b_n$$

the measure

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} Q(x_1, t_1)\, Q(x_2 - x_1, t_2 - t_1) \ldots Q(x_n - x_{n-1}, t_n - t_{n-1})\, dx_1 \ldots dx_n,$$

where

$$Q(x, t) = \frac{1}{\pi}\, \frac{t}{x^2 + t^2}.$$

It has been shown ([3], [4]) that when this measure is extended it has the property that almost every $x(t)$ has at most a denumerable number of discontinuities, all of the first kind (jump discontinuities). Also if a is fixed, almost all $x(\nu)$ are continuous at $\nu = a$. Consequently almost every $x(\nu)$ has the property

$$\underset{0 \leqslant \nu \leqslant t}{\text{l.u.b.}}\ |x(\nu)| = \lim_{n \to \infty}\ \max_{k < nt}\ |x(k/n)|,$$

and thus $\underset{0 \leqslant \nu \leqslant t}{\text{l.u.b.}}\ |x(\nu)|$ is a measurable function. Moreover,

$$\text{Prob}\left\{\underset{0 \leqslant \nu \leqslant t}{\text{l.u.b.}}\ |x(\nu)| < a\right\} = \lim_{n \to \infty} \text{Prob}\left\{\max_{k < nt}\ |x(k/n)| < a\right\}. \tag{2.1}$$

Since

$$\text{Prob}\left\{\max_{k < nt}\ |x(k/n)| < a\right\} = \text{Prob}\left\{\max_{k < nt}\ |s_k| < an\right\} \tag{2.2}$$

the existence of $p(a, t)$, as defined by (1.2) is established. To see that (2.2) is satisfied we write

$$x\left(\frac{k}{n}\right) = \sum_{l=0}^{k-1} \left\{x\left(\frac{l+1}{n}\right) - x\left(\frac{l}{n}\right)\right\}$$

and note that from the definition of the Cauchy measure it follows that the increments $x\left(\frac{l+1}{n}\right) - x\left(\frac{l}{n}\right)$ are independent and all have the density $Q\left(x, \frac{1}{n}\right)$. Thus, expressing both sides of (2.2) as multiple integrals we see that they are identical.

3. First reduction of the problem. We have

$$\text{Prob}\left\{\max_{1\leqslant k\leqslant m} |s_k| < an\right\}$$

$$= \left(\int_{-a}^{a}\right)^m Q(x_1, \tau)Q(x_2 - x_1, \tau)\ldots Q(x_m - x_{m-1}, \tau)dx_1 \ldots dx_m,$$

where $\tau = 1/n$. Now let

$$\varphi_m(x) = \left(\int_{-a}^{a}\right)^{m-1} Q(x_1, \tau)Q(x_2 - x_1, \tau)\ldots Q(x - x_{m-1}, \tau)dx_1 \ldots dx_{m-1} \tag{3.1}$$

and

$$\varphi_0(x) = Q(x, \tau) = \frac{1}{\pi}\frac{\tau}{x^2 + \tau^2} = \frac{1}{\pi}\frac{n}{n^2x^2 + 1};$$

note that

$$\varphi_m(x) = \int_{-a}^{a} \varphi_{m-1}(y)\varphi(x - y;\tau)dy.$$

Furthermore let

$$\psi_\tau(x) = \tau \sum_{m=0}^{\infty} e^{-ms\tau}\varphi_m(x), \qquad s > 0.$$

We shall show that

$$\lim_{\tau\to\infty} \int_{-a}^{a} \psi_\tau(x)\, dx = \int_0^\infty e^{-st}\, \text{Prob}\left\{\underset{0\leqslant\nu\leqslant t}{\text{l.u.b.}} |x(\nu)| < a\right\} dt. \tag{3.2}$$

Letting

$$f_\tau(t) = \int_{-a}^{a} \varphi_m(x)\, dx \qquad m\tau < t \leqslant (m + 1)\,\tau,$$

we see that $0 < f_\tau(t) \leqslant 1$, and by (2.1), (2.2) that

$$\lim_{n\to\infty} \text{Prob}\left\{\max_{1\leqslant k\leqslant m} |s_k| < an\right\} = \lim_{\tau\to 0} f_\tau(t) = \text{Prob}\left\{\underset{0\leqslant\nu\leqslant t}{\text{l.u.b.}} |x(\nu)| < a\right\},$$

where m satisfies, of course, the inequality $nt - 1 \leqslant m < nt$.

Thus

$$\lim_{\tau\to 0} \int_0^\infty e^{-st} f_\tau(t)\, dt = \int_0^\infty e^{-st}\, \text{Prob}\left\{\underset{0\leqslant\nu\leqslant t}{\text{l.u.b.}} |x(\nu)| < a\right\} dt.$$

On the other hand

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$$\int_{-a}^{a} \psi_\tau(x)dx = \frac{s\tau}{1-e^{-st}}\int_0^\infty e^{-st} f_\tau(t)\, dt$$

$$= \frac{s\tau}{1-e^{-s\tau}} \sum_{m=0}^{\infty} \int_{m\tau}^{(m+1)\tau} e^{-st}\, dt \int_{-a}^{a} \varphi_m(x)dx,$$

and (3.2) follows.

Note also that the definitions of $\varphi_m(k)$ and $\psi_\tau(x)$ imply that

(3.3) $$\frac{e^{s\tau}-1}{\tau}\psi_\tau(x) - e^{s\tau}Q(x, \tau) = \frac{1}{\tau}\left\{\int_{-a}^{a} \psi_\tau(y)Q(x-y, \tau)dy - \psi_\tau(x)\right\}.$$

4. Passage to the limit. It follows from (3.1) that

$$0 < \varphi_m(x) < \left(\int_{-\infty}^{\infty}\right)^{m-1} Q(x_1, \tau)\ldots Q(x - x_{m-1}, \tau)dx_1, \ldots, dx_{m-1}$$

$$= Q(x,(m+1)\,\tau) = \frac{1}{\pi}\,\frac{(m+1)\tau}{x^2+(m+1)^2\tau^2}$$

and hence that

$$0 < \psi_\tau(x) < \frac{\tau}{\pi} \sum_{m=0}^{\infty} e^{-ms\tau}\,\frac{(m+1)\tau}{x^2+(m+1)^2\tau^2}\,.$$

For $p > 1$ define

$$||f|| = \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p}$$

Then

$$||\psi_\tau(x)|| < \frac{\tau}{\pi}\sum_{m=0}^{\infty} e^{-ms\tau}(m+1)\tau\, ||[x^2+(m+1)^2\tau^2]^{-1}||$$

$$\leqslant A\,\tau^{1/p}\sum_{m=0}^{\infty}\frac{e^{-ms\tau}}{(m+1)^{1-1/p}}\,.$$

Since

$$\sum_{m=0}^{n}\frac{1}{(m+1)^{1-1/p}} \sim Bn^{1/p} \qquad n\to\infty$$

it follows by an Abelian argument that

$$\sum_{m=0}^{\infty}\frac{e^{-ms\tau}}{(m+1)^{1-1/p}} \sim C\tau^{-1/p}\,, \qquad \tau\to 0.$$

Hence

(4.1) $$\left(\int_{-a}^{a} |\psi_\tau(x)|^p\, dx\right)^{1/p} \leqslant ||\psi_\tau(x)|| < C(p, s),$$

where C is independent of τ. By a theorem of Banach and Saks [5] a subsequence $\{\psi_{\tau_k}(x)\}$ can be chosen so that $\tau_k \to 0$ and

$$\underset{k\to\infty}{\overset{(p)}{\text{l.i.m.}}} \frac{1}{k} \sum_{1}^{k} \psi_{\tau_j}(x) = \psi(x) \tag{4.2}$$

in the sense of mean convergence in $L^p(-a, a)$. Strong convergence will be necessary to derive formula (5.2). From (3.2) we infer that

$$\int_{-a}^{a} \psi(x)\,dx = \int_{0}^{\infty} e^{-st} \operatorname{Prob}\{ \underset{0\leqslant\nu\leqslant t}{\text{l.u.b.}} |x(\nu)| < a\}\,dt,$$

and consequently we may concentrate on calculating $\int_{-a}^{a} \psi(x)\,dx$.

5. Introduction of the infinitesimal generator. We shall use the following lemma [6, 386]:

LEMMA 5.1. *If $g(x)$ and $g'(x)$ belong to $L^2(-a, a)$ and $g(-a) = g(a) = 0$, then*

$$\underset{\tau\to 0}{\overset{(2)}{\text{l.i.m.}}} \frac{1}{\tau}\left\{\frac{\tau}{\pi}\int_{-a}^{a} \frac{g(y)}{\tau^2+(x-y)^2}\,dy - g(x)\right\} = \frac{1}{\pi}\,\text{P.V.}\int_{-a}^{a} \frac{g'(y)}{y-x}\,dy. \tag{5.1}$$

Now *let $g(x)$ be any function satisfying the conditions of the lemma.* Multiply both sides of (3.3) by $g(x)$ and integrate from $-a$ to a to obtain

$$\frac{e^{s\tau}-1}{\tau}\int_{-a}^{a} \psi_\tau(x)g(x)dx - e^{s\tau}\int_{-a}^{a} Q(x,\tau)g(x)dx = \int_{-a}^{a} \psi_\tau(y)\,h_\tau(y)dy,$$

where $$h_\tau(x) = \frac{1}{\tau}\left\{\frac{\tau}{\pi}\int_{-a}^{a} \frac{g(y)dy}{\tau^2+(y-x)^2} - g(x)\right\}.$$

Letting $\tau \to 0$, through the sequence τ_k, we infer from (4.2) with $p = 2$ and. our lemma that

$$s\int_{-a}^{a} \psi(x)\,g(x)dx - g(0) = \int_{-a}^{a} \psi(y)\,h(y)\,dy, \tag{5.2}$$

where

$$h(x) = \frac{1}{\pi}\,\text{P.V.}\int_{-a}^{a} \frac{g'(y)}{y-x}\,dy.$$

It is on the equation (5.2) that we shall base our determination of $p(a, t)$.

6. Solution of the problem. For the sake of simplicity we assume that $a = 1$. Let m be odd and set

$$g(x) = \sin m(\text{arc cos } x) - m\sqrt{1-x^2}. \tag{6.1}$$

It is easy to verify that $g(x)$ satisfies the conditions of Lemma 5.1. Moreover it is known [7, 8] that

$$\text{P.V.} \int_{-1}^{1} \frac{\cos m(\text{arc cos } y)}{(1-y^2)^{1/2}} \frac{dy}{y-x} = \frac{\pi}{m} \frac{\sin m\,(\text{arc cos } x)}{(1-x^2)^{1/2}} \tag{6.2}$$

and

$$\text{P.V.} \int_{-1}^{1} \frac{y}{(1-y^2)^{1/2}} \frac{dy}{y-x} = \pi. \tag{6.3}$$

Substituting (6.1) in (5.2), and using (6.2) and (6.3), we get after a rearrangement

$$\begin{aligned} \frac{s}{m}\int_{-1}^{1} \psi(x) \sin m\,(\text{arc cos } x)\, dx + \int_{-1}^{1} \frac{\psi(x) \sin m\,(\text{arc cos } x)}{(1-x^2)^{1/2}}\, dx \\ = \int_{-1}^{1} \psi(x)\{1 + s(1-x^2)^{1/2}\}\, dx - 1 + \frac{\sin m\pi/2}{m}. \end{aligned} \tag{6.4}$$

The functions

$$\left(\frac{2}{\pi}\right)^{1/2} \frac{\sin m\,(\text{arc cos } x)}{(1-x^2)^{1/4}}$$

form an orthonormal set on $(-1,1)$ and since $\psi \in L^p\,(-1, 1)$ for every $p > 1$ we see that

$$\frac{\psi(x)}{(1-x^2)^{1/4}} \in L^2$$

and, in particular

$$\lim_{m\to 0} \int_{-1}^{1} \psi(x) \frac{\sin m\,(\text{arc cos } x)}{(1-x^2)^{1/2}}\, dx = 0.$$

Also

$$\lim_{m\to 0} \frac{1}{m} \int_{-1}^{1} \psi(x) \sin m\,(\text{arc cos } x)\, dx = 0,$$

and consequently it follows from (6.4) that

$$\int_{-1}^{1} \psi(x)\{1 + s(1-x^2)^{1/2}\}\, dx = 1.$$

Thus for odd m

$$(6.5)\quad \frac{s}{m}\int_{-1}^{1}\psi(x)\sin m\,(\arc\cos x)\,dx+\int_{-1}^{1}\psi(x)\,\frac{\sin m\,(\arc\cos x)}{(1-x^2)^{1/2}}\,dx = \frac{\sin m\pi/2}{m},$$

and since $\psi(x)$ is even we have also

$$(6.6)\quad \frac{s}{m}\int_{-1}^{1}\psi(x)\sin m\,(\arc\cos x)\,dx+\int_{-1}^{1}\psi(x)\,\frac{\sin m\,(\arc\cos x)}{(1-x^2)^{1/2}}\,dx = 0 = \frac{\sin m\pi/2}{m}$$

for even m. It follows from (6.5) and (6.6) that

$$(6.7)\quad \begin{aligned} s\int_{-1}^{1}\psi(x)\sum_{m=1}^{n}\frac{\sin m\,(\arc\cos x)\sin m\,(\arc\cos y)}{m}\,dx \\ +\sum_{m=1}^{n}\sin m\,(\arc\cos y)\int_{-1}^{1}\psi(x)\,\frac{\sin m\,(\arc\cos x)}{(1-x^2)^{1/2}}\,\mathrm{d}x \\ = \sum_{m=1}^{n}\frac{\sin m\,(\arc\cos 0)\sin m\,(\arc\cos y)}{m}. \end{aligned}$$

Now, for fixed x,

$$\begin{aligned} \operatorname*{l.i.m.}^{(2)}_{n\to\infty}\sum_{m=1}^{n}\frac{\sin m(\arc\cos x)\sin m\,(\arc\cos y)}{m} &= K(x,y) \\ = \sum_{m=1}^{\infty}\frac{\sin m\,(\arc\cos x)\sin m(\arc\cos y)}{m} \\ = \tfrac{1}{4}\log\frac{1-xy+[(1-x^2)(1-y^2)]^{1/2}}{1-xy-[(1-x^2)(1-y^2)]^{1/2}},\ * \\ \operatorname*{l.i.m.}^{(2)}_{n\to\infty}\sum_{m=1}^{n}\frac{\sin m\,(\arc\cos 0)\sin m\,(\arc\cos y)}{m} &= K(0,y), \end{aligned}$$

and since $\psi(x)\,(1-x^2)^{-\frac{1}{2}}\in L^2$

$$\operatorname*{l.i.m.}^{(2)}_{n\to\infty}\sum_{m=1}^{n}\sin m\,(\arc\cos y)\int_{-1}^{1}\psi(x)\,\frac{\sin m\,(\arc\cos x)}{(1-x^2)^{1/2}}\,dx = \frac{\pi}{2}\psi(y).$$

Thus letting $n\to\infty$ in (6.7) we obtain

*This kernel is known to aerodynamicists under the name of the "Betz Kernel". See e.g. Söhngen [11].

$$s \int_{-1}^{1} K(x, y)\, \psi(x)\, dx + \frac{\pi}{2} \psi(y) = K(0, y). \tag{6.8}$$

Consider the kernel

$$K(x, y) = \sum_{m=1}^{\infty} \frac{\sin m \,(\arc \cos x) \sin m \,(\arc \cos y)}{m}$$

It is symmetric and

$$\int_{-1}^{1} \int_{-1}^{1} K^2(x, y)\, dx\, dy < \infty.$$

It is clearly positive definite, and hence its eigenfunctions $g_j(x)$ form a complete set. It also follows almost at once that the eigenfunctions $g_j(x)$ are continuous. Multiplying (6.8) by $g_j(y)$ and integrating we obtain

$$\int_{-1}^{1} \psi(x)\, g_j(x)\, dx = \frac{g_j(0)}{s + \pi/2\lambda_j},$$

where λ_j is the eigenvalue of $K(x, y)$ which corresponds to the normalized eigenfunction $g_j(y)$. Thus

$$\psi(x) \sim \sum_{j=1}^{\infty} \frac{g_j(0)}{s + \pi/2\lambda_j} g_j(x)$$

and consequently

$$\int_{-1}^{1} \psi(x)\, dx = \sum_{j=1}^{\infty} \frac{g_j(0)}{s + \pi/2\lambda_j} \int_{-1}^{1} g_j(x)\, dx.$$

For the general values of a the result is

$$\int_{-a}^{a} \psi(x)\, dx = \frac{1}{a} \sum_{j=1}^{\infty} \frac{1}{s + \pi/2a\lambda_j}\, g_j(0) \int_{-a}^{a} g_j(x/a)\, dx,$$

where the g_j and λ_j retain their previous meanings. The last series is a Stieltjes transform, and consequently an iterated Laplace transform [10, 334]. Hence we can invert the series term by term to obtain

$$p(a, t) = \frac{1}{a} \sum_{j=1}^{\infty} e^{-\pi t/2a\lambda_j}\, g_j(0) \int_{-a}^{a} g_j(x/a)\, dx. \tag{6.9}$$

This is the Cauchy analogue of formula (1.1). Although we are unable to reduce (6.9) to a more explicit form we can draw significant conclusions about the mean time before absorption in a Cauchy process. If we define $T(a)$ as the greatest lower bound of those t's for which

$$\underset{0 \leqslant \nu \leqslant t}{\text{l.u.b.}} \;|x(\nu)| < a$$

we see that

$$p(t, a) = \text{Prob}\left\{ \underset{0 \leqslant \nu \leqslant t}{\text{l.u.b.}} |x(\nu)| < a \right\} = \text{Prob}\left\{ T(a) > t \right\} = 1 - \sigma(t).$$

Consequently,

$$\int_0^\infty t^k p(t, a)\, dt = \int_0^\infty t^k \,\text{Prob}\left\{ T(a) > t \right\} dt$$

$$= \int_0^\infty t^k(1 - \sigma(t))\, dt = \frac{1}{k+1} \int_0^\infty t^{k+1}\, d\sigma(t) = \frac{1}{k+1} E\left\{ T^{k+1}(a) \right\}.$$

In particular,

$$E\{T(a)\} = \int_0^\infty p(t, a)\, dt$$

and

$$E\{T^2(a)\} = 2 \int_0^\infty t\, p(t, a)\, dt.$$

From (6.9) we get

$$\int_0^\infty p(t, a)\, dt = \frac{2a}{\pi} \sum_{j=1}^{\infty} \lambda_j g_j(0) \int_{-1}^{1} g_j(x)\, dx = \frac{2a}{\pi} \int_{-1}^{1} K(0, x)\, dx = a.$$

Likewise, formula (1.3) for $E\{T^2(a)\}$ can be obtained in terms of the iterated kernel $K^{(2)}(0, x)$. Calculation of higher moments of $T(a)$ involves, of course, higher iterates of the kernel K.

Only slight modifications of the theory are needed in order to calculate

$$p(a, b, t) = \text{Prob}\left\{ -b < \underset{0 \leqslant \nu \leqslant t}{\text{g.l.b.}}\ x(\nu) \leqslant \underset{0 \leqslant \nu \leqslant t}{\text{l.u.b.}}\ x(\nu) < a \right\}.$$

Defining $T(a, b, t)$ in a way analogous to $T(a, t)$ it is then easy to show that

$$E\{T(a, b, t)\} = \sqrt{ab}.$$

In the Gaussian case $\sqrt{ab}$ is to be replaced by ab.

Finally, let us mention that from (6.8) we can determine the behaviour of $\psi(y)$ for y near 0. In fact, since $\psi \in L^2$, $s \int_{-1}^{1} K(x, y)\, \psi(x)\, dx$ is bounded for all y and it follows that

$$\psi(y) \sim \frac{2}{\pi} K(0, y) \sim \frac{1}{\pi} \log \frac{1}{|y|} + \text{const}.$$

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