Problem Corner

Solutions are invited to the following problems. They should be addressed to Chris Starr, (e-mail: czqstarr@gmail.com) c/o Kintail, Longmorn, Elgin IV30 8RJ and should arrive not later than 10th December 2025. Proposals for problems are equally welcome. They should also be sent to Chris Starr at the above address and should be accompanied by solutions and any relevant background information.

109.A (Stan Dolan)

Show that for $k > 1$, the Fibonacci sequence modulo 2^k is cyclic with period 3.2^{k-1}.

For example, the first 12 Fibonacci numbers modulo 4 are 1,1,2,3,1,0,1,1,2,3,1,0…

109.B (Seán Stewart)

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} \operatorname{sech}^2\left(\frac{\pi}{2^{n+1}}\right) = \frac{4}{\pi^2} - \operatorname{cosech}^2\left(\frac{\pi}{2}\right).$$

109.C (Narendra Bhandari)

Prove :

$$\int_0^1 \int_0^1 \frac{(1 + xy) \log x (\log y)^2}{(2x - 2)(1 - y)(1 - xy)^2} dx\, dy = \zeta(2) + \zeta(3)$$

109.D (Dorin Marghidanu)

Prove that if $x_1, x_2, x_3, \dots, x_n > 0$, $p_1, p_2, p_3, \dots, p_n > 0$, with $p_1 + p_2 + p_3 + \dots + p_n = 1$, and $n, r \in \mathbb{N}$, then

$$\frac{x_1 + x_2 + \dots + x_n}{n}$$

$$\leqslant \sqrt[r]{\frac{(p_1x_1 + p_2x_2 + \dots + p_nx_n)^r + (p_2x_1 + p_3x_2 + \dots + p_1x_n)^r + \dots + (p_nx_1 + p_1x_2 + \dots + p_{n-1}x_n)^r}{n}}$$

$$\leqslant \sqrt[r]{\frac{x_1^r + x_2^r + \dots + x_n^r}{n}}.$$

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Solutions and comments on **108.E**, **108.F**, **108.G**, **108.H** (July 2024).

108.E (Ovidiu Gabriel Dinu)

Find all positive integers such that each of n, $n + 2$, $n + 6$, $n + 8$ and $n + 14$ is a prime number.

Answer: $n = 5$.

Solvers of this popular problem tended to argue along similar lines to the proposer, O. G. Dimu:

If $n = 5k$, then $5k$ must be prime, giving $n = 5$ as the only possibility. The other values are therefore 7, 11, 13 and 19 which are indeed all prime.

However, if $n = 5k + 1$, $5k + 2$, $5k + 3$, $5k + 4$, then $n + 14$, $n + 8$, $n + 2$, $n + 1$ respectively are all composite. Thus, the only possibility is that the primes are 5, 7, 11, 13 and 19 as stated.

Prithwijit De offered a more original solution as follows:

All primes are of the form $6k \pm 1$, but if $n = 6k + 1$ then $n + 2 = 6k + 3$ which is composite, therefore $n = 6k - 1$. In this case, the primes would be $6k - 1$. $6k + 1$, $6k + 5$, $6k + 7$, $6k + 13$. However, if $k \equiv 1, 4, 0, 3, 2 \bmod 5$ respectively, these are all composite apart from the case when $k = 1$.

Correct solutions were received from: S. Abbott, M.V. Channakeshava, N. Curwen, H. L. Das, P. De, S. Dolan, V. Galandarli, M. Hennings, K.W. Lau, K. McClean, R. Mortini and R. Rupp, J.A. Mundie, Z. Retkes, H. Ricardo, G. Strickland, R. Summerson, L. Wimmer and the proposer, O. G. Dimu.

108.F (Peter Shiu)

Let $z = x + iy$ and $w = u + iv$ be complex numbers satisfying $z^2 + w^2 = r^2$, where $r > 0$. Show that if (x, y) runs over an ellipse with foci $\pm r$, then (u, v) runs over the same ellipse.

Solution:

Solvers employed a variety of techniques for this problem, and appreciated the opportunity to revisit the geometry of the ellipse, such as the following based on the solution offered by J. D. Mahoney:

Without loss of generality, we may define the semi-major axis of the ellipse described by (x, y) as unity, and its eccentricity may be defined as $\sin \alpha$. Therefore, the equation of the ellipse is $\frac{x^2}{1} + \frac{y^2}{b^2} = 1$, and using the definition for eccentricity, $e = \sqrt{1 - \frac{b^2}{a^2}}$ with $e = \sin \alpha$, $b = 1$ we can rewrite this as $\frac{x^2}{1} + \frac{y^2}{\cos^2 \alpha} = 1$. Furthermore, the foci are given by the

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standard formula $r = \pm\sqrt{a^2 - b^2} = \pm\sin\alpha$. This has parametrisation $(\cos\theta, \cos\alpha\sin\theta)$, so as z runs over the ellipse it has the form $z = \cos\theta + i\cos\alpha\sin\theta$. Letting $w = u + iv$, we substitute these into the given relationship $z^2 + w^2 = r^2$ to obtain:

$$(\cos\theta + i\cos\alpha\sin\theta)^2 + (u + iv)^2 = \sin^2\alpha.$$

We may now expand and use basic trigonometric identities, and we can compare real and imaginary coefficients to give $u^2 - v^2 = \sin^2\theta - \cos^2\alpha\cos^2\theta$ and $uv = -\cos\theta\cos\alpha\sin\theta$. These have the solution $u = \mp\sin\theta$, and $v = \pm\cos\alpha\cos\theta$. By direct substitution, these can be shown to satisfy $\frac{u^2}{1} + \frac{v^2}{\cos^2\alpha} = 1$, which is the same ellipse as before.

A very different approach that caught my eye was offered by R. Mortini and R. Rupp:

An ellipse with foci $\pm r$, $r > 0$ in the z-plane is given by $|z - r| + |z + r| = 2c$ for some $c > 0$. Using $z^2 + w^2$, we have $|z^2 - r^2| = w^2$ and $|w^2 - r^2| = z^2$.

Through squaring, we obtain

$$\begin{aligned}(|z-r| + |z+r|)^2 &= |z-r|^2 + |z+r|^2 + 2|z-r||z+r| \\ &= 2|z|^2 + 2r^2 + 2|z^2 - r^2| \\ &= 2|z|^2 + 2r^2 + 2|w|^2.\end{aligned}$$

Since the right-hand side is symmetric in z and w. we can therefore write:

$$(|z - r| + |z + r|)^2 = (|w - r| + |w + r|)^2$$

and since all terms are positive, we have $|z - r| + |z + r| = |w - r| + |w + r|$, so both z and w run over the same ellipse. Furthermore, P.F. Johnson was able to show, by considering $w = \pm i\sqrt{z^2 - r^2}$ that both z and w move round the ellipse in the same direction, with w 'ahead' of z.

Correct solutions were received from: N. Curwen, S. Dolan, M. Hennings, P.F. Johnson, J.D. Mahoney, R. Mortini and R. Rupp, J.A. Mundie, Z. Retkes, G. Strickland, R. Summerson, and the proposer, P. Shiu.

108.G (Seán M. Stewart)

Let $I_n = \int_0^{\pi/2} \sin^n x\, dx$ where n is a positive integer. Evaluate:

$$\lim_{n\to\infty} n\left(\prod_{k=1}^{n} I_k\right)^{\frac{2}{n}}.$$

Answer: $\frac{1}{2}\pi e$

Solvers used careful arguments to establish this result, such as the following which is based on the solution provided by the proposer, Seán Stewart.

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Using the half-angle identity $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$, we may rewrite I_k in the following form:

$$I_k = \int_0^{\frac{1}{2}\pi} \sin^k x\,dx = \frac{1}{2}\cdot 2\int_0^{\frac{1}{2}\pi} \sin^{2\left(\frac{k+1}{2}\right)-1}x.\cos^{2\left(\frac{1}{2}\right)-1}x\,dx = \frac{1}{2}B\left(\frac{n+1}{2},\frac{1}{2}\right)$$

where $B(x, y)$ represents the beta function. Using the well-known result $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we may rewrite this as

$$I_k = \frac{1}{2}\frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)}.$$

Therefore

$$\prod_{k=1}^{n} I_k = \frac{\sqrt{\pi}}{2}\frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} \times \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} \times \ldots \times \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \times \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

$$= \frac{\pi^{\frac{1}{2}n}}{2^n\Gamma\left(\frac{n+2}{2}\right)} = \frac{\pi^{\frac{1}{2}n}}{2^n\left(\frac{n}{2}\right)!}.$$

Note here that we have used the results $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

We may now use Stirling's approximation $m! \sim \sqrt{2\pi m}\left(\frac{m}{e}\right)^m$ with $m = \frac{n}{2}$ to give:

$$\lim_{n\to\infty} n\left(\prod_{k=1}^{n} I_k\right)^{2/n} = \lim_{n\to\infty} n\left(\frac{\pi^{n/2}}{2^n\sqrt{\pi n}}\left(\frac{2e}{n}\right)^{n/2}\right)^{2/n} = \lim_{n\to\infty} n\left(\frac{\pi e}{2n\pi^{1/n}n^{1/n}}\right) = \frac{\pi e}{2}\lim_{n\to\infty}\frac{1}{(\pi n)^{1/n}}.$$

If we denote the remaining limit as L, then

$$\log L = \log\lim_{n\to\infty}\frac{1}{(\pi n)^{1/n}} = -\lim_{n\to\infty}\frac{\log \pi n}{n} = -\lim_{n\to\infty}\frac{1}{n} = 0$$

where l'Hôpital's rule has been used for the final step. Thus $L = e^0 = 1$, and we obtain $\lim_{n\to\infty} n\left(\prod_{k=1}^{n} I_k\right)^{2/n}$ as required.

Rob Summerson investigated $p_n = n\left(\prod_{k=1}^{n} I_k\right)^{2/n}$ for various values of n, and noted that the convergence is slow, as demonstrated by the following table:

n	10	25	50	100	150	limit
p_n	3.395738	3.584092	3.858638	4.031145	4.098136	4.269867

Correct solutions were received from: U. Abel, N. Curwen, H.L. Das, S. Dolan, M. Hennings, P.F. Johnson, R. Mortini and R. Rupp, J.A. Mundie, Z. Retkes, R. Summerson, S. Riccarelli and the proposer, S.M. Stewart.

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108.H (Mark Hennings)

Prove the following:

(a) $$\int_0^{\pi} \frac{x(\pi - x)}{\sin x} dx = 7\zeta(3),$$

(b) $$\int_{-\infty}^{\infty} \tan^{-1}(e^x) \tan^{-1}(e^{-x}) \, dx = \frac{7}{4}\zeta(3).$$

Solution:

There was a variety of different techniques used to establish part (a), but the proposer, Mark Hennings, dealt with it in the following manner:

Consider the integral $\int_0^{\pi} x(\pi - x) e^{-(2r+1)xi}$, $r \geqslant 0$. This can be shown to be $-\frac{4i}{(2r + 1)^3}$ upon integrating by parts twice. We therefore deduce that

$$\int_0^{\pi} \frac{x(\pi - x)}{\sin x} \left(1 - e^{-2kix}\right) dx = 2i \int_0^{\pi} x(\pi - x) \frac{1 - e^{-2kix}}{e^{ix} - e^{-ix}} dx$$
$$= 2i \sum_{r=0}^{k-1} \int_0^{\pi} x(\pi - x) e^{-(2r+1)ix} dx$$
$$= \sum_{r=0}^{k-1} \frac{8}{(2r + 1)^3},$$

for any $k \in \mathbb{N}$. Since the function is Lebesgue integrable over $[0, \pi]$, the Riemann-Lebesgue lemma tells us that

$$\int_0^{\pi} \frac{x(\pi - x)}{\sin x} dx = \lim_{k \to \infty} \int_0^{\pi} \frac{x(\pi - x)}{\sin x} \left(1 - e^{-2kix}\right) dx$$
$$= \lim_{k \to \infty} \sum_{r=0}^{k-1} \frac{8}{(2r + 1)^3} = \sum_{r=0}^{\infty} \frac{8}{(2r + 1)^3}.$$

Now, since

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{1}{1^2} + \frac{1}{3^2} + \ldots + \frac{1}{2^2} + \frac{1}{4^2} +$$
$$= \frac{1}{1^2} + \frac{1}{3^2} + \ldots + \frac{1}{8}\left(\frac{1}{1^2} + \frac{1}{3^2} + \ldots\right),$$

we have

$$\sum_{r=0}^{\infty} \frac{8}{(2r + 1)^3} = 8\zeta(3) - 8 \cdot \frac{1}{8}\zeta(3) = 7\zeta(3).$$

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H. Ricardo and J.A. Mundie pointed out that a related result,

$$\int_0^{\pi/2} \frac{x(\pi - x)}{\sin x} dx = \frac{7}{2}\zeta(3)$$

could be found in [1], and since the integrand is symmetric about the line $x = \frac{1}{2}\pi$, the result immediately follows. J.A. Mundie also stated a related formula found in [2]:

$$\int_0^{\pi/2} \frac{x(\pi - x)(\pi^2 - \pi x - x^2)}{\sin x} dx = 93\zeta(5).$$

Part (b) was dealt with neatly by Stan Dolan. If we use the substitution $\tan\frac{1}{2}\theta = e^x$, then $\frac{1}{2}\sec^2\frac{1}{2}\theta\frac{d\theta}{dx} = e^x = \tan\frac{1}{2}\theta$, so $\frac{d\theta}{dx} = \sin\theta$, and hence, using the result from part (a):

$$\int_{-\infty}^{\infty} \tan^{-1}(e^x)\tan^{-1}\left(\frac{1}{e^x}\right)dx = \int_0^{\pi} \frac{\theta}{2}\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\frac{1}{\sin\theta}d\theta = \frac{1}{4}\int_0^{\pi} \frac{\theta(\pi - \theta)}{\sin\theta}d\theta = \frac{7}{4}\zeta(3).$$

References

1. I. S. Gradshteyn & I.M. Ryzhik, *Tables of Integrals, Series & Products* (7th Edn.), Academic Press 2007.
2. Seán M Stewart, A Catalan Constant Inspired Integral Odyssey, *Math. Gaz*. **104** (November 2020) pp. 449-459.

Correct solutions were received from: N. Curwen, S. Dolan, P.F. Johnson, J.D. Mahoney, R. Mortini and R.Rupp, J.A. Mundie, Z. Retkes, H. Ricardo, R. Summerson, and the proposer, M. Hennings.

Erratum from Problem 108.D (November 2024).

The integrals I and J were defined as

$$I = \int_{-\infty}^{\infty} \sin^2(\tan x)\frac{\sin^2 x}{x^2}dx \text{ and } J = \int_{-\infty}^{\infty} \cos^2(\tan x)\frac{\cos^2 x}{x^2}dx.$$

They should have been defined as

$$I = \int_{-\infty}^{\infty} \sin^2(\tan x)\frac{\cos^2 x}{x^2}dx \text{ and } J = \int_{-\infty}^{\infty} \cos^2\left(\tan x\right)\frac{\sin^2 x}{x^2} dx.$$

My apologies for this transcription error.

10.1017/mag.2025.36 C. STARR

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