Can. J. Math., Vol. XXV, No. 3, 1973, pp. 611–620

GENERALIZATIONS OF NOSHIRO'S THEOREM AND THEIR APPLICATIONS

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Meier [**8**, Hauptsatz] proved a remarkable theorem concerning the boundary behavior of functions meromorphic in the upper half plane; but his techniques are very complicated. So Noshiro [**10**, p. 72–73] proved an analogous (but somewhat weaker) result to Meier's by a simple method using the theorem of Gross and Iversen.

In this paper, we sharpen and generalize Noshiro's theorem in some directions by making use of the notion "porosity", and we state some applications.

1. Notations and definitions. In the following, we denote the unit disc $\{z; |z| < 1\}$ by D, the unit circle $\{z; |z| = 1\}$ by Γ and the extended w-plane by W.

Suppose a set $A \subset \Gamma$ and a point $\zeta = e^{i\theta} \in \Gamma$ are given. For $\epsilon > 0$, we denote an arc $\{e^{i\theta'}; \theta - \epsilon < \theta' < \theta + \epsilon\}$ by $\Gamma(\epsilon, \zeta)$. Let $\gamma(\zeta, \epsilon, A)$ be the largest length of arcs contained in $\Gamma(\epsilon, \zeta)$ and not intersecting with A. The set A is *of porosity of the order* μ, $0 < \mu \leqq 1$ (or simply *of porosity* (μ)) *at* ζ, if

$$\overline{\lim_{\epsilon \to 0}} \frac{1}{\epsilon} \{\gamma(\zeta, \epsilon, A)\}^{\mu} > 0.$$

A set A is *of porosity* (μ) *on* Γ if it is so at each $\zeta \in A$. A set which is a countable sum of sets of porosity (μ) on Γ is said to be *of* σ-*porosity* (μ) *on* Γ. (The notion of "porosity" was defined originally in [**6**] for the case $\mu = 1$ and in [**14**] for the general case $0 < \mu \leqq 1$.)

A set of σ-porosity (μ) on Γ is of the first Baire category on Γ.

A set of σ-porosity (1) on Γ has no points of density with respect to outer measure (i.e., no points of outer density), hence is of measure 0 (see [**11**, p. 129, Theorem (10.2)]. However, there exists a set, which is of measure 0 and not of σ-porosity (1) on Γ (see [**5**, p. 75]).

For $0 \leqq q < \infty$, $0 < \alpha < \infty$ and $\zeta = e^{i\theta} \in \Gamma$, we denote a q-curve $\{z; \alpha|\arg(z) - \theta|^{q+1} = 1 - |z|, \arg(z) > \theta\}$ terminating at ζ by $t^+(\alpha, q)(\zeta)$. For $0 \leqq q < \infty$, $0 < \alpha < \beta < \infty$, $0 < \delta < 1$ and $\zeta \in \Gamma$, we define a *right* q-*angle* $\nabla^+(\alpha, \beta, \delta, q)(\zeta)$ at ζ as the open region lying between the q-curves $t^+(\alpha, q)(\zeta)$ and $t^+(\beta, q)(\zeta)$, and lying outside the circle $\{z; |z| = \delta\}$ of radius δ, sufficiently near to 1. The *left* q-*angle* at ζ with parameters (α, β, δ), denoted by $\nabla^-(\alpha, \beta, \delta, q)(\zeta)$, is the reflection of $\nabla^+(\alpha, \beta, \delta, q)(\zeta)$ with respect to the

Received February 23, 1972 and in revised form, May 25, 1972.

https://doi.org/10.4153/CJM-1973-062-9 Published online by Cambridge University Press

radius at ζ. When convenient, we use the shorter notations $\nabla^+(q)(\zeta)$, $\nabla^-(q)(\zeta)$ or $\nabla(q)(\zeta)$ without specifying whether it be right or left.

For $0 \leqq q < \infty$, $0 < \alpha < \infty$, $0 < \delta < 1$ and $\zeta \in \Gamma$, we define a *q-cycle* $\Omega(\alpha, \delta, q)(\zeta)$ at ζ as the open region lying between the q-curves $t^+(\alpha, q)(\zeta)$ and $t^-(\alpha, q)(\zeta)$, lying outside the circle $\{z; |z| = \delta\}$ of radius δ, sufficiently near to 1. When convenient, we use the shorter notation $\Omega(q)(\zeta)$ without specifying α. (The notion of "q-angle" was introduced originally in [**12**].)

We denote a Stolz angle terminating at $\zeta \in \Gamma$ by $\Delta(\zeta)$. The circle of radius r, $0 < r < 1$, internally tangent to Γ at $\zeta \in \Gamma$ is called a horocycle at ζ and is denoted by $h_r(\zeta)$. Given a horocycle $h_r(\zeta)$ at a point $\zeta \in \Gamma$, the region interior to $h_r(\zeta)$ is called an *oricycle* at ζ and is denoted by $K_r(\zeta)$. If $0 < r_1 < r_2 < 1$, $0 < r_3 < 1$, and if r_3 is so large that the circle $\{z; |z| = r_3\}$ intersects both horocycles at ζ with radii r_1 and r_2, then each of the two regions lying between the two horocycles $h_{r_1}(\zeta)$, $h_{r_2}(\zeta)$ at ζ as well as in the exterior of the circle $\{z; |z| = r_3\}$ is called a *horocyclic angle* at ζ and is denoted by $H_{r_1,r_2,r_3}(\zeta)$. When convenient, we use the shorter notation $H(\zeta)$ (or $K(\zeta)$) without specifying r_1, r_2 and r_3 (or r). (The notion of "horocyclic angle" was introduced originally in [**3**] which began systematic investigations of tangential boundary behaviors.)

It is easily seen that for each Stolz angle $\Delta(\zeta)$ (or each horocyclic angle $H(\zeta)$), there exists a 0-cycle $\Omega(0)(\zeta)$ (or a 1-cycle $\Omega(1)(\zeta)$) and a 0-angle $\nabla(0)(\zeta)$ (or a 1-angle $\nabla(1)(\zeta)$) satisfying $\Omega(0)(\zeta) \supset \Delta(\zeta) \supset \nabla(0)(\zeta)$ (or $\Omega(1)(\zeta) \supset H(\zeta) \supset \nabla(1)(\zeta)$) and for each oricycle $K(\zeta)$, there exists a 1-cycle $\Omega(1)(\zeta)$ satisfying $\Omega(1)(\zeta) \supset K(\zeta)$.

For a function $f(z)$ defined in D, we define, in the usual manner, the cluster sets at ζ relative to the sets $D, \nabla^+(\alpha, \beta, \delta, q)(\zeta), \nabla^-(\alpha, \beta, \delta, q)(\zeta)$ or $\Omega(\alpha, \delta, q)(\zeta)$, and denote them by $C_D(f, \zeta)$, $C_{\nabla^+(\alpha,\beta,\delta,q)(\zeta)}(f, \zeta)$, $C_{\nabla^-(\alpha,\beta,\delta,q)(\zeta)}(f, \zeta)$ or $C_{\Omega(\alpha,\beta,q)(\zeta)}(f, \zeta)$, respectively.

A point $\zeta \in \Gamma$ is said to be a *q-angular Plessner point* (or a *q-cyclic Plessner point*) of $f(z)$ provided that

$$C_{\nabla^+(q)(\zeta)}(f, \zeta) = W \quad \text{and} \quad C_{\nabla^-(q)(\zeta)}(f, \zeta) = W \quad (\text{or } C_{\Omega(q)(\zeta)}(f, \zeta) = W)$$

for each right and left q-angle (or q-cycle) at ζ. The set of all q-angular Plessner points of $f(z)$ is denoted by $I_q(f)$. (These are the notions analogous to "horocyclic angular Plessner point" in [**3**, p. 6]. See [**15**, p. 112].)

By an arc at $\zeta = e^{i\theta} \in \Gamma$ we mean a continuous curve terminating at ζ: $\Lambda(\zeta) = \{z; z = z(t)\ (t_0 \leqq t < 1),\ z(t) \to \zeta \text{ as } t \to 1\}$. By an *admissible q-arc* $\Lambda_q(\zeta)$ $(0 \leqq q < \infty)$ at ζ we mean an arc at ζ having a positive number $\rho_{\Lambda_q(\zeta)}$ such that the limit

$$\lim_{t \to 1} \frac{|\arg(z(t)) - \theta|^{q+1}}{1 - |z(t)|}$$

exists and is equal to $\rho_{\Lambda_q(\zeta)}$ (see [**13**, p. 134]). We denote the cluster set of

$f(z)$ at ζ relative to an admissible q-arc $\Lambda_q(\zeta)$ (or a chord $X(\zeta)$) at ζ by $C_{\Lambda_q(\zeta)}(f, \zeta)$ (or $C_{X(\zeta)}(f, \zeta)$).

We set

$$\Pi_{T_q}(f, \zeta) = \bigcap_{\Lambda_q(\zeta)} C_{\Lambda_q(\zeta)}(f, \zeta) \quad (\text{or } \Pi_X(f, \zeta) = \bigcap_{X(\zeta)} C_{X(\zeta)}(f, \zeta)),$$

where the intersection is taken over all admissible q-arcs $\Lambda_q(\zeta)$ (or $X(\zeta)$) at ζ. A point $\zeta \in \Gamma$ is said to be a *Meier point* of $f(z)$ provided

$$\Pi_{T_0}(f, \zeta) = C_D(f, \zeta) \subsetneqq W.$$

(Meier [**9**, p. 329] considered points $\zeta \in \Gamma$ such that $\Pi_X(f, \zeta) = C_D(f, \zeta) \subsetneqq W$. And in [**2**, p. 422] these points are called "Meier points" of $f(z)$.) We set

$$C_{A_q}(f, \zeta) = \bigcup_{\nabla(q)(\zeta)} C_{\nabla(q)(\zeta)}(f, \zeta),$$

where the union is taken over all q-angles $\nabla(q)(\zeta)$. A point $\zeta \in \Gamma$ is said to be a *pre-Meier point of order* q of $f(z)$ provided

$$\Pi_{T_q}(f, \zeta) = C_{A_q}(f, \zeta) \subsetneqq W.$$

(This notion of "pre-Meier point" was introduced originally in [**7**, p. 75] for the horocyclic case and in [**16**] for the general case.)

2. Main results.

LEMMA 1. *Let A be a subset of Γ and $0 \leqq q_1 \leqq q_2$. Suppose the set A is not of porosity $(q_1 + 1)/(q_2 + 1)$ at $\zeta \in \Gamma$. Then for arbitrarily fixed numbers α, α', β', δ', $\Omega(\alpha, \delta, q_2)(\zeta)$ is covered by the set $M = \bigcup_{\xi \in A} \nabla(\alpha', \beta', \delta', q_1)(\xi)$ where δ is sufficiently near to* 1 (see also [**15**, Lemma 1 and Lemma 2]).

Proof. Without loss of generality, we may assume that $\zeta = 1$. Now we suppose that there exists a sequence $\{z_\nu\}$, $z_\nu = r_\nu e^{i\theta_\nu}$ ($\nu = 1, 2, 3, \cdots$) such that $z_\nu \in \Omega(\alpha, \delta, q_2)(1)$, $z_\nu \notin M$ and $z_\nu \to 1$. For each z_ν, points $R_1(z_\nu)$, $S_1(z_\nu)$, $R_2(z_\nu)$ and $S_2(z_\nu)$ on Γ are decided as follows.

$R_1(z_\nu)$ (or $S_1(z_\nu)$) is the point on Γ such that the point z_ν lies on the q_1-curve $t^+(\alpha', q_1)(R_1(z_\nu))$ (or $t^+(\beta', q_1)(S_2(z_\nu))$).

$R_2(z_\nu)$ (or $S_2(z_\nu)$) is the point on Γ such that the point z_ν lies on the q_1-curve $t^-(\alpha', q_1)(R_2(z_\nu))$ (or $t^-(\beta', q_1)(S_2(z_\nu))$).

We immediately have that

$\overline{R_1(z_\nu)S_1(z_\nu)}$ (the arc length connecting $R_1(z_\nu)$ and $S_1(z_\nu)$)

$$= \overline{R_2(z_\nu)S_2(z_\nu)} = \left(\frac{1 - r_\nu}{\alpha'}\right)^{1/(q_1+1)} - \left(\frac{1 - r_\nu}{\beta'}\right)^{1/(q_1+1)},$$

$$\overline{R_i(z_\nu)\,1} = \theta_\nu + (-1)^i \left(\frac{1 - r_\nu}{\alpha'}\right)^{1/(q_1+1)} \qquad (i = 1, 2).$$

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Since $z_\nu \in \Omega(\alpha, \delta, q_2)(1)$, we have

$$|\theta_\nu| < \left(\frac{1 - r_\nu}{\alpha}\right)^{1/(q_2+1)}.$$

We set

$$\epsilon_\nu = \max\{\overline{R_1(z_\nu)\,1}, \overline{R_2(z_\nu)\,1}\}.$$

Then we have

$$\lim_{\nu\to\infty} \frac{\overline{R_1(z_\nu)S_1(z_\nu)}}{(1 - r_\nu)^{1/(q_1+1)}} > 0 \text{ and } \epsilon_\nu = O((1 - r_\nu)^{1/(q_2+1)}) \quad \text{as } \nu \to \infty.$$

Since

$(\{R_1(z_\nu)S_1(z_\nu)$ (the arc connecting $R_1(z_\nu)$ and $S_1(z_\nu))\} \cup \{R_2(z_\nu)S_2(z_\nu)\}) \cap A = \emptyset$,

we have $\gamma(1, \epsilon_\nu, A) \geqq \overline{R_1(z_\nu)S_1(z_\nu)}$. Thus we obtain

$$\begin{aligned}\overline{\lim_{\epsilon\to 0}} \frac{1}{\epsilon} \{\gamma(1, \epsilon, A)\}^{(q_1+1)/(q_2+1)} &\geqq \overline{\lim_{\nu\to\infty}} \frac{1}{\epsilon_\nu} \overline{R_1(z_\nu)S_1(z_\nu)}^{(q_1+1)/(q_2+1)} \\ &\geqq \overline{\lim_{\nu\to\infty}} \frac{(1 - r_\nu)^{1/(q_2+1)}}{\epsilon_\nu} \left(\frac{\overline{R_1(z_\nu)S_1(z_\nu)}}{(1 - r_\nu)^{1/(q_1+1)}}\right)^{(q_1+1)/(q_2+1)} > 0,\end{aligned}$$

and this contradicts the assumption that $\zeta = 1$ is not a point of porosity $(q_1 + 1)/(q_2 + 1)$ for A. Therefore, for δ sufficiently near to 1, $\Omega(\alpha, \delta, q_2)(\zeta)$ is covered by the set $M = \cup_{\xi\in A} \nabla(\alpha', \beta', \delta', q_1)(\xi)$.

From Lemma 1, we obtain:

LEMMA 2. *Let $w = f(z)$ be an arbitrary function defined in D and let $0 \leqq q_1 \leqq q_2$. Let E be a subset of Γ such that for every $\zeta \in E$, $f(z)$ assumes a fixed value $w \in W$ only finitely often in a q_1-angle $\nabla(\alpha', \beta', \delta', q_1)(\zeta)$, where α', β', δ' can vary with ζ. Then for every $\zeta \in E$ except on a set S of σ-porosity $(q_1 + 1)/(q_2 + 1)$ on Γ, $f(z)$ assumes the value $w \in W$ only finitely often in any q_2-cycle $\Omega(\alpha, \delta, q_2)(\zeta)$ where $\delta = \delta(\alpha, \zeta)$ depends on α and ζ.*

Remark 1. If we consider the contraposition of Lemma 2, we obtain the following result.

(i) *Let E be a subset of Γ. Suppose for each $\zeta \in E$ there is a number α, $0 < \alpha < \infty$, such that $f(z)$ assumes a fixed value $w \in W$, infinitely often in a q_2-cycle $\Omega(\alpha, \delta, q_2)(\zeta)$, however small the value of $1 - \delta > 0$. Then for every $\zeta \in E$ except on a set S of σ-porosity $(q_1 + 1)/(q_2 + 1)$ on Γ, $f(z)$ assumes the value w infinitely often in any q_1-angle $\nabla(q_1)(\zeta)$.*

Hence, for the case $q_1 = q_2 = 0$ in (i), we have:

(ii) *Let E be a subset of Γ such that for every $\zeta \in E$, $f(z)$ assumes a fixed value $w \in W$, infinitely often in a Stolz angle $\Delta(\zeta) \cap \{z; |z| > \delta\}$, however small the value of $1 - \delta > 0$, (hence in a 0-cycle $\Omega(0)(\zeta)$), where $\Delta(\zeta)$ (hence $\Omega(0)(\zeta)$) can vary with ζ. Then for every $\zeta \in E$ except on a set of σ-*

porosity (1) *on* Γ, $f(z)$ *assumes the value* w *infinitely often* (*in any* 0-*angle* $\nabla(0)(\zeta)$, *hence*) *in any Stolz angle* $\Delta(\zeta)$.

And for the case $q_1 = q_2 = 1$ in (i), we have:

(iii) *Let* E *be a subset of* Γ *such that for every* $\zeta \in E$, $f(z)$ *assumes a fixed value* $w \in W$, *infinitely often in a Stolz angle* $\Delta(\zeta) \cap \{z; |z| > \delta\}$, *however small* $1 - \delta > 0$, (*hence, in an* 1-*cycle* $\Omega(1)(\zeta)$), *where* $\Delta(\zeta)$ (*hence*, $\Omega(1)(\zeta)$) *can vary with* ζ. *Then for every* $\zeta \in E$ *except on a set of* σ-*porosity* (1) *on* Γ, $f(z)$ *assumes the value* w *infinitely often* (*in any* 1-*angle* $\nabla(1)(\zeta)$, *hence*) *in any horocyclic angle* $H(\zeta)$.

Proof of Lemma 2. Let $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_i\}_{i=1}^{\infty}$ and $\{\delta_i\}_{i=1}^{\infty}$ be sequences of rational numbers satisfying $0 < \alpha_i, \beta_i < \infty$, $0 < \delta_i < 1$. At each point $\zeta \in E$ we can choose positive integers k, l, m such that $\nabla(\alpha_k, \beta_l, \delta_m, q_1)(\zeta) \subset \nabla(\alpha', \beta', \delta', q_1)(\zeta)$ and $f(z)$ does not assume the value $w \in W$ in the q_1-angle $\nabla(\alpha_k, \beta_l, \delta_m, q_1)(\zeta)$.

Let $S_{k,l,m}(w)$ be the set of point $\zeta \in E$ such that $f(z)$ does not assume the value $w \in W$ in the q_1-angle $\nabla(\alpha_k, \beta_l, \delta_m, q_1)(\zeta)$. Then we have

$$E = \bigcup{}_{k,l,m} S_{k,l,m}(w).$$

Let $S_{k,l,m}'(w)$ be the set of points $\zeta \in E$ such that the set $S_{k,l,m}(w)$ is not of porosity $(q_1 + 1)/(q_2 + 1)$ at ζ. Then $S_{k,l,m}(w)$ is of porosity $(q_1 + 1)/(q_2 + 1)$ at each point $\zeta \in S_{k,l,m}(w) - S_{k,l,m}'(w)$, and also $S_{k,l,m}(w) - S_{k,l,m}'(w)$ is of porosity $(q_1 + 1)/(q_2 + 1)$ at each point $\zeta \in S_{k,l,m}(w) - S_{k,l,m}'(w)$. Hence, the set $S_{k,l,m}(w) - S_{k,l,m}'(w)$ is of porosity $(q_1 + 1)/(q_2 + 1)$ on Γ.

In each point $\zeta \in S_{k,l,m}'(w)$, a q_2-cycle $\Omega(\alpha, \delta, q_2)(\zeta)$ for arbitrarily fixed α is covered by the set

$$M = \bigcup_{\xi \in S_{k,l,m}(w)} \nabla(\alpha_k, \beta_l, \delta_m, q_1)(\xi)$$

where δ is sufficiently near to 1 (depending on α and ζ, by Lemma 1). Since $f(z)$ does not assume the value $w \in W$ in the q_1-angle $\nabla(\alpha_k, \beta_l, \delta_m, q_1)(\xi)$, $\xi \in S_{k,l,m}(w)$, $f(z)$ does not assume the value w in the q_2-cycle $\Omega(\alpha, \delta, q_2)(\zeta)$. Therefore for every $\zeta \in E$ except on the set

$$S = \bigcup_{k,l,m} (S_{k,l,m}(w) - S_{k,l,m}'(w))$$

of σ-porosity $(q_1 + 1)/(q_2 + 1)$ on Γ, $f(z)$ does not assume the value $w \in W$ in a q_2-cycle $\Omega(\alpha, \delta, q_2)(\zeta)$, where δ may vary with α, ζ, and is sufficiently near to 1. Thus for every $\zeta \in E - S$, $f(z)$ assumes the value $w \in W$, only finitely often in any q_2-cycle $\Omega(\alpha, \delta, q_2)(\zeta)$, where $\delta = \delta(\alpha, \zeta)$ depends on α and ζ.

Let $w = f(z)$ be a function defined in D and let $0 \leqq q_1 \leqq q_2$. We say that the value $w \in W$ *belongs to* $R_{q_1,q_2}^*(f, \zeta)$, if there are two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ terminating at ζ such that there exists a q_1-angle $\nabla(q_1)(\zeta)$ between $\Lambda_1(\zeta)$,

$\Lambda_2(\zeta)$ and a q_2-cycle $\Omega(q_2)(\zeta)$ containing $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ and

$$w \notin \{C_{\Lambda_1(\zeta)}(f, \zeta) \cup C_{\Lambda_2(\zeta)}(f, \zeta)\}.$$

THEOREM 1. *Let $w = f(z)$ be a meromorphic function in D and let $0 \leqq q_1 \leqq q_2$, $0 \leqq q \leqq q_2$. Let E be a subset of Γ such that for every $\zeta \in E$ $f(z)$ assumes a fixed value $w \in W$ belonging to $R_{q_1,q_2}{}^*(f, \zeta)$, only finitely often in a q-angle $\nabla(\alpha, \beta, \delta, q)(\zeta)$, where α, β and δ can vary with ζ. Then $E \cap I_{q_1}(f)$ is of σ-porosity $(q + 1)/(q_2 + 1)$ on Γ.*

Remark 2. If we consider the case $q_1 = q_2 = q = 0$, then Theorem 1 implies the Noshiro result [**10**, p. 72, Lemma 2]. It is remarkable that Noshiro's proof depends on a function-theoretical method, but our proof of Theorem 1 depends on the set-theoretical Lemma 2.

Proof. According to Lemma 2, for every $\zeta \in E$ except on a set S of σ-porosity $(q + 1)/(q_2 + 1)$ on Γ, $f(z)$ assumes the given value $w \in R_{q_1,q_2}{}^*(f, \zeta)$, only finitely often in any q_2-cycle $\Omega(q_2)(\zeta)$.

We shall show that the set $(E - S) \cap I_{q_1}(f)$ is at most countable.

For each point $\zeta \in (E - S) \cap I_{q_1}(f)$, we consider the domain $G(\zeta)$ bounded by the two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$, and lying outside the circle $\{z; |z| = \rho_\zeta\}$, where ρ_ζ is sufficiently near to 1. Since there exists a q_1-angle $\nabla(q_1)(\zeta)$ satisfying $\nabla(q_1)(\zeta) \subset G(\zeta)$ and ζ is a q_1-angular Plessner point of $f(z)$, we have $C_{G(\zeta)}(f, \zeta) = W$. Furthermore, since there exists a q_2-cycle $\Omega(q_2)(\zeta)$ satisfying $\Omega(q_2)(\zeta) \supset G(\zeta)$, $f(z)$ does not assume the value w in the domain $G(\zeta)$. Hence by the well-known theorem of Gross and Iversen (Noshiro [**10**, p. 14, Theorem 1]), there exists an asymptotic path $L(\zeta)$ inside $G(\zeta)$ terminating at ζ along which $f(z)$ converges to w. Thus we have

$$C_{\Lambda_1(\zeta)}(f, \zeta) \cap C_{L(\zeta)}(f, \zeta) = \emptyset$$

and ζ is an ambiguous point of $f(z)$. Since the set of ambiguous points of $f(z)$ is at most countable by Bagemihl's Theorem [**1**], the set $(E - S) \cap I_{q_1}(f)$ is at most countable. Since a countable subset of Γ is clearly of σ-porosity $(q + 1)/(q_2 + 1)$ on Γ, the set $E \cap I_{q_1}(f)$ is also of σ-porosity $(q + 1)/(q_2 + 1)$ on Γ.

The phrase "almost every" means the exception of a set of linear Lebesgue measure 0.

COROLLARY 1. *Let $w = f(z)$ be a meromorphic function in D and $0 \leqq q$. Let E be a subset of Γ such that for every $\zeta \in E$, $R_{0,q}{}^*(f, \zeta)$ contains a fixed value $w \in W$. Then for almost every $\zeta \in E$, either $f(z)$ has an angular limit or $f(z)$ assumes the value w infinitely often in any q-angle $\nabla(q)(\zeta)$.*

Remark 3. In the case $q = 0$, Corollary 1 is the analogue to Noshiro's result [**10**, p. 72, Theorem 3]; but the arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ in the definition of $R_{0,0}{}^*(f, \zeta)$ need not necessarily be chords.

Proof. Let E' be a subset of E such that for every $\zeta \in E'$ $f(z)$ assumes the given value w, only finitely often in a q-angle $\nabla(\alpha, \beta, \delta, q)(\zeta)$, where α, β and δ can vary with ζ. Then $E' \cap I_0(f)$ is of σ-porosity (1) on Γ by Theorem 1. Hence for almost every $\zeta \in E'$, $f(z)$ has an angular limit by Plessner's theorem.

From this Corollary, we can obtain the generalization of Meier's result [**8**, Satz 2].

COROLLARY 2. *Let $w = f(z)$ be a meromorphic function in D and E be a subset of Γ. Suppose that for every $\zeta \in E$, there are two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ terminating at ζ for which there exist two Stolz angles $\Delta'(\zeta)$ and $\Delta''(\zeta)$ such that $\Delta'(\zeta)$ lies between $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ and $\Delta''(\zeta)$ contains $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ and $f(z)$ is bounded on the two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$. Then, for almost every $\zeta \in E$, either $f(z)$ has a finite angular limit or $f(z)$ assumes ∞ infinitely often in any Stolz angle $\Delta(\zeta)$ and in any horocyclic angle $H(\zeta)$.*

Proof. If we put $q = 0$ and $w = \infty$ in Corollary 1, then for almost every $\zeta \in E$, either $f(z)$ has an angular limit or $f(z)$ assumes the value ∞ infinitely often in any 0-angle $\nabla(0)(\zeta)$, hence in any Stolz angle $\Delta(\zeta)$. Noting (iii) of Remark 1, we obtain the desired result.

As a horocyclic version of Meier's Theorem [**8**, Satz 2], we obtain from Corollary 1 the following corollary.

COROLLARY 3. *Let $w = f(z)$ be a meromorphic function in D and let E be a subset of Γ. Suppose that for every $\zeta \in E$, there are two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ terminating at ζ for which there exist a Stolz angle $\Delta(\zeta)$ and an oricycle $K(\zeta)$ such that $\Delta(\zeta)$ lies between $\Lambda_1(\zeta)$ and $\Lambda_2(\zeta)$, $K(\zeta)$ contains $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$, and $f(z)$ is bounded on the two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$. Then, for almost every $\zeta \in E$, either $f(z)$ has a finite angular limit or $f(z)$ assumes ∞ infinitely often in any horocyclic angle $H(\zeta)$.*

Proof. Put $q = 1$ in Corollary 1 and the result follows.

3. In connection with Bagemihl's Remark [**4**, p. 55, Remark], we prove two further theorems from Theorem 1.

LEMMA 3. *Let $w = f(z)$ be an arbitrary function defined in D. Then, a 1-angular Plessner point of $f(z)$ is a 0-angular Plessner point of $f(z)$ except on a set of σ-porosity $(\frac{1}{2})$ on Γ* (see [**15**, Theorem 7]).

LEMMA 4. *Let $w = f(z)$ be a meromorphic function in D and $0 \leqq q$. Then a every point $\zeta \in \Gamma$ is either a pre-Meier point of order q of $f(z)$ or a q-angular Plessner point of $f(z)$ except on a set of σ-porosity* (1) *on* Γ (see [**16**, Theorem 1]).

THEOREM 2. *Let $w = f(z)$ be a meromorphic function in D. Let E be a subset of Γ such that for every $\zeta \in E$ there are two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$, separated by a*

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Stolz angle $\Delta'(\zeta)$ or a horocyclic angle $H(\zeta)$, contained in an oricycle $K(\zeta)$ and satisfying

$$w \notin \{C_{\Lambda_1(\zeta)}(f, \zeta) \cup C_{\Lambda_2(\zeta)}(f, \zeta)\}$$

for a fixed $w \in W$. Then for every $\zeta \in E$ except on a set of σ-porosity $(\frac{1}{2})$ on Γ, either ζ is a pre-Meier point of order 1 of $f(z)$ or $f(z)$ assumes the value w infinitely often in any Stolz angle $\Delta(\zeta)$.

Remark. Since a pre-Meier point of order 1 of $f(z)$ is a pre-Meier point of order 0 of $f(z)$ except on a set of σ-porosity (1) by [**16**, Theorem 5], it is possible in Theorem 2 to replace "a pre-Meier point of order 1 of $f(z)$" by "a pre-Meier point of order 0 of $f(z)$".

Proof. Let E_0 (or E_1) be a subset of E such that for each $\zeta \in E_0$ (or E_1) there exists a Stolz angle $\Delta'(\zeta)$ (or a horocyclic angle $H(\zeta)$) separating the two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ and there exists a Stolz angle $\Delta''(\zeta)$ where $f(z)$ assumes the value w only finitely often. For each $\zeta \in E_0$ (or E_1) we can choose a 0-angle $\nabla'(0)(\zeta)$ (or a 1-angle $\nabla(1)(\zeta)$), a 1-cycle $\Omega(1)(\zeta)$ and a 0-angle $\nabla''(0)(\zeta)$ satisfying $\nabla'(0)(\zeta) \subset \Delta'(\zeta)$ (or $\nabla(1)(\zeta) \subset H(\zeta)$), $\Omega(1)(\zeta) \supset K(\zeta)$ and $\nabla''(0)(\zeta) \subset \Delta''(\zeta)$. Thus, we consider the case $q_1 = 0$, $q_2 = 1$, $q = 0$ in Theorem 1 and we find that $E_0 \cap I_0(f)$ is of σ-porosity $(\frac{1}{2})$ on Γ. Further, by Lemma 3, we find that $E_0 \cap I_1(f)$ is also of σ-porosity $(\frac{1}{2})$ on Γ. Next, considering the case $q_1 = 1$, $q_2 = 1$, $q = 0$ in Theorem 1, we find that $E_1 \cap I_1(f)$ is of σ-porosity $(\frac{1}{2})$ on Γ. Hence, for every $\zeta \in E \cap I_1(f)$ except on a set of σ-porosity $(\frac{1}{2})$ on Γ, $f(z)$ assumes the value w infinitely often in any Stolz angle $\Delta(\zeta)$.

Thus by applying Lemma 4 in the case $q = 1$, we obtain the desired result.

LEMMA 5. *Let $w = f(z)$ be a meromorphic function in D. Then every point $\zeta \in \Gamma$ is either a Meier point of $f(z)$ or a Plessner point of $f(z)$ except on a set of the first Baire category on Γ.*

Remark. Since $\Pi_{T_0}(f, \zeta) \subseteq \Pi_X(f, \zeta)$, Lemma 5 implies Meier's result [**9**, p. 330, Satz 5].

Proof. This is the case $q = 0$ in [**16**, Theorem 2].

THEOREM 3. *Let $w = f(z)$ be a meromorphic function in D. Let E be a subset of Γ such that for every $\zeta \in E$ there are two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$, separated by a Stolz angle $\Delta'(\zeta)$ or a horocyclic angle $H(\zeta)$, satisfying*

$$w \notin \{C_{\Lambda_1(\zeta)}(f, \zeta) \cup C_{\Lambda_2(\zeta)}(f, \zeta)\}$$

for a fixed $w \in W$ and there exists a q-cycle $\Omega(\alpha, q)(\zeta)$, for a sufficiently large positive integer q, containing the two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$, where α and q can vary with ζ. Then for every $\zeta \in E$ except on a set of the first Baire category on Γ, either ζ is a Meier point of $f(z)$ or $f(z)$ assumes the value w, infinitely often in any Stolz angle $\Delta(\zeta)$.

Proof. Let E_0^n (or E_1^n) be a subset of E such that for every $\zeta \in E_0^n$ (or E_1^n) there exist a Stolz angle $\Delta'(\zeta)$ (or a horocyclic angle $H(\zeta)$) separating the two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ and an n-cycle $\Omega(n)(\zeta)$ $(n \geqq 1)$ containing the two arcs $\Lambda_1(\zeta), \Lambda_2(\zeta)$ and there exists a Stolz angle $\Delta''(\zeta)$ where $f(z)$ assumes the value w only finitely often. For each $\zeta \in E_0^n$ (or E_1^n) we choose a 0-angle $\nabla'(0)(\zeta)$ (or a 1-angle $\nabla(1)(\zeta)$) and a 0-angle $\nabla''(0)(\zeta)$ satisfying $\nabla'(0)(\zeta) \subset \Delta'(\zeta)$ (or $\nabla(1)(\zeta) \subset H(\zeta)$), $\nabla''(0)(\zeta) \subset \Delta''(\zeta)$. Thus, considering the case $q_1 = 0$, $q_2 = n, q = 0$ (or $q_1 = 1, q_2 = n, q = 0$) in Theorem 1, we find that $E_0^n \cap I_0(f)$ (or $E_1^n \cap I_1(f)$) is of σ-porosity $1/(n+1)$ on Γ, hence of the first Baire category on Γ. By Lemma 3, $E_1^n \cap I_0(f)$ is also of σ-porosity $1/(n+1)$ on Γ, hence of the first Baire category on Γ. If we set

$$E' = \bigcup_n (E_0^n \cup E_1^n)$$

the set $E' \cap I_0(f)$ is also of the first Baire category on Γ. Therefore, by Lemma 5, every $\zeta \in E'$ except for a set of the first Baire category on Γ is a Meier point of $f(z)$.

On the other hand, for every $\zeta \in E - E'$, $f(z)$ assumes the fixed value w, infinitely often in any Stolz angle $\Delta(\zeta)$. Thus we obtain the desired result.

Remark 4. In view of Theorem 2 and Theorem 3, in order to treat such a problem as Bagemihl states in his Remark [**4**, p. 55], we would be required to note connections between the exceptional sets and the restrictions imposed on two arcs $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ (for example, $\Lambda_1(\zeta)$ and $\Lambda_2(\zeta)$ would have to be contained in some domain, e.g., q-cycle $\Omega(\alpha, q)(\zeta)$).

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