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THE EXACT NON-NULL DISTRIBUTION OF WILKS' Λ CRITERION IN THE BIVARIATE COLLINEAR CASE

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It is well-known that Wilks' Λ criterion is distributed as the product of p independent beta variables in the p-variable null-case [3]. In the collinear case, Λ is still distributed as the product of p independent beta variables, one of them following a non-central beta density. Thus when $p=2$, the exact non-null distribution of Λ in the collinear case is given by the product of two independent beta variables, one central and the other having non-centrality parameter λ. Therefore, if we let Λ be denoted by the random variable w, its distribution function is

$$F(w) = \int_{xy<w} f(x, y)\, dx\, dy \tag{1}$$

where

$$\begin{aligned} f(x, y) &= f_1(x)f_2(y) \\ &= \frac{x^{a-3/2}(1-x)^{b-1}}{\beta(a-1/2, b)} \sum_{i=0}^{\infty} \frac{y^{a-1}(1-y)^{b+i-1}}{\beta(a, b+i)} e^{-\lambda/2} \frac{(\lambda/2)^i}{i!}, \end{aligned} \tag{2}$$

$$a = (N-n)/2 > 1/2,\ b = (n-1)/2 > 0, \qquad 0 \le x, y \le 1,$$

$2a$ and $2b$ being the degrees of freedom for the error and for the hypothesis respectively.

Malik [4] uses the Mellin transform to derive the distribution of the product of two independent non-central beta variables. The distribution of w here, however, cannot be obtained from his formula, since the non-centrality is imposed on $1-y$ and not on y. We use the technique of Mellin transform as in [4], to obtain our result.

The Mellin transform $g(s)=\int_0^\infty t^{s-1}f(t)\, dt$ in our case yields

$$\begin{aligned} g_1(s) &= \frac{1}{\beta(a-1/2, b)} \int_0^1 x^{s+a-5/2}(1-x)^{b-1}\, dx \\ &= \frac{\beta(s+a-3/2, b)}{\beta(a-1/2, b)} \end{aligned} \tag{3}$$

$$\begin{aligned} g_2(s) &= \sum_{i=0}^{\infty} \int_0^1 \frac{y^{s+a-2}(1-y)^{b+i-1}}{\beta(a, b+i)} \frac{(\lambda/2)^i}{i!} e^{-\lambda/2}\, dy \\ &= \sum_{i=0}^{\infty} \frac{\beta(s+a-1, b+i)}{\beta(a, b+i)} \frac{(\lambda/2)^i}{i!} e^{-\lambda/2}. \end{aligned} \tag{4}$$

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Since the Mellin transform of the density function of the product of two independent random variables is the product of their individual Mellin transforms [2], the Mellin transform of the density function of $w=xy$ is

$$(5)\qquad g(s)=\sum_{i=0}^{\infty}\frac{\beta(s+a-3/2,\, b)\beta(s+a-1,\, b+i)}{\beta(a-1/2,\, b)\beta(a,\, b+i)}\frac{(\lambda/2)^i}{i!}e^{-\lambda/2}$$
$$=\sum_{i=0}^{\infty}\frac{\Gamma(a+b-1/2)\Gamma(a+b+i)}{\Gamma(a-1/2)\Gamma(a)}M_i\frac{(\lambda/2)^i}{i!}e^{-\lambda/2},$$

where

$$M_i=\frac{\Gamma(s+a-1)\Gamma(s+a-3/2)}{\Gamma(s+a+b+i-1)\Gamma(s+a+b-3/2)}.$$

In order to obtain the density function of w we need to find the inverse Mellin transform $f(t)=(1/2\pi i)\int_{c-i\infty}^{c+i\infty} t^{-s}g(s)\, ds$ of each term in (5). We use [1],

$$(6)\qquad M^{-1}[M_i]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}\frac{\Gamma(s+u)\Gamma(s+v)}{\Gamma(s+u+m)\Gamma(s+v+n)}\, ds$$
$$=\frac{x^u(1-x)^{m+n-1}}{\Gamma(m+n)}F(n,\, u-v+m;\, m+n;\, 1-x)$$

where $F(\alpha,\beta;\gamma;x)$ is a hypergeometric function ${}_2F_1(\cdot)$. Letting $u=a-1$, $v=a-\frac{3}{2}$, $m=b+i$, $n=b$ and $x=w$ in (6), we obtain

$$M^{-1}[M_i]=\frac{w^{a-1}(1-w)^{2b+i-1}}{\Gamma(2b+i)}F(b,\, b+i+1/2;\, 2b+i;\, 1-w).$$

Hence, the density function of w is

$$(7)\qquad f(w)=\sum_{i=0}^{\infty}\frac{\Gamma(a+b-1/2)\Gamma(a+b+i)}{\Gamma(a-1/2)\Gamma(a)\Gamma(2b+i)}w^{a-1}(1-w)^{2b+i-1}\frac{(\lambda/2)^i}{i!}e^{-\lambda/2}$$
$$\times F(b,\, b+i+1/2;\, 2b+i;\, 1-w),\qquad 0\le w\le 1.$$

In the null-case, $\lambda=0$ and (7) reduces to the product of two independent beta densities.

References

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