

ANALOGUES OF THE LIOUVILLE THEOREM FOR LINEAR FRACTIONAL RELATIONS IN BANACH SPACES

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Consider a bounded linear operator T between Banach spaces \mathcal{B} , \mathcal{B}' which can be decomposed into direct sums $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$, $\mathcal{B}' = \mathcal{B}'_1 \oplus \mathcal{B}'_2$. Such linear operator can be represented by a 2×2 operator matrix of the form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where $T_{ij} \in \mathcal{L}(\mathcal{B}_j, \mathcal{B}'_i)$, $i, j = 1, 2$. (By $\mathcal{L}(\mathcal{B}_j, \mathcal{B}'_i)$ we denote the space of bounded linear operators acting from \mathcal{B}_j to \mathcal{B}'_i ($i, j = 1, 2$.) The map G_T from $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ into the set of closed affine subspaces of $\mathcal{L}(\mathcal{B}'_1, \mathcal{B}'_2)$, defined by

$$G_T(X) = \{Y \in \mathcal{L}(\mathcal{B}'_1, \mathcal{B}'_2) : T_{21} + T_{22}X = Y(T_{11} + T_{12}X)\}$$

is called a *linear fractional relation* associated with T .

Such relations can be considered as a generalisation of linear fractional transformations which were studied by many authors and found many applications. Many traditional and recently discovered areas of application of linear fractional transformations would benefit from a better understanding of the behaviour of linear fractional relations. The present paper is devoted to analogues of the Liouville theorem “a bounded entire function is constant” for linear fractional relations.

1. INTRODUCTION

The subject of the present paper belongs to both linear and non-linear analysis: it considers some problems of infinite-dimensional holomorphic analysis of multivalued non-linear maps, constructed via linear bounded operators between Banach spaces.

Consider a bounded linear operator T between Banach spaces \mathcal{B} , \mathcal{B}' which can be decomposed into direct sums $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$, $\mathcal{B}' = \mathcal{B}'_1 \oplus \mathcal{B}'_2$. Such linear operator can be represented by a 2×2 operator matrix of the form

$$(1) \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

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where $T_{i,j} \in \mathcal{L}(\mathcal{B}_j, \mathcal{B}'_i)$, $i, j = 1, 2$. (By $\mathcal{L}(\mathcal{B}_j, \mathcal{B}'_i)$ we denote the space of bounded linear operators acting from \mathcal{B}_j to \mathcal{B}'_i ($i, j = 1, 2$)).

With each such matrix T one can associate a map (defined on some, possibly empty, part of $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$) by the formula

$$(2) \quad H_T(X) = (T_{21} + T_{22}X)(T_{11} + T_{12}X)^{-1}.$$

Such maps are called (operator) *linear fractional transformations*.

Krein [16, 17] discovered that linear fractional transformations can serve as a powerful tool in the study of operators on indefinite metric spaces. The theory of linear fractional transformations, with their comparatively simple algebraic and complicated analytic properties, is an interesting *subject* of investigation which attracted many prominent mathematicians (Helton, Iokhvidov, Krein, Langer, Shmulian, and others, see [2, 7, 8, 18, 19, 20]). Operator linear fractional transformations found applications in non-linear holomorphic analysis in Banach spaces (see, for example, [3, 4, 6, 15] and references therein), to Koenigs embedding problem, Abel-Schröder equations, composition operators on Hardy and Bergman spaces, theory of generators of non-linear semigroups, and to many other problems (see [1, 3, 4, 6, 13, 14, 22], and references therein). In most of these applications the requirement that $(T_{11} + T_{12}X)$ is invertible (needed to define a linear fractional transformation) is not natural and is quite restrictive. In this connection it became important to generalise results of the theory of linear fractional transformations to the case when $(T_{11} + T_{12}X)$ is not invertible. Work in this direction has been done in [9, 10, 11, 12]. In these papers results of the theory of linear fractional transformations were generalised through the study of *multivalued* maps defined in the following way.

DEFINITION 1: The map G_T from $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ into the set of closed affine subspaces of $\mathcal{L}(\mathcal{B}'_1, \mathcal{B}'_2)$, defined by

$$(3) \quad G_T(X) = \{Y \in \mathcal{L}(\mathcal{B}'_1, \mathcal{B}'_2) : T_{21} + T_{22}X = Y(T_{11} + T_{12}X)\},$$

is called a *linear fractional relation* (associated with T).

DEFINITION 2: A linear fractional relation G_T is said to be *defined* at X if $G_T(X) \neq \emptyset$. The set of all $X \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ at which G_T is defined is called the *domain* of G_T and is denoted by $\text{dom } G_T$.

An interesting (though somewhat vague) problem is: how should one define and check “holomorphic” properties of multivalued maps? In this paper we mainly consider, for linear fractional relations (which without doubt can be considered as “holomorphic” multivalued maps and expected to have the corresponding behaviour) one such property - the validity of analogues of Liouville’s theorem “a bounded entire function is constant”. Clearly for single-valued holomorphic (in any reasonable sense) maps between Banach

spaces the direct analogue is valid. In the multivalued case one should first define constant and bounded maps. We use the following definitions. Let G be a multivalued map from an arbitrary set S into a Banach space \mathcal{B} . We say that G is *constant* on a subset U of S if there is an element y belonging to $G(s)$ for all $s \in U$. The map G is called *bounded* on U if there is $C > 0$ such that $\inf_{y \in G(s)} \|y\| \leq C$ for each $s \in U$.

It was noted in [12] that even in the case when \mathcal{B} and \mathcal{B}' are Hilbert spaces, there are non-constant and non-linear linear fractional relations defined on the whole $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$. We shall give such an example for the convenience of the reader, because in [12] it was not clearly written out.

Let T_{11} be an isometry with the image $Z \subset \mathcal{B}'_1$ of infinite codimension. Let T_{12} be a compact operator whose image is orthogonal to Z . Operators T_{2i} can be arbitrary. Then, for each X , the operator $T_{11} + T_{12}X$ has trivial kernel and a closed image. Since \mathcal{B}_1 is a Hilbert space, it follows that $G_T(X)$ is non-empty. The map G_T is clearly non-constant if, for example, T_{22} is a non-compact operator.

We are going to study the following problem on analogues of the Liouville theorem for linear fractional relations: can maps G_T be defined and bounded on $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ without being constant? The answer is surprising: it depends on the geometry of the spaces. Namely, for reflexive spaces the answer is negative (Theorem 1), but in general the answer is affirmative (Theorem 2).

Theorem 1 actually proves (for spaces complemented in their second duals) much more than an analogue of the Liouville theorem: if a linear fractional transformation is bounded on its domain, then it is constant. For linear fractional transformations this "strong Liouville theorem" (without any restrictions on the Banach spaces involved) will be established in Theorem 3.

In the last section of the work we consider similar problems for Banach (mostly C^* -) algebras.

2. PRELIMINARY RESULTS

For linear fractional relations the notion of constant maps introduced above can be written in the following way.

DEFINITION 3: A linear fractional relation G_T is called *constant* on $A \subset \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ if $A \subset \text{dom } G_T$ and there exists $W \in \mathcal{L}(\mathcal{B}'_1, \mathcal{B}'_2)$ such that $W \in G_T(X)$ for each $X \in A$. If G_T is constant on $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ we simply say that G_T is *constant*.

It turns out that if G_T is constant on a *rich* set (see the definition below), then the second row of T is an operator multiple of the first row (see (ii) in Proposition 1). For Hilbert space operators this was observed in [12]. Below we show that the same argument works for Banach space operators.

DEFINITION 4: A subset A of $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is called *rich* if the subspace of \mathcal{B}_2 spanned by the union of all subspaces of the form $(K_1 - K_2)(\mathcal{B}_1)$, where $K_1, K_2 \in A$, is dense in \mathcal{B}_2 .

PROPOSITION 1. Let $A \subset \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ be rich. For a matrix T the following conditions are equivalent:

- (i) G_T is constant on A .
- (ii) There exists an operator W in $\mathcal{L}(\mathcal{B}'_1, \mathcal{B}'_2)$ such that

$$(4) \quad T = \begin{pmatrix} T_{11} & T_{12} \\ WT_{11} & WT_{12} \end{pmatrix}.$$

- (iii) G_T is constant.

PROOF: We start by proving (i) \Rightarrow (ii). Let G_T be constant on a rich set A and let W be an operator satisfying

$$W \in G_T(K) \quad \forall K \in A.$$

Let $K_1, K_2 \in A$. Then

$$(5) \quad T_{21} + T_{22}K_1 = W(T_{11} + T_{12}K_1)$$

and

$$(6) \quad T_{21} + T_{22}K_2 = W(T_{11} + T_{12}K_2)$$

Subtracting (6) from (5) we get

$$T_{22}(K_1 - K_2) = WT_{12}(K_1 - K_2).$$

Therefore $T_{22}x = WT_{12}x$ for each $x \in (K_1 - K_2)(\mathcal{B}_1)$. Since A is rich, it implies $T_{22} = WT_{12}$.

Now we derive $T_{21} = WT_{11}$ from either (5) or (6).

The implications (ii) \Rightarrow (iii) \Rightarrow (i) are obvious. □

REMARK. In general, if G_T is constant on its domain, it does not imply that the second row of T is a multiple of the first. As we shall see in Theorem 1, the only example of this kind is

$$T = \begin{pmatrix} 0 & 0 \\ 0 & T_{22} \end{pmatrix}$$

with $T_{22} \neq 0$.

3. MAIN RESULTS

In this section we find analogues of the Liouville Theorem for linear fractional relations. The restriction of the general definition of boundedness (mentioned above) to the case of linear fraction relations is:

DEFINITION 5: A linear fractional relations G_T is called *bounded* if

$$\sup_{X \in \text{dom}(G_T)} \inf_{Y \in G_T(X)} \|Y\| < \infty.$$

Our first result is the following analogue of the Liouville Theorem for linear fraction relations between Banach space operators.

THEOREM 1. Let \mathcal{B}'_2 be such that the canonical image of \mathcal{B}'_2 is complemented in $(\mathcal{B}'_2)^{**}$. If $\text{dom } G_T \neq \emptyset$ and G_T is bounded, then either G_T is constant, or T is of the form

$$(7) \quad T = \begin{pmatrix} 0 & 0 \\ 0 & T_{22} \end{pmatrix}$$

with $T_{22} \neq 0$. In the latter case G_T is defined at X if and only if $T_{22}X = 0$, and, for such X , $G_T(X) = \mathcal{L}(\mathcal{B}'_1, \mathcal{B}'_2)$. Therefore in both cases G_T is constant on $\text{dom } G_T$.

PROOF: First we prove the theorem in the case $T_{21} = 0$. In such a case $0 \in \text{dom } G_T$, $0 \in G_T(0)$, and the equation in (3) becomes

$$(8) \quad T_{22}X = Y(T_{11} + T_{12}X).$$

Our first purpose is to prove that in this case $\text{dom } G_T$ contains all operators of finite rank. Let R be the set of operators X of rank one in $\text{dom } G_T$, such that $G_T(X)$ contains a rank one operator. An operator $X = u \otimes f$ (this means that $Xx = f(x)u$ for all x) belongs to R if and only if there is an operator $Y = p \otimes g$ satisfying (8). This condition can be written as

$$(9) \quad p \otimes (T_{11}^*g + g(T_{12}u) \cdot f) = T_{22}u \otimes f.$$

It is clear that this condition is satisfied for arbitrary u and g satisfying $g(T_{12}u) \neq 1$, if we let $f = (1 - g(T_{12}u))^{-1}T_{11}^*g$ and $p = T_{22}u$. In particular, the condition (9) is satisfied if $g(T_{12}u) = 0$, $f = T_{11}^*g$, and $p = T_{22}u$.

For $x \in \mathcal{B}_1$ we denote by $W(x)$ the set of all vectors $y \in \mathcal{B}_2$ which can be written in the form $y = Xx$, for some $X \in R$.

CASE 1. $T_{11} \neq 0$. Suppose that $T_{11}x \neq 0$. Let $X = u \otimes f$, we have $Xx = f(x)u$. Take $g_0 \in (\mathcal{B}'_1)^*$ with $g_0(T_{11}x) \neq 0$, $g = \lambda g_0$, and $f = (\lambda / (1 - \lambda g(T_{12}u))) T_{11}^*g_0$. Then

$$Xx = \frac{\lambda}{1 - \lambda g(T_{12}u)} \cdot g_0(T_{11}x)u$$

can be arbitrarily close to u (if one choose λ in a proper way). This means that $W(x)$ is dense in B_2 if $T_{11}x \neq 0$. Therefore for $T_{11} \neq 0$ the set Q of all pairs (x, y) satisfying $y \in W(x)$ is dense in $B_1 \times B_2$.

Let

$$C = \sup_{X \in \text{dom } G_T} \inf_{Y \in G_T(X)} \|Y\|.$$

Then

$$(10) \quad \|T_{22}Xx\| \leq C\|(T_{11} + T_{12}X)x\|$$

for all $X \in R$ and all x . Hence

$$(11) \quad \|T_{22}y\| \leq C\|T_{11}x + T_{12}y\|$$

for all $(x, y) \in Q$. Since Q is dense, we may assume that (11) holds for all x, y .

In particular, the inequality (10) holds for each X and all x . So, for a fixed finite rank operator X , setting $Y(T_{11} + T_{12}X)x = T_{22}Xx$ we define a bounded operator Y on the linear subspace $(T_{11} + T_{12}X)B_1$. Being finite rank it extends to whole B_2 and clearly belongs to $G_T(X)$. Thus $\text{dom } G_T$ contains all operators of finite rank.

Now we show that for each triple (M, N, ε) , where M and N are finite dimensional subspaces in B_1 and B_2 , respectively, $\varepsilon > 0$; there exists an operator $Q_{M,N,\varepsilon} : B'_1 \rightarrow B'_2$ such that $\|Q_{M,N,\varepsilon}\| \leq C$ and the following two conditions are satisfied:

$$(12) \quad Q_{M,N,\varepsilon}T_{11}x = 0 \text{ for each } x \in M,$$

$$(13) \quad \|Q_{M,N,\varepsilon}T_{12}y - T_{22}y\| \leq \varepsilon \text{ for each } y \in B_N$$

(by B_B we denote the unit ball of a Banach space B).

Let $X \in \mathcal{L}(B_1, B_2)$ be an operator of finite rank, such that

$$(14) \quad X|_M = 0$$

and

$$(15) \quad X(B_{B_1}) \supset \left(\frac{C\|T_{11}\|}{\varepsilon}\right)B_N.$$

(Here we use the assumption that the space B_1 is infinite dimensional.)

Since X is in the domain of G_T , there exists an operator $Q_{M,N,\varepsilon}$ such that $\|Q_{M,N,\varepsilon}\| \leq C$ and

$$Q_{M,N,\varepsilon}(T_{11} + T_{12}X) = T_{22}X$$

or

$$(16) \quad Q_{M,N,\varepsilon}T_{12}X - T_{22}X = -Q_{M,N,\varepsilon}T_{11}.$$

By (14) the condition (16) immediately implies (13).

Also (16) implies

$$\|Q_{M,N,\varepsilon}T_{12}X - T_{22}X\| \leq C \|T_{11}\|.$$

By condition (15) it implies

$$\left(\frac{C \|T_{11}\|}{\varepsilon}\right) \|Q_{M,N,\varepsilon}T_{12}y - T_{22}y\| \leq C \|T_{11}\|$$

for each $y \in B_N$. The condition (13) follows.

We endow the set of all triples (M, N, ε) with the following ordering:

$$(M_1, N_1, \varepsilon_1) \succ (M_2, N_2, \varepsilon_2) \text{ if and only if } M_1 \supset M_2, N_1 \supset N_2, \text{ and } \varepsilon_1 \leq \varepsilon_2.$$

Let \mathcal{U} be an ultrafilter majorising this ordering. The set of all linear operators from B'_1 into $(B'_2)^{**}$ with norm $\leq C$ is compact in the pointwise weak* topology. Hence the image of the ultrafilter \mathcal{U} under the map $(M, N, \varepsilon) \mapsto Q_{M,N,\varepsilon}$ is convergent in this set. Let $\tilde{Q} = w^* - \lim_{\mathcal{U}} Q_{M,N,\varepsilon}$ be the corresponding limit. Let $P : (B'_2)^{**} \rightarrow B'_2$ be a bounded linear projection (whose existence is one of the conditions of Theorem 1). We let $Q = P\tilde{Q}$. It is easy to check that $QT_{12} = T_{22}$ and $QT_{11} = 0$. By Proposition 1, the linear fractional relation G_T is constant.

CASE 2. $T_{11} = 0$. In this case the equation has the form: $YT_{12}X = T_{22}X$. Take $X = u \otimes f$, then it is easy to see that $X \in \text{dom } G_T$, if $T_{12}u \neq 0$. Hence for $T_{12} \neq 0$, $W(x)$ is dense in B_2 and $\|T_{22}y\| \leq C\|T_{12}y\|$ for all $y \in B_2$. This means that $T_{22} = QT_{12}$, for some $Q \in \mathcal{L}(B'_1, B'_2)$, and G_T is constant.

It remains to consider the case when both $T_{11} = 0$ and $T_{12} = 0$. In this case T is of the form given in the statement of Theorem 1. It is easy to see that all relevant statements of Theorem 1 are valid. We have proved Theorem 1 in the case when $T_{21} = 0$.

Now we consider the general case. Let $X_0 \in \text{dom } G_T$ and $Y_0 \in G_T(X_0)$. Then

$$T_{21} + T_{22}X_0 = Y_0(T_{11} + T_{12}X_0).$$

Subtracting this equation from (3) and using simple algebraic transformations, we get

$$(T_{22} - Y_0T_{12})(X - X_0) = (Y - Y_0)((T_{11} + T_{12}X_0) + T_{12}(X - X_0)).$$

Hence the equation (3) is equivalent to

$$(17) \quad \tilde{T}_{22}\tilde{X} = \tilde{Y}(\tilde{T}_{11} + \tilde{T}_{12}\tilde{X}),$$

where $\tilde{X} = X - X_0$, $\tilde{Y} = Y - Y_0$, $\tilde{T}_{12} = T_{12}$, $\tilde{T}_{11} = T_{11} + T_{12}X_0$, and $\tilde{T}_{22} = T_{22} - T_{12}Y_0$. The equation (17) describes $G_{\tilde{T}}$ for

$$\tilde{T} = \begin{pmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ 0 & \tilde{T}_{22} \end{pmatrix}.$$

It is clear that $0 \in \text{dom } G_{\tilde{T}}$ and that $G_{\tilde{T}}$ is bounded. Therefore the argument above implies that either $G_{\tilde{T}}$ is constant, or \tilde{T} satisfies

$$(18) \quad \tilde{T}_{11} = \tilde{T}_{12} = \tilde{T}_{21} = 0.$$

It is easy to see that “ $G_{\tilde{T}}$ is constant” implies “ G_T is constant”, and that (18) implies that T is of the form (7). □

In particular, the result holds when \mathcal{B}'_2 is a reflexive space.

The following lemma shows that in the reflexive case it is enough to require boundedness of G_T on a weakly dense subspace in $\text{dom } G_T$ only. (The basic facts about the weak operator topology and the strong operator topology which we use below can be found in [5, Chapter V].)

LEMMA 1. *Suppose that the space \mathcal{B}'_2 is reflexive. A bounded linear fractional relation G_T on a subspace \mathcal{R} of $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ can be extended to a bounded linear fractional relation on the closure \mathcal{W} of \mathcal{R} in the weak operator topology.*

PROOF: Since the closures of a linear subspace in the weak operator topology and the strong operator topology coincide, for each $X \in \mathcal{W}$ there is a net $X_\alpha \in \mathcal{R}$ which converges to X in the strong operator topology. Let $Y_\alpha \in G_T(X_\alpha)$; by the boundedness and reflexivity conditions we may assume that the net Y_α converges to some $Y \in \mathcal{W}$ in the weak operator topology. Then

$$Y(T_{11} + T_{12}X) - (T_{21} + T_{22}X) = (Y - Y_\alpha)(T_{11} + T_{12}X) + Y_\alpha T_{12}(X - X_\alpha) + T_{22}(X_\alpha - X) \rightarrow 0$$

in the weak operator topology. □

At this moment it is not clear to what extent the restriction on \mathcal{B}'_2 in Theorem 1 can be relaxed. Our next result shows that some restrictions on \mathcal{B}'_2 are necessary for an analogue of the Liouville theorem for linear fractional relations to be valid. Below we assume that all direct sums are in ℓ_∞ sense. It means that $\|(x, y)\| = \max\{\|x\|, \|y\|\}$ for $(x, y) \in X \oplus Y$. Necessary background in Banach space theory can be found in [21].

THEOREM 2. *If $\mathcal{B}_1 = \mathcal{B}'_1 = \ell_\infty \oplus c_0$ and $\mathcal{B}_2 = \mathcal{B}'_2 = \ell_\infty \oplus c_0 \oplus \ell_1(\Gamma)$, where Γ has the cardinality of continuum, then there exists T of the form (1) such that the linear fractional relation G_T is bounded, $\text{dom } G_T = \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$, but G_T is not constant.*

PROOF: We need the following properties of the introduced objects

- (1) The space \mathcal{B}_1 is isometric to $\mathcal{B}_1 \oplus \mathcal{B}_1$ (because the direct sums are in ℓ_∞ sense).
- (2) \mathcal{B}_1 is isometric to an uncomplemented subspace of \mathcal{B}_1 . This fact immediately follows from the well-known fact: the canonical image of c_0 in ℓ_∞ is uncomplemented (see [21, Section II.4.n]). Let $A : \mathcal{B}_1 \rightarrow \mathcal{B}_1$ be such an isometry.

- (3) Each operator from \mathcal{B}_1 into $\ell_1(\Gamma)$ is compact (see [21, Section I.1.b]).
- (4) For each finite set Z in \mathcal{B}_1 and each $\varepsilon > 0$ there exists a finite-dimensional subspace M in \mathcal{B}_1 , which is isometric to ℓ_∞^n for some $n \in \mathbb{N}$ and satisfies $\text{dist}(z, M) < \varepsilon$ for each $z \in Z$ (see [21, Section II.5.b]).
- (5) There exists a quotient map $\varphi : \ell_1(\Gamma) \rightarrow \mathcal{B}_1$ (see [21, p. 37]).

Now we define operators T_{ij} . In all these definitions we use representations:

$$\begin{aligned} \mathcal{B}_1 &= \mathcal{B}_1 \oplus \mathcal{B}_1, \\ \mathcal{B}_2 &= \mathcal{B}_1 \oplus \mathcal{B}_1 \oplus \ell_1(\Gamma), \end{aligned}$$

where in (19) we use the existence of isometry from (1). When we write vectors of \mathcal{B}_1 as pairs, and vectors of \mathcal{B}_2 as triples, we mean the decompositions (19) and (19).

Let $T_{11} : \mathcal{B}_1 \rightarrow \mathcal{B}_1 \oplus \mathcal{B}_1$ be defined by $T_{11}(x) = (x, 0)$ (that is, T_{11} is isometry of \mathcal{B}_1 onto its ‘half’).

$$\begin{aligned} T_{12}(z_1, z_2, z_3) &= (\varphi(z_3), A\varphi(z_3)), \\ T_{22}(z_1, z_2, z_3) &= (\varphi(z_3), \varphi(z_3), 0), \end{aligned}$$

and

$$T_{21}(x) = (x, 0, 0).$$

LEMMA 2. *The linear fractional relation G_T is not constant.*

PROOF: Assume the contrary. By Proposition 1, there exists $Q \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ such that $QT_{12} = T_{22}$ and $QT_{11} = T_{21}$. By the definitions of T_{11} and T_{21} , we get

$$Q(x, 0) = (x, 0, 0) \quad \forall x \in \mathcal{B}_1.$$

Using the definitions of T_{12} and T_{22} , and the fact that φ is a quotient map, we get

$$Q(x, Ax) = (x, x, 0).$$

Therefore

$$Q(0, Ax) = (0, x, 0).$$

Consider $R : \mathcal{B}_2 \rightarrow \mathcal{B}_1$ defined by

$$R(u, x, z) = (0, Ax).$$

Then RQ is a projection of $\mathcal{B}_1 \oplus \mathcal{B}_1$ onto $0 \oplus A\mathcal{B}_1$, the existence of such projection contradicts the fact that $A\mathcal{B}_1$ is uncomplemented in \mathcal{B}_1 . □

It remains to show that $\text{dom } G_T = \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ and that G_T is bounded. That is, we need to find for each $K : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ an operator $\alpha(K) : \mathcal{B}_1 \rightarrow \mathcal{B}_2$, such that

$$(19) \quad (T_{21} + T_{22}K) = \alpha(K)(T_{11} + T_{12}K)$$

and

$$\sup_K \|\alpha(K)\| < \infty.$$

Let $Kb = (K_1b, K_2b, K_3b)$ according to the three components of \mathcal{B}_2 . The condition (19) can be written as

$$\alpha(K)(b + \varphi(K_3b), A\varphi(K_3b)) = (b + \varphi(K_3b), \varphi(K_3b), 0).$$

We need to establish the existence of such “moderate-norm” operator $\alpha(K)$ (no matter how large the norm of K is).

The operator $\alpha(K)$ should map pairs (according to the decomposition (19)) onto triples (according to the decomposition (19)). It is easy to determine the first component of $\alpha(K)$, and to suggest the most natural third component, namely

$$\alpha(K)(x, y) = (x, ?, 0).$$

It remains to determine the operator which should replace the question mark.

The operator K_3 is compact by the condition (3) above. Hence, $A\varphi K_3(B_{\mathcal{B}_1})$ is a compact set. Let $\{x_i\}_{i=1}^n$ be an ε -net in it. By the property (4) we can find a finite-dimensional subspace $M \in \mathcal{B}_1$ isometric to ℓ_∞^m for some m such that $\text{dist}(x_i, M) < \varepsilon \forall i$. Let P be a projection of norm 1 onto M and let $x \in A\varphi K_3(B_{\mathcal{B}_1})$. There exists $y \in M$ such that $\|x - y\| \leq 2\varepsilon$. Hence $\|Px - Py\| \leq 2\varepsilon$. Since $y = Py$, we get $\|x - Px\| \leq 4\varepsilon$. In other words

$$\|(I - P)A\varphi K_3\| \leq 4\varepsilon.$$

On the other hand, it is clear that P can be considered as an operator defined on the whole space $\mathcal{B}_1 \oplus \mathcal{B}_1$, we let $P(x, 0) = 0$. We consider an auxiliary operator $U : \mathcal{B}_1 \oplus \mathcal{B}_1 \rightarrow \mathcal{B}_1 \oplus \mathcal{B}_1$ given by $U(x, y) = (x - A^{-1}Py, Py)$. The norm of this operator is ≤ 2 . Another useful property of this operator is that

$$(20) \quad U(T_{11} + T_{12}K)b = (b + (I - \tilde{P})\varphi K_3b, PA\varphi K_3b),$$

where $\tilde{P} = A^{-1}PA$. The first operator in the right-hand side of (20) is a small perturbation of the identity, namely

$$(21) \quad (1 - 4\varepsilon) \|b\| \leq \|b + (I - \tilde{P})\varphi K_3b\| \leq (1 + 4\varepsilon) \|b\|.$$

We let

$$\alpha(K)(x, y) = (x, A^{-1}Py + D(I - A^{-1}P)x, 0),$$

where D is an operator satisfying

$$D(b + (I - \tilde{P})\varphi K_3b) = \varphi K_3b - \tilde{P}\varphi K_3b.$$

Such a “moderate-norm” operator D exists because of (21) and because $(I - \tilde{P})\varphi K_3$ is a “moderate-norm” operator. □

Our next purpose is to prove an analogue of Theorem 1 for linear fractional transformations $H_T : \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) \rightarrow \mathcal{L}(\mathcal{B}'_1, \mathcal{B}'_2)$ of the form

$$(22) \quad H_T(X) = (T_{21} + T_{22}X)(T_{11} + T_{12}X)^{-1}.$$

The domain of H_T is defined by $\text{dom } H_T = \{X \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) : T_{11} + T_{12}X \text{ is invertible}\}$. It is clear that $\text{dom } H_T \neq \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ for each T with $T_{12} \neq 0$. For linear fractional transformations we prove the following analogue of Theorem 1 (without any restrictions on the geometry of Banach spaces involved).

THEOREM 3. *If a linear fractional transformation H_T is such that $\text{dom } H_T \neq \emptyset$ and H_T is bounded on $\text{dom } H_T$, then H_T is constant.*

PROOF: It is convenient to start with a change of the variable. Suppose $X_0 \in \text{dom } H_T$. Let

$$Y = (X - X_0)(T_{11} + T_{12}X_0)^{-1}.$$

Then

$$\begin{aligned} H_T(X) &= (T_{21} + T_{22}X_0 + T_{22}(X - X_0))(T_{11} + T_{12}X_0 + T_{12}(X - X_0))^{-1} \\ &= ((T_{21} + T_{22}X_0)(T_{11} + T_{12}X_0)^{-1} + T_{22}Y)(I + T_{12}Y)^{-1}. \end{aligned}$$

Hence $H_T(X) = H_S(Y)$, where S is a matrix given by

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} I & T_{12} \\ H_T(X_0) & T_{22} \end{pmatrix}.$$

Therefore

$$(23) \quad H_S(Y) = (S_{21} + S_{22}Y)(I + S_{12}Y)^{-1}.$$

We shall prove that $S_{22} = S_{21}S_{12}$. This equality immediately implies that H_S and H_T are constant.

If $S_{12} = 0$, then S_{22} should also be equal to 0 (otherwise $H_S(Y)$ is unbounded). So we suppose that $S_{12} \neq 0$.

Let $y \in \mathcal{B}_2 \setminus \ker S_{12}$ and let $x = S_{12}y$. Let $e \in \mathcal{B}'_1$ be such that $e(x) = \|x\| \|e\| = 1$. For $Y = e \otimes y$ we have $Yx = y$ and $S_{12}Y = e \otimes x$, so $\|S_{12}Y\| = 1$, $S_{12}Yx = x$.

Therefore the form of the denominator in (23) implies that $\lambda Y \in \text{dom } H_S$ for $\lambda \in (-1, 1)$. Hence

$$\|S_{21}S_{12}y + \lambda S_{22}y\| = \|(S_{21} + \lambda S_{22}Y)x\| \leq k\|(I + \lambda S_{12}Y)x\| = k\|(1 + \lambda)x\|,$$

where

$$k = \sup_{Y \in \text{dom } H_S} \|H_S(Y)\|.$$

Letting $\lambda \searrow -1$ we get $S_{21}S_{12}y = S_{22}y$.

Since $\mathcal{B}_2 \setminus \ker S_{12}$ is dense in \mathcal{B}_2 (we have assumed that $S_{12} \neq 0$), the same is true for all $y \in \mathcal{B}_2$. Hence $S_{21}S_{12} = S_{22}$. □

4. LINEAR FRACTIONAL RELATIONS IN C^* -ALGEBRAS

Our next purpose is to find analogues of the Liouville Theorem for linear fractional relations on Banach algebras. (Theorem 1 in the case $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}'_1 = \mathcal{B}'_2$ may be considered as a result of this type.)

A C^* -algebra is called *primitive* if it has a faithful (= injective) irreducible representation. The class of primitive algebras is quite wide: it includes all simple algebras and many others.

Recall that the multiplier algebra $M(\mathcal{A})$ of a C^* -algebra \mathcal{A} may be realised as a subalgebra of the universal enveloping von Neumann algebra $W(\mathcal{A})$ consisting of all $T \in W(\mathcal{A})$ such that $T\mathcal{A} \subset \mathcal{A}$ and $\mathcal{A}T \subset \mathcal{A}$. So any representation of \mathcal{A} extends to $M(\mathcal{A})$.

We say that a C^* -algebra \mathcal{A} has property (D) if any element of \mathcal{A} is a product AV where $A \in \mathcal{A}$ is non-negative and $V \in M(\mathcal{A})$ is invertible. The property (D) holds for the algebra $\mathcal{K}(\mathcal{H})$ of all compact operators and for its unital extension. Also it holds for all finite and all purely infinite von Neumann algebras.

THEOREM 4. *Suppose that a primitive C^* -algebra \mathcal{A} has the property (D). Then a bounded linear fractional relation defined at each point of \mathcal{A} is constant.*

PROOF: Since \mathcal{A} is primitive, we may realise it as an irreducible (hence dense in the weak operator topology) subalgebra of $\mathcal{L}(\mathcal{H})$. By Lemma 1, G_T extends to a bounded linear fractional relation on $\mathcal{L}(\mathcal{H})$. By Theorem 1, there is $Q \in \mathcal{L}(\mathcal{H})$ such that $T_{21} = QT_{11}$, $T_{22} = QT_{12}$. It remains to prove that such an operator Q can be found in \mathcal{A} .

By our assumption, $T_{11} = NV$, where $N \geq 0$ and V is an invertible element of $M(\mathcal{A})$. By the remark above, V is an invertible element of $\mathcal{L}(\mathcal{H})$. We claim that for $X_0 = T_{12}^*V$, the subspace $\text{cl}((T_{11} + T_{12}X_0)\mathcal{H})$ contains both $T_{11}\mathcal{H}$ and $T_{12}\mathcal{H}$. Since V is an invertible element of $\mathcal{L}(\mathcal{H})$, we have $(T_{11} + T_{12}X_0)\mathcal{H} = (N + T_{12}T_{12}^*)\mathcal{H}$. Hence the orthogonal complement to $(T_{11} + T_{12}X_0)\mathcal{H}$ is the kernel of $N + T_{12}T_{12}^*$, which coincides with the intersection of the kernels of N and $T_{12}T_{12}^*$; the latter coincides with the kernel of T_{12}^* .

By definition, there is $Y_0 \in G_T(X_0)$ such that $Y_0(T_{11} + T_{12}X_0) = T_{21} + T_{22}X_0$. Since the same is true for Q , we get that Y_0 coincides with Q on the range of $T_{11} + T_{12}X_0$. Hence they coincide on the closure of this subspace. By the above, it follows that they coincide

on $T_{11}\mathcal{H}$ and $T_{12}\mathcal{H}$. Hence $Y_0T_{11} = T_{21}$ and $Y_0T_{12} = T_{22}$. This means that $Y_0 \in G_T(X)$ for each $X \in \mathcal{L}(\mathcal{H})$. □

Now we consider the opposite extreme and prove an analogue of the Liouville theorem for linear fractional relations on commutative C^* -algebras, that is, on algebras of continuous functions on compacta. Let Ω be a compactum, by $C(\Omega)$ we denote the space of all continuous functions on Ω with the supremum norm.

THEOREM 5. *Let $\mathcal{A} = C(\Omega)$, and let $T_{ij} \in \mathcal{A}$ ($i, j = 1, 2$). Suppose that for each $X \in \mathcal{A}$ there is $Y = Y_X \in \mathcal{A}$ such that*

$$Y_X(T_{11} + T_{12}X) = T_{21} + T_{22}X$$

and

$$(24) \quad \|Y_X\| \leq f(\|X\|), \text{ where } f(a) = o(a) \text{ as } a \rightarrow \infty.$$

Then there exists $Q \in \mathcal{A}$ such that $QT_{12} = T_{22}$, $QT_{11} = T_{21}$.

PROOF: Consider functions $X_n(\omega) \equiv n$, $n = 0, 1, 2, \dots$. Let $Y_n = Y_{X_n}$. Then

$$(25) \quad Y_0T_{11} = T_{21}.$$

For $n \geq 1$ we can rewrite the defining identity for Y_n in the form

$$Y_nT_{12} - T_{22} = -\frac{Y_n}{n}T_{11} + \frac{T_{21}}{n}.$$

By the condition (24) we get

$$\lim_{n \rightarrow \infty} (Y_n(\omega)T_{12}(\omega) - T_{22}(\omega)) = 0 \quad \forall \omega \in \Omega.$$

It follows that there exists a function Y_∞ on Ω such that

$$(26) \quad Y_\infty(\omega)T_{12}(\omega) = T_{22}(\omega).$$

Observe that the argument above does not imply that Y_∞ is continuous.

Let $Z_i = \{\omega : T_{1i}(\omega) = 0\}$, $i = 1, 2$, and let $Z = Z_1 \cap Z_2$.

We shall need the following observations.

- (α) $Y_n(\omega) = Y_0(\omega)$ for $\omega \in Z_2 \setminus Z_1$.
- (β) $Y_n(\omega) = Y_\infty(\omega)$ for $\omega \in Z_1 \setminus Z_2$, $n \geq 1$.

Statements (α) and (β) follow immediately from the definitions.

- (γ) $Y_0(\omega) = Y_\infty(\omega)$ for $\omega \in \Omega \setminus (Z_1 \cup Z_2)$.

PROOF OF (γ): Consider $\omega_0 \in \Omega \setminus (Z_1 \cup Z_2)$ and let X be the constant function $-(T_{11}(\omega_0))/(T_{12}(\omega_0))$. Then

$$Y_X(\omega) \left(T_{11}(\omega) - T_{12}(\omega) \cdot \frac{T_{11}(\omega_0)}{T_{12}(\omega_0)} \right) = T_{21}(\omega) - T_{22}(\omega) \cdot \frac{T_{11}(\omega_0)}{T_{12}(\omega_0)}.$$

Evaluating both sides at ω_0 we get

$$0 = T_{21}(\omega_0) - T_{22}(\omega_0) \cdot \frac{T_{11}(\omega_0)}{T_{12}(\omega_0)}.$$

Hence $Y_\infty(\omega_0) = Y_0(\omega_0)$. □

Now we turn to definition of Q . Observe that the equations (25) and (26) imply that $T_{11}(\omega) = T_{12}(\omega) = T_{21}(\omega) = T_{22}(\omega) = 0$ for $\omega \in Z$. Since Ω is compact (and hence each continuous function on its closed subset has a continuous extension to Ω), it is enough to define Q satisfying the conditions of Theorem 5 on $\text{cl}(\Omega \setminus Z)$.

We let

$$Q(\omega) = \begin{cases} \frac{T_{21}(\omega)}{T_{11}(\omega)} = Y_0(\omega) & \text{for } \omega \notin Z_1 \\ \frac{T_{22}(\omega)}{T_{12}(\omega)} = Y_\infty(\omega) & \text{for } \omega \in (Z_1 \setminus Z_2) \end{cases}$$

and try to extend it to $\text{cl}(\Omega \setminus Z)$.

This approach does not work if and only if there exists a point $\alpha \in \text{cl}(\Omega \setminus Z)$ such that $\lim_{\mathcal{U}_\alpha} Q$ does not exist, where \mathcal{U}_α is the filter on $\Omega \setminus Z$ given by

$$\mathcal{U}_\alpha = \{U \setminus Z : U \text{ is a neighbourhood of } \alpha\}.$$

This, in turn, can happen if and only if one of the following is true

- (1) There exists an ultrafilter \mathcal{V} majorising \mathcal{U}_α such that $\lim_{\mathcal{V}} |Q| = \infty$ or;
- (2) There exist two ultrafilters \mathcal{V} and \mathcal{W} majorising \mathcal{U}_α such that

$$\lim_{\mathcal{V}} Q \neq \lim_{\mathcal{W}} Q.$$

In the case (1) we may assume that $Z_1 \setminus Z_2 \in \mathcal{V}$. Hence we get for Y_1 (defined above)

$$\lim_{\mathcal{V}} Y_1 = \lim_{\mathcal{V}} \frac{T_{22}}{T_{12}} = \lim_{\mathcal{V}} Q.$$

A contradiction.

As for (2): without loss of generality we may assume that one of the following is true

- (a) $Z_1 \setminus Z_2$ belongs to both \mathcal{V} and \mathcal{W} .
- (b) $Z_1 \setminus Z_2 \in \mathcal{V}$ and $\Omega \setminus Z_1 \in \mathcal{W}$.

In the case (a) we get a contradiction in a straightforward way: it implies that $\lim_{\mathcal{V}} Y_1 \neq \lim_{\mathcal{W}} Y_1$, this contradicts the continuity of Y_1 .

In the case (b), let

$$K_n = \{\omega : T_{11}(\omega) + T_{12}(\omega)n \neq 0\}.$$

The statement (γ) implies that $Y_n(\omega) = Y_0(\omega) = Y_\infty(\omega)$ provided $\omega \in K_n \cap (\Omega \setminus (Z_1 \cup Z_2))$.

The case (b) contains the following subcases

(b₀) $Z_2 \setminus Z_1 \in \mathcal{W}$.

In this case

$$\lim_{\mathcal{W}} Y_1 = \lim_{\mathcal{W}} \frac{T_{21}}{T_{11}} = \lim_{\mathcal{W}} Q \neq \lim_{\mathcal{V}} Q = \lim_{\mathcal{V}} \frac{T_{22}}{T_{12}} = \lim_{\mathcal{V}} Y_1,$$

a contradiction.

(b_n) $n \geq 1$. $K_n \cap (\Omega \setminus (Z_1 \cup Z_2)) \in \mathcal{W}$.

In such a case, by the observation above

$$\lim_{\mathcal{W}} Y_n = \lim_{\mathcal{W}} Q \neq \lim_{\mathcal{V}} Q = \lim_{\mathcal{V}} \frac{T_{22}}{T_{12}} = \lim_{\mathcal{V}} Y_n,$$

a contradiction.

It remains to show that at least one of the cases (b_i) ($i = 0, 1, 2$) occurs. Assume the contrary. Using the basic properties of ultrafilters we get that \mathcal{W} contains

$$(\Omega \setminus (Z_1 \cup Z_2)) \cap \{\omega : T_{11}(\omega) + T_{12}(\omega) = 0\}$$

and

$$(\Omega \setminus (Z_1 \cup Z_2)) \cap \{\omega : T_{11}(\omega) + T_{12}(\omega) \cdot 2 = 0\}.$$

Hence \mathcal{W} contains

$$(\Omega \setminus (Z_1 \cup Z_2)) \cap \{\omega : T_{11}(\omega) + T_{12}(\omega) = 0\} \cap \{\omega : T_{11}(\omega) + T_{12}(\omega) \cdot 2 = 0\}.$$

Since this set is empty, we get a contradiction. □

REMARK. The result extends to many other Banach algebras of functions, for example the algebra $A(D)$ of bounded continuous functions on a compact $D \subset \mathbb{C}^n$ analytic in all inner points of D . For the proof it suffices to apply Theorem 5 and to note that if a continuous function f coincides with an analytic function g outside the nullset of a non-zero analytic function, then $f = g$.

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