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# SOLVABLE AND NILPOTENT SUBGROUPS OF $GL(n, q^m)$

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**0. Introduction.** Let $V \neq 0$ be a vector space of dimension $n$ over a finite field $\mathscr{F}_1$ of order $q^m$ for a prime $q$. Of course, $GL(n, q^m)$ denotes the group of $\mathscr{F}$-linear transformations of $V$. With few exceptions, $GL(n, q^m)$ is non-solvable. How large can a solvable subgroup of $GL(n, q^m)$ be? The order of a Sylow-$q$-subgroup $Q$ of $GL(n, q^m)$ is easily computed. But $Q$ cannot act irreducibly nor completely reducibly on $V$.

Suppose that $G$ is a solvable, completely reducible subgroup of $GL(n, q^m)$. Huppert ([**9**], Satz 13, Satz 14) bounds the order of a Sylow-$q$-subgroup of $G$, and Dixon ([**5**], Corollary 1) improves Huppert's bound. Here, we show that $|G| \leqq q^{3nm} = |V|^3$. In fact, we show that

$$|G| \leqq |V|^{\alpha}/(24)^{1/3}$$

where

$$\alpha = (3 \log (48) + \log (24))/3 \log (9).$$

We also show that a bound of the form $D|V|^{\delta}$ exists only when $\delta \geqq \alpha$. Since $11/5 < \alpha < 9/4$, a quadratic bound is not possible.

Suppose now that $G$ is a nilpotent, completely reducible subgroup of $GL(V)$. We show that $|G| \leqq |V|^{\beta}/2$ where $\beta = \log (32)/\log (9)$ (note that $3/2 < \beta < 8/5$). This last problem deals with Burnside's "other" $p^a q^b$ theorem, which states:

THEOREM A. *Let $G$ be a group of order $p^a q^b$ for distinct primes $p$ and $q$ and for positive integers $a$ and $b$. If $p^a > q^b$, then $\mathbf{O}_p(G) \neq 1$ unless*

(i) *$p$ is a Mersenne prime and $q = 2$;*

(ii) *$p = 2$ and $q$ is a Fermat prime; or*

(iii) *$p = 2$ and $q = 7$.*

Coates, Dwan, and Rose [**3**] noted that Burnside's proof [**2**] was incorrect and gave a correct proof. In fact, Burnside omitted exception (iii). This exception is necessary, because an elementary abelian group of order $7^8$ is acted upon faithfully by a group of order $2^{23}$. Here, we show that if $p^a > q^{b\beta}/2$, then $\mathbf{O}_p(G) \neq 1$. This handles the exceptional cases.

In a minimal counter-example to Theorem A, $G$ has a normal elementary-abelian Sylow-$q$-subgroup, that is acted upon faithfully by a Sylow-$p$-subgroup of $G$. Consequently, Theorem A is equivalent to a number-

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theoretic statement about prime divisors of certain integers. The method of proof in [**3**] is essentially number-theoretic. We will show that by a slight strengthening of Theorem A (i.e., replacing $p^a > q^b$ by $p^a > q^b/2$), the result may be proven by a short group-theoretic proof. This improvement and method is known for both $p$ and $q$ odd (see Exercise 6.16 of [**11**]). We also mention that Glauberman [**7**] takes a different approach to Burnside's other $p^a q^b$ theorem by comparing the orders of class 2 nilpotent subgroups of $G$.

All groups considered here are finite.

**1. Burnside's other $p^a q^b$ theorem.** A 2-group $S$ is dihedral (quaternion, semi-dihedral) if $S$ contains a cyclic subgroup $A = \langle a \rangle$ of index 2 and order $2^n$ with $n$ at least 2 (2, 3 respectively), and if there is an element $y \in S$ of order 2 (4, 2 respectively) such that $a^y = a^m$ where $m = -1$ ($-1$, $-1 + 2^{n-1}$ respectively). The following theorem is due to P. Hall.

1.1. THEOREM. *Let $P \neq 1$ be a $p$-group for a prime $p$ and assume that every characteristic abelian subgroup of $P$ is cyclic. Let $S \leqq \mathbf{Z}(P)$ with $|Z| = p$. Then there exist $F$, $S \leqq P$ such that*

(i) *$S \leqq \mathbf{C}_S(F)$, $FS = P$, and $F \cap S = Z$;*

(ii) *$S$ is cyclic, or $p = 2$ and $S$ is dihedral, quaternion, or semi-dihedral;*

(iii) *$F = Z$ or $F$ is extra-special; and*

(iv) *$\exp(F) = p$ or $p = 2$.*

*Proof.* Note that $Z$ is unique. See Satz III. 13.10 of [**10**] for a proof.

The following may easily be proved as a corollary of Theorem 1.1 (see Theorem 5.4.10 of [**8**]).

1.2 THEOREM. *Let $P$ be a $p$-group for a prime $p$. Assume that every normal abelian subgroup of $P$ is cyclic. Then $P$ is cyclic; or $p = 2$ and $S$ is dihedral, quarternion, or semi-dihedral.*

A prime $q$ is a *Fermat prime* (*Mersenne prime*, respectively) if there exists an integer $n \geqq 1$ such that $q = 2^n + 1$ ($q = 2^n - 1$, respectively). Proposition 1.3 is well-known and the proof is omitted.

1.3 PROPOSITION. *Let $p$ and $q$ be primes, and let $m$ and $n$ be positive integers. If $q^n - 1 = p^m$, then*

(i) *$p = 2$ and $q$ is a Fermat prime; or*

(ii) *$q = 2$ and $p$ is a Mersenne prime.*

1.4 PROPOSITION. *Let $q$ be a prime, and let $m$ and $n$ be positive integers. If $q^m - 1 = 2^n \cdot 3$, then*

(i) *$m = 1$; or*

(ii) *$m = 2$ and $q$ is 5 or 7.*

*Proof.* Assume that $m > 1$ and note that $q$ is odd. Let $t = 1 + \ldots + q^{m-1}$, so that $t$ divides $2^n \cdot 3$. If $m$ is odd, then $t$ is odd and thus $t = 3$. This is a contradiction. Hence $m = 2k$ for an integer $k$. Then $2^n \cdot 3 = (q^k - 1)(q^k + 1)$. Since $4 \nmid (q^k - 1, q^k + 1)$ and since $q$ is odd, it follows that $6 = q^k \pm 1$. Conclusion (ii) now holds.

1.5 *Definition.* We say that the ordered pair $(p, q)$ of primes $p$ and $q$ *satisfy Property* B if none of the following occurs:

(i) $p$ is a Mersenne prime and $q = 2$;
(ii) $p = 2$ and $q$ is a Fermat prime; or
(iii) $p = 2$ and $q = 7$.

For the remainder of this section, we will let $\beta = \log(32)/\log(9)$ (i.e., $9^\beta = 32$).

1.6 THEOREM. *Let $V \neq 0$ be a faithful, completely-reducible $\mathscr{F}[G]$-module for a nilpotent group $G$ and a field $\mathscr{F}$ of characteristic $q \neq 0$. Then*

(a) $|G| \leqq |V|^\beta/2$; *and*
(b) *If $G$ is a $p$-group and if $(p, q)$ satisfies property* B, *then* $|G| \leqq |V|/2$.

*Proof.* We may assume that $|V| < \infty$, and we work by induction on $|G||V|$.

*Step* 1. $V$ is an irreducible $G$-module.

Otherwise, write $V = V_1 \oplus \ldots \oplus V_m$ for irreducible $G$-modules $V_i \neq 0$ with $m \geqq 2$. Let $C_i = \mathbf{C}_G(V_i)$ for $1 \leqq i \leqq m$. Induction yields that

$$|G/C_i| \leqq |V_i|^\beta/2 \text{ for } 1 \leqq i \leqq m,$$

and in part (b) that

$$|G/C_i| \leqq |V_i|/2 \text{ for } 1 \leqq i \leqq m.$$

Since $\cap C_i = 1$, $G$ is isomorphic to a subgroup of $G/C_1 \times \ldots \times G/C_m$. Then

$$|G| \leqq \Pi|G/C_i| \leqq \Pi|V_i|^\beta/2 \leqq |V|^\beta/2^m < |V|^\beta/2.$$

Similarly, we find that $|G| \leqq |V|/2^m < |V|/2$ for part (b).

*Step* 2. $V_N$ is homogeneous for all $N \trianglelefteq G$.

If not, choose $N \vartriangleleft G$ maximal such that $V$ is not a homogeneous $N$-module. Write $V_N = W_1 \oplus \ldots \oplus W_n$ where $n \geqq 2$ and the $W_i$ are the homogeneous components of $V$ as an $N$-module. Let $M/N$ be a chief factor of $G$. Since $V_M$ is homogeneous, Clifford's Theorem (3.4.1. of [**8**]) yields that $M/N$ transitively permutes the $W_i$. Since $M/N$ is an abelian chief factor of $G$, we must have that $M/N$ acts regularly on the $W_i$. Thus $n = |M:N|$. Let $I$ be the stabilizer in $G$ of $W_i$, so that $MI = G$ and $M \cap I = N$. Let $C = \mathbf{C}(M/N)$ and $B = C \cap I \vartriangleleft MI = G$. Since $B \leqq I$ and $B$ centralizes $M/N$, each $V_i$ is left invariant by $B$. Since $N \leqq B$, we have that $V$ is not a homogeneous $B$-module. The maximality of $N$ yields that $B = N$ and $C = M$.

Since $G$ is nilpotent and $M/N$ is a chief factor in $G$, we have that $M/N \leqq \mathbf{Z}(G/N)$. But $\mathbf{C}_G(M/N) = C = M$. Thus $G = M$. By Clifford's Theorem, each $W_i$ is a completely reducible $N$-module. As in Step 1, induction yields that $|N| \leqq |V|^\beta/2^n$. For part (b), induction also yields that $|N| \leqq |V|/2^n$. Since $|G| = n|N|$ and since $2^{x-1} \geqq x$ for $x \geqq 2$, Step 2 is now completed.

*Step* 3. Conclusion.

Since $V_N$ is homogeneous for all $N \trianglelefteq G$; it follows that every normal abelian subgroup of $G$ is cyclic (see Theorem 3.2.3 of [**8**]). Since $G$ is nilpotent, Theorem 1.2 implies that the Sylow-subgroups of $G$ are cyclic, dihedral, quaternion, or semi-dihedral. In any case, there exists a cyclic $1 \neq U \trianglelefteq G$ with index 1 or 2. Furthermore, $G = U$ if and only if $G$ is cyclic. Since $V$ is a faithful homogeneous $U$-module, $U$ permutes the non-zero elements of $V$ in $k$ orbits of size $|U|$ for some integer $k$. In particular, $|V| - 1 = k|U|$.

Since $x^{3/2} - 2x + 2 \geqq 0$ for $x \geqq 2$ and since $\beta > 3/2$, we have that

$$|U| \leqq |V| - 1 \leqq |V|^{3/2}/2 \leqq |V|^\beta/2.$$

To prove part (a), we may assume that $G$ is not cyclic, that $|G:U| = 2$, and that $2||U|$. Since $x^{3/2} - 4x + 4 \geqq 0$ for $x \geqq 16$, we have that $|V| < 16$ or that

$$|G| \leqq 2|U| \leqq 2(|V| - 1) \leqq |V|^{3/2}/2 \leqq |V|^\beta/2.$$

To prove (a), we may assume that $|V| < 16$. Since $2||U|$ and $|V| - 1 = k|U|$, we have that $|V|$ is odd. Because $GL(V)$ is not cyclic and because $|V| < 16$, we must have that $|V| = 9$. But then $|U||8$ and $|G| = 2|U| \leqq 16 = |V|^\beta/2$. This proves part (a).

We now have that $G$ is a $p$-group, that $(p, q)$ satisfies property B, that $|G:U| \leqq 2$, and that $|V| - 1 = k|U|$. Write $|V| = q^m$ and $|U| = p^n$ for some positive integers $n$ and $m$. If $k = 1$, then $p^n = q^m - 1$ and Proposition 1.3 yields a contradiction to the hypotheses. Thus $k \geqq 2$. If $G = U$, then

$$|G| = |U| = (|V| - 1)/k \leqq |V|/2.$$

We may assume that $p = 2 = |G:U|$. If $k = 2$, then $q^m - 1 = |V| - 1 = |G|$. Then $q$ is a Fermat prime, contradicting the hypotheses. If $k \geqq 4$, then

$$|G| = 2|U| = 2(|V| - 1)/k \leqq |V|/2.$$

We may assume that $k = 3$. Then $q^m - 1 = |V| - 1 = 3|U|$. Since the hypotheses imply that $q$ is not 5 nor 7, Proposition 1.4 yields that $m = 1$.

But then $GL(V)$ is cyclic. This implies that $G = U$, a contradiction. This completes the proof.

We next show that the bound in part (a) of Theorem 1.6 is in some sense "a best bound".

1.7 PROPOSITION. *Suppose that $|G| \leqq C|V|^\gamma$ whenever $V$ and $G$ satisfy the hypotheses of Theorem 1.6. Then $\beta \leqq \gamma$. Furthermore, if $\beta = \gamma$, then $1/2 \leqq C$.*

*Proof.* Given a group $H$ and a faithful irreducible $H$-module $V$, we define a group $H^*$ and a $H^*$-module $V^*$ as follows. We let $V^* = V \oplus V$ and let $H^*$ be the semi-direct product of $H \times H$ with a cyclic group of order 2 that permutes the two copies of $H$. Then $H^*$ acts faithfully and irreducible on $V$ in the obvious action.

Let $V_0$ be a vector space of dimension two over a field of order 3. Then let $H_0 \in \mathrm{Syl}_2(GL(V_0))$, so that $|H_0| = 16$ and $H_0$ acts irreducibly on $V_0$. For each $n \geqq 1$, we inductively define a group $H_n$ and an irreducible $H_n$-module $V_n$ by letting $H_n = (H_{n-1})^*$ and $(V_n) = (V_{n-1})^*$. For each $n$, $V_n$ is a faithful irreducible $H_n$-module. Then

$$\log(|V_n|) = 2^n \log(9) \text{ and}$$

$$\log(|H_n|) = 2^n \log(16) + (1 + 2 + \ldots + 2^{n-1}) \log(2).$$

For any positive number $\delta$, we have that

$$\log(|H_n|/|V_n|^\delta) = 2^n \log(32/9^\delta) + \log(1/2).$$

If $9^\delta < 32$, then $\log(|H_n|/|V_n|^\delta) \to \infty$ as $n \to \infty$. Since $9^\beta = 32$, the proposition follows.

The proof of Proposition 1.7 shows that the bound in Theorem 1.6(a) occurs for infinitely many values of $|V|$ (namely, for $\log(|V|) = 2^n \log(9)$ and $n \geqq 1$).

Burnside's other $p^a q^b$ theorem now follows as an easy corollary of Theorem 1.6(b). Also, an alternative bound can be given to handle the exceptional cases.

1.8 COROLLARY. *Assume that $G$ is a group of order $p^a q^b$ for distinct primes $p$ and $q$ and positive integers $a$ and $b$. Then*

(a) *If $p^a > q^{b\beta}/2$, then $\mathbf{O}_p(G) \neq 1$; and*

(b) *If $p^a > q^b/2$ and if $(p, q)$ satisfies property $B$, then $\mathbf{O}_p(G) \neq 1$.*

*Proof.* Assume that $\mathbf{O}_p(G) = 1$. By Burnside's "well-known" $p^a q^b$ theorem [**1**], $G$ is solvable. Thus $\mathbf{O}_q(G) \neq 1$. Let $Q = \mathbf{O}_q(G)$ and $C = \mathbf{C}_G(Q) \trianglelefteq G$. Then $\mathbf{O}_{qp}(CQ) = Q \times P_0$ for a normal $p$-subgroup $P_0$ of $G$. Since $\mathbf{O}_p(G) = 1$, we have that $P_0 = 1$ and $\mathbf{C}_G(Q) \leqq Q$. Thus a Sylow-$p$-subgroup $P$ of $G$ acts faithfully on $Q$. Thus $P$ acts faithfully on

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$Q/\Phi(Q)$, where $\Phi(Q)$ is the Fitting subgroup of $Q$ (see Theorem 5.1.4 of [**8**]). But $Q/\Phi(Q)$ is elementary abelian, and thus may be viewed as a faithful $P$-module over the field of $q$-elements. Since $(|P|, q) = 1$, $Q/\Phi(Q)$ is a completely reducible $P$-module by Maschke's Theorem (3.3.1 of [**8**]). Theorem 1.6 now implies that

$$|P| \leqq |Q/\Phi(Q)|^{\beta}/2 \leqq q^{b\beta}/2;$$

and for part (b) that $|P| \leqq q^{b}/2$.

**2. The Fitting subgroup.** An irreducible $\mathscr{F}[G]$-module $W$ is *quasi-primitive* if $W$ is a homogeneous $N$-module for all $N \trianglelefteq G$. In this section, we look at the structure of a solvable group $G$ that has a faithful quasi-primitive module. Some of the structure of $G$ is known, particularly when $\mathscr{F}$ is algebraically closed (see [**14**] or Chapter 4 of [**6**]). The following is Lemma 6.5 of [**12**]. We include a proof, since the lemma is used frequently.

2.1 LEMMA. *Assume that $Z \leqq \mathbf{Z}(E)$, that $Z$ is cyclic, and that $E/Z$ is abelian. Let $A \leqq \operatorname{Aut}(E)$ with $[E, A] \leqq Z$ and $[Z, A] = 1$. Then $|A|$ divides $|E:Z|$.*

*Proof.* We view Hom $(E/Z, Z)$ as a group with multiplication defined pointwise. For $a \in A$, define $\phi_a: E/Z \rightarrow Z$ by $\phi_a(Zx) = [x, a]$. Since $[Z, A] = 1$ and since $[E, A] \leqq Z \leqq \mathbf{Z}(E)$, $\phi_a$ is well-defined and $\phi_a \in$ Hom $(E/Z, Z)$. Furthermore, $a \rightarrow \phi_a$ is an isomorphism of $A$ into Hom $(E/Z, Z)$, because $[E, E, A] = 1$ and $A$ acts faithfully on $E$. It suffices to show that

$$|\operatorname{Hom}(E/Z, Z)| \mid |E/Z|.$$

Write $E/Z = D_1 \times \ldots \times D_n$ for cyclic groups $D_i$. Then

$$\operatorname{Hom}(E/Z, Z) = \Pi \operatorname{Hom}(D_i, Z).$$

It suffices to show $|\operatorname{Hom}(D_1, Z)| \mid |D_1|$. Since $D_1$ and $Z$ are cyclic, it follows that $|\operatorname{Hom}(D_1, Z)| = (|D_1|, |Z|)$.

We need some elementary facts about certain 2-groups. We denote the Frattini subgroup of $G$ by $S(G)$.

2.2 PROPOSITION. *Let $P$ be a dihedral, quarternion, or semi-dihedral 2-group. Then*

(a) *$|P/\Phi(P)| = 4$ and $|\mathbf{Z}(P)| = 2$; and*

(b) *If $P$ is not isomorphic to the quaternion group $Q_8$ of order 8, then $P$ has a characteristic, cyclic subgroup of index 2.*

*Proof.* If $P$ is dihedral (quaternion, semi-dihedral), then, by definition, $P$ has a cyclic subgroup $A = \langle a \rangle$ of index 2 and order $2^n$ with $n$ at least 2 (2, 3 respectively). Furthermore, there is an element $y \in P$ of order 2

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(4, 2 resp.) such that $a^y = a^m$ where $m = -1$ $(-1, -1 + 2^{n+1}$ resp.). Evidently $\mathbf{Z}(P) \leqq A$ and $|\mathbf{Z}(P)| = 2$. Since $\Phi(P)$ is the smallest $U \trianglelefteq P$ such that $P/U$ is elementary abelian, it follows that $\Phi(P) = \langle a^2 \rangle$. This yields part (a).

Direct computation shows that each element of $P - A$ has order at most 2 (4, 4 resp.). Assume that $P$ is not isomorphic to $Q_8$. Then $A$ contains (and is generated by) all elements of $P$ with order at least 4 (8, 8 resp.). Thus $A$ is characteristic in $P$.

2.3 LEMMA. *Assume that every abelian normal subgroup of $G$ is cyclic. Let $1 \neq P$ be a normal-$p$-subgroup of $G$ for a prime $p$. If $p = 2$, then assume that $G$ is solvable. Let $Z \leqq \mathbf{Z}(P)$ with $|Z| = p$. Then there exist $E, T \trianglelefteq G$ such that*

(i) *$ET = P$, $E \cap T = Z$, and $E \leqq \mathbf{C}_G(T)$;*

(ii) *$E = Z$ or $E$ is extra-special;*

(iii) *$p = 2$ or* exp $(E) = p$;

(iv) *$T$ is cyclic, or $p = 2$ and $T$ is dihedral, quaternion, or semi-dihedral;*

(v) *If $T$ is not cyclic, then there exists $U \trianglelefteq G$ with $U$ cyclic, $U \leqq T$, and $|T:U| = 2$;*

(vi) *If $Z \leqq D \leqq E$ with $D/Z$ chief in $G$, then $D$ is non-abelian; and*

(vii) *If $E > Z$, there exist $E_1, \ldots, E_n \trianglelefteq G$ such that each $E_i/Z$ is a chief factor of $G$, $E_i \leqq \mathbf{C}_G(E_j)$ for $i \neq j$, and*

$$E/Z = E_1/Z \times \ldots \times E_n/Z.$$

*Proof.* We use induction on $|P|$. We note that $Z$ is unique, since $\mathbf{Z}(P) \trianglelefteq G$.

Assume conclusions (i)–(vi). Here we prove (vii). Assume that $E > Z$ and let $E_1/Z$ be a chief factor in $G$. Since $E_1$ is non-abelian, $Z = \mathbf{Z}(E_1)$. Let $A = \mathbf{C}_G(Z) \geqq P$, so that $A \trianglelefteq G$. Let $C_1 = \mathbf{C}_A(E_1/Z)$ and let $B_1 = \mathbf{C}_A(E_1)$. Then $C_1/B_1$ acts on $E_1$ and centralizes both $E_1/Z$ and $Z$. By Lemma 2.1, $|C_1/B_1| \leqq |E_1/Z|$. But $E_1 \leqq C_1$ and $B_1 \cap E_1 = Z$. Thus $E_1B_1 = C_1$. If $E = E_1$ we are done. Thus $E_1 < E \leqq C_1$ and $E \cap B_1 > Z$. Choose a chief factor $E_2/Z$ of $G$ with $E_2 \leqq E \cap B_1$. Let $C_2 = \mathbf{C}_A(E_1E_2/Z)$ and $B_2 = \mathbf{C}_A(E_1E_2)$, so that $B_2 \cap E_1E_2 = Z$. By Lemma 2.1, $|C_2/B_2| \leqq |E_1E_2/Z|$. Since $E_1E_2 \leqq C_2$ and $E_1E_2 \cap B_2 = Z$, we have that $E_1E_2B_2 = C_2$. If $E = E_1E_2$, we are done. Otherwise $E_1E_2 < E \leqq C_2$ and $E \cap B_2 > Z$. Choose $E_3/Z$ chief in $G$ with $E_2 \leqq B_2$. Part (vii) is proved by repetition of the above arguments.

We will now prove conclusions (i)–(vi). The hypotheses imply that every characteristic abelian subgroup of $P$ is cyclic. Thus Theorem 1.1 applies. Namely, there exist $F, S \leqq P$ such that $S \leqq \mathbf{C}_G(F)$, such that $F = Z$ or $F$ is extra-special, such that exp $(F) = p$ or $p = 2$, and such that $S$ is cyclic, quaternion, dihedral, or semi-dihedral.

Assume that $p \neq 2$. Then $S = \mathbf{Z}(P)$ and $F = \{x \in P | x^p = 1\}$. Thus

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$F$ and $S$ are characteristic in $P$ and $F, S \trianglelefteq G$. If $D/Z$ is chief in $G$ and if $D \leqq F$, then $D$ is not cyclic since $\exp(D) = p$. The hypotheses imply that $D$ is non-abelian. We finish if $p \neq 2$ by setting $E = F$ and $T = S$.

We have that $p = 2$ and that $G$ is solvable. If $S$ is non-abelian of order 8, then $P$ is extra-special (see Theorem 5.5.2 of [**8**]). Thus, it involves no loss of generality to assume that $|S| \geqq 16$ if $S$ is non-abelian. Note that $Z \leqq \mathbf{Z}(G)$.

W may assume that $F > Z$. Otherwise we let $E = F = Z$ and $T = S = P$, and then finish by Proposition 2.2(b).

We may assume that $|P| \geqq 16$. For if $P$ is not cyclic and if $|P| < 16$, then we must have that $|P| = 8$ and that $P$ is either dihedral or quaternion. If there exists $R \leqq P$ with $R \trianglelefteq G$ and $|R| = 4$, we let $E = Z$ and $T = P$. Otherwise, we let $E = P$ and $T = Z$.

First assume that $S$ is non-abelian; so that $|S| \geqq 16$ and $S$ is quaternion, dihedral, or semi-dihedral. By Proposition 2.2, $Z = \mathbf{Z}(P)$ and thus $Z \leqq \Phi(P)$. By Proposition 2.2, $|S:\Phi(S)| = 4$. Since $F/Z$ is elementary abelian and $S \leqq \mathbf{C}_G(F)$, it follows that $\Phi(P) = \Phi(S)$. Since $S$ has a cyclic subgroup $U = \langle u \rangle$ of index 2, we must have that $\Phi(P) = \Phi(S) = \langle u^2 \rangle$. Now $F\langle u \rangle = \mathbf{C}_P(\Phi(P)) \trianglelefteq G$. We next apply induction and Proposition 2.2(a) to $\mathbf{C}_P(\Phi(P))$. Since $\langle u \rangle = \mathbf{Z}(\mathbf{C}_P(\Phi(P))$ and $|\langle u \rangle| \geqq 8$, we have that there exists $E \trianglelefteq G$ such that $E$ is extra-special, $E\langle u \rangle = F\langle u \rangle$, and $E \cap \langle u \rangle = Z$. Furthermore, if $Z \leqq D \leqq E$ with $D/Z$ chief in $G$, then $D$ is non-abelian. Let $C = \mathbf{C}_G(E/Z)$ and $B = \mathbf{C}_G(E)$. Then $C/B$ centralizes both $E/Z$ and $Z$. By Lemma 2.1, $|C/B| \leqq |E/Z|$. But $E \leqq C$ and $B \cap E = Z(E) = Z$. Thus $EB = C$. Since

$$C = \mathbf{C}_G(E\langle u \rangle/\langle u \rangle) = \mathbf{C}_G(F\langle u \rangle/\langle u \rangle),$$

we have that $P \leqq C$. Let $T = P \cap B \trianglelefteq G$. Since $|P:E\langle u \rangle| = 2$ and since $EB = C$ it follows that $\langle u \rangle \leqq T$ and $|T:\langle u \rangle| = 2$. Since $T \leqq \mathbf{C}_P(E)$ and $Z = \mathbf{Z}(P)$, we have that $\mathbf{Z}(T) = Z$. Since $|T:\langle u \rangle| = 2$, the hypotheses and Theorem 1.2 yield that $T$ is dihedral, quaternion, or semi-dihedral. Furthermore, $\langle u \rangle \trianglelefteq G$, since $\langle u \rangle = \mathbf{Z}(E\langle u \rangle)$. We are done if $S$ is non-abelian.

Suppose that $S = Z$, so that $F = P \trianglelefteq G$. If $D$ is non-abelian whenever $P \geqq D > Z$ with $D \trianglelefteq G$, we finish by setting $E = P$ and $T = Z$. Hence, we may choose $W \trianglelefteq G$ with $Z \leqq W$ and $W$ cyclic of order 4. Since $Z = \mathbf{Z}(P)$, it follows that $|G:\mathbf{C}_G(W)| = |P:\mathbf{C}_P(W)| = 2$. We apply induction and Proposition 2.2(a) to $\mathbf{C}_P(W)$. Since $W \leqq \mathbf{Z}(\mathbf{C}_P(W))$ since $|P| > 8$, and since $|W| = 4$; we have that there exists an extra-special group $E \trianglelefteq G$ such that $EW = \mathbf{C}_P(W)$, $W \cap E = Z$. Furthermore, $D$ is non-abelian whenever $Z < D < E$ and $D \trianglelefteq G$. Since $|P:\mathbf{C}_P(W)| = 2$, this case can now be handled in a manner similar to that in the last paragraph. Again $T = \mathbf{C}_P(E)$.

Hence we must have that $S$ is cyclic and that $S > Z$. We let $T = S$. Since $T = \mathbf{Z}(P)$, we have that $T \trianglelefteq G$. We let $Y \leqq T$ with $|Y| = 4$. Then

$$YF/Z = \{x \in P/Z | x^2 = 1\},$$

so that $YF/Z$ is characteristic in $P/Z$ and $YF \trianglelefteq G$. If $Y < H \leqq YF$ with $H \trianglelefteq G$, then $H$ is not cyclic and thus non-abelian because $\exp(YF) = 4$. Repetition of the argument in the second paragraph of this proof yields that there exist $H_1, H_2, \dots, H_m \trianglelefteq G$ with $H_i \leqq \mathbf{C}_G(H_j)$ for $i \neq j$, with each $H_i/Y$ a chief factor in $G$, such that $FY/Y = H_1/Y \times \dots \times H_m/Y$.

Let $H/Y$ be a chief factor of $G$ with $H \leqq FY$. Let $I = \mathbf{C}_G(Y)$, so that $P \leqq I \trianglelefteq G$. Let $C = \mathbf{C}_I(H/Y)$ and note that $P \leqq C \trianglelefteq G$. Let $B = \mathbf{C}_G(H) \trianglelefteq G$ so that $B \leqq I$. By Lemma 2.1, $|C/B| \leqq |H/Y|$. Since $H/Y$ is a chief factor of $G$ and $H$ is non-abelian, $Y = \mathbf{Z}(H) = B \cap H$. Since $H \leqq C$, it follows that $HB = C$. If $C = G$, then $H/Y \leqq \mathbf{Z}(G/Y)$ and $H/Y$ is cyclic. This is a contradiction, since $Y = \mathbf{Z}(H)$. Hence $C < G$. Since $I = \mathbf{C}_G(Y)$ and $|Y| = 4$, we have that $|G:I| \leqq 2$. If $C = I$, then $\mathbf{C}_{H/Y}(G/C) \neq 1$, a contradiction as $H/Y$ is a chief factor of $G$ and $C < G$. Thus $C < I$ and we may choose a chief factor $M/C$ of $G$ with $M \leqq I$. Since $\mathbf{C}_{H/Y}(M/C) = 1$, we have that $M/C$ is a $q$-group for a prime $q \neq 2$. Choose $Q/B \in \mathrm{Syl}_q(M/B)$. Since $CQ = M \leqq I$, we must have that $\mathbf{C}_{H/Y}(Q) = 1$ and $\mathbf{C}_H(Q/B) = Y$. Since $H/Y$ is elementary abelian and $Y \leqq \mathbf{Z}(H)$, it follows from Theorem 2.2.1 of [**8**] that $|H'| = 2$ and thus $H' = Z$. By Fitting's Lemma (Theorem 5.2.3 of [**8**]),

$$H/Z = Y/Z \times E_0/Z$$

where

$$E_0/Z = [H/Z, Q/B] = [H/Z, Q].$$

Since $B \leqq Q$, we have that

$$E_0 \trianglelefteq H \cdot \mathbf{N}_G(Q) = HBQ\mathbf{N}(Q) = CQ\mathbf{N}_G(Q) = M\mathbf{N}_G(Q).$$

Since $Q/B \in \mathrm{Syl}_q(M/B)$, the Frattini argument yields that $G = MN_G(Q)$. Thus $E_0 \trianglelefteq G$. Since $H$ is non-abelian, since $H/Y$ is a chief factor of $G$, and since $H = YE_0$, it follows that $E_0$ is non-abelian and $E_0/Z$ is a chief factor of $G$. In particular, $Z = \mathbf{Z}(E_0)$.

We have $H_1, \dots, H_m \trianglelefteq G$ such that $H_i \leqq \mathbf{C}_G(H_j)$ for $i \neq j$, such that $H_1/Y$ is a chief factor in $G$, and such that $FY/Y = H_1/Y \times \dots \times H_m/Y$. Thus, it follows from the last paragraph that there exist $E_1, \dots, E_m$ such that $E_iY = H_i$ for each $i$, that $E_i/Z$ is a chief factor in $G$ for each $i$, $Z = \mathbf{Z}(E_i)$ for each $i$, and $E_i \leqq \mathbf{C}_G(E_j)$ for $i \neq j$. In particular,

$$E_1E_2 \dots E_m/Z = E_1/X \times \dots \times E_m/Z.$$

We set $E = E_1E_2 \ldots E_m$. Since $E/Z$ is elementary abelian and since $Z = \mathbf{Z}(E)$, it follows that $E$ is an extra-special 2-group. Now

$$ET = (E_1 \ldots E_m)YT = H_1 \ldots H_mT = HT = P.$$

Also

$$Z \leqq E \cap T \leqq \mathbf{Z}(E) = Z,$$

so that $E \cap T = Z$. Assume that $Z < D \leqq E$ with $D \trianglelefteq G$ and $D$ abelian. The hypotheses imply that $DT$ and $DT/Z$ are cyclic. This is impossible, since $T > Z$. This completes the proof.

In the above lemma, we have that $A = \mathbf{C}_G(Z) \geqq P$. Each $E_i/Z$ may be viewed as a (not necessarily faithful) $A/P$-module over $Z_P$. We note that $A/P$ preserves a symplectic form on $E_i/Z$. We let $\mathbf{F}(G)$ denote the Fitting subgroup of $G$ (i.e., the largest normal nilpotent subgroup of $G$).

2.4 COROLLARY. *Suppose that every normal, abelian subgroup of $G$ is cyclic. Assume that $G \neq 1$ and that $G$ is solvable. Let $p_1, \ldots, p_n$ be the distinct prime divisors of $|\mathbf{F}(G)|$, and let $Z \leqq (\mathbf{F}(G))$ with $|Z| = p_1, \ldots p_n$. Let $A = \mathbf{C}_G(Z)$. Then there exist $E$, $T \lhd G$ such that*

(i) *$ET = \mathbf{F}(G)$ and $E \cap T = Z$;*

(ii) *Each Sylow subgroup of $T$ is either cyclic, dihedral, quaternion, or semi-dihedral;*

(iii) *If $T$ is not cyclic, then $T$ has a cyclic subgroup $U$ of index 2 with $U \trianglelefteq G$;*

(iv) *Each Sylow subgroup of $E$ is either cyclic of prime order, or is extra-special of prime exponent or exponent four;*

(v) *$G$ is nilpotent if and only if $G = T$;*

(vi) *$T = \mathbf{C}_G(E)$ and $\mathbf{F}(G) = \mathbf{C}_A(E/Z)$;*

(vii) *Each Sylow-subgroup of $E/Z$ is elementary abelian, and is a completely reducible (not necessarily faithful) $A/\mathbf{F}(G)$-module.*

*Proof.* Since $G \neq 1$ is solvable, we have that $\mathbf{F}(G) \neq 1$. Note that $Z$ is unique. Parts (i)–(iv) follow from Lemma 2.3. Also, $T \leqq \mathbf{C}_G(E)$. Furthermore, if $Z < D \leqq E$ with $D \trianglelefteq G$, then $D$ is non-abelian.

One direction of part (v) is trivial. Assume that $G$ is nilpotent, so that $G = \mathbf{F}(G)$. It suffices to show that $E = Z$. If not, choose $Z < Y \leqq E$ with $|Y/Z|$ prime. Then $Y$ is abelian and $Y \trianglelefteq G$. This contradiction proves part (v).

Let $B = \mathbf{C}_G(E)$ and $C = \mathbf{C}_A(E/Z)$, so that $B \leqq C \leqq A$. By Lemma 2.1, $|C/B| \leqq E/Z$. Since $E \leqq \mathbf{F}(G) \leqq C$ and $B \cap E = \mathbf{Z}(E)$, it follows that $BE = C$. From above, we have that $T \leqq B$. To prove (vi), it suffices to show that $T = B$. Assume not and choose $M/T$ chief in $G$ with $M \leqq B$. Then $M/T$ is a $q$-group for a prime $q$. Let $Q \in \mathrm{Syl}_q\,(M)$. Suppose that $p \neq q$ is a prime divisor of $T$ and that $P \in \mathrm{Syl}_p(T)$. It follows from part (iii) that $P$ has a cyclic subgroup $P_1 \neq 1$ of index 1 or 2 with $P_1 \trianglelefteq G$. Since $Q \leqq A$, $Q$ centralizes $Z \cap P_1$. Since $Z \cap P_1 \neq 1$, since $P_1$ is cyclic, and since $p \neq q$, we must have that $Q$ centralizes $P_1$. Since $Q$ centralizes both $P/P_1$ and $P_1$, and since $p \neq q$, it follows that $Q$ centralizes $P$. Hence $M$ is nilpotent. Thus

$$M \leqq \mathbf{F}(G) \cap B = \mathbf{C}_{\mathbf{F}(G)}(E) = T.$$

This contradiction completes part (vi).

Let $D$ be a Sylow-subgroup of $E/Z$ for some prime. By Lemma 2.3, $D$ is elementary abelian and $D$ is a completely reducible $G/\mathbf{F}(G)$-module. Since $A \trianglelefteq G$, part (vii) follows from Clifford's Theorem (Theorem 3.4.1 of [**8**]).

A solvable group $G$ that has a faithful quasiprimitive module will satisfy the hypotheses of Corollary 2.4 (see Theorem 3.2.3 of [**8**]).

2.5. LEMMA. *Assume that $E$, $U$, and $Z$ satisfy the conclusion of Corollary* 2.4. *Let $V$ be a faithful irreducible* $\mathscr{F}[EU]$-module *for a finite field* $\mathscr{F}$. *Let $W \neq 0$ be an irreducible $U$-submodule of $V$ and let $e = |E:Z|^{1/2}$. Then* $\dim(V) = me \dim(W)$ *for an integer $m$.*

*Proof.* Since $E$ is nilpotent and since the Sylow-subgroups of $E$ are extra-special or of prime order, $e$ is an integer. Since $V$ is faithful and irreducible and since $EU$ is nilpotent, we have that $\mathrm{char}(\mathscr{F}) \nmid |EU|$. Let $\mathscr{D}$ be the centralizer of $EU$ in $\mathrm{Hom}_{\mathscr{F}}(V, V)$, so that $\mathscr{F}$, $U \subseteq \mathscr{D}$. By Schur's Lemma (1.5 of [**11**]), $\mathscr{D}$ is a division ring. Since $|\mathscr{D}|$ is finite, $\mathscr{D}$ is a field. Then $V$ is an irreducible $\mathscr{D}[EU]$-module. Since $\mathscr{D}$ is the centralizer of $EU$ in $\mathrm{Hom}_{\mathscr{D}}(V, V)$, we have that $V$ is an absolutely irreducible $\mathscr{D}[EU]$-module. Since $\mathrm{char}(\mathscr{D}) \nmid |EU|$, it follows that $\dim_{\mathscr{D}}(V) = e$ (see Satz V.16.14 of [**10**]). Let $Y \neq 0$ be an irreducible $D[U]$-submodule. Since $U \subseteq \mathscr{D}$, we have that $\dim_{\mathscr{D}}(Y) = 1$. We also have that $Y$ is an $\mathscr{F}[U]$-submodule and that

$$\dim_{\mathscr{F}}(V)/\dim_{\mathscr{F}}(Y) = \dim_{\mathscr{D}}(V)/\dim_{\mathscr{D}}(Y) = e.$$

Since $U \leqq Z(EU)$, it follows via Clifford's Theorem that $V$ is a direct sum of isomorphic $\mathscr{F}[U]$-submodules. Hence, we may assume that $W \leqq Y$. We finish by setting $m = \dim_{\mathscr{F}}(Y)/\dim_{\mathscr{F}}(W)$.

It is in fact true that $m = 1$ above, but it is not needed here.

**3. Solvable groups.** Here we obtain a bound for the order of a solvable, completely reducible subgroup of $GL(n, q)$. Throughout this section, we let

$$\alpha = (3 \log (48) + \log (24))/3 \log (9)$$

so that $9^\alpha = 48 \cdot (24)^{1/3}$. Note that $11/5 < \alpha < 9/4$.

3.1 THEOREM. *Let $V \neq 0$ be a faithful and completely reducible $\mathscr{F}[G]$-module for some field $\mathscr{F}$ and a solvable group $G$. Then*

(a) $|G| \leqq |V|^{\alpha}/(24)^{1/3}$; *and*

(b) *If $|G|$ is odd, then $|G| \leqq |V|^2/2$.*

*Proof.* We will prove part (a). The proof for part (b) is similar to that of (a). We may assume that $|V|$ is finite. We let $\lambda = (24)^{1/3}$. To prove (a), we use induction on $|G||V|$.

*Step* 1. $V$ is an irreducible $G$-module.

Mimic Step 1 of Theorem 1.6.

*Step* 2. $V$ is an homogeneous $N$-module for all $N \trianglelefteq G$.

If not, choose $N \trianglelefteq G$ maximal such that $V_N$ is not homogeneous. Write $V = W_1 \oplus \ldots \oplus W_n$ with $n \geqq 2$ and the $W_i$ as the homogeneous components of $V_N$. Use induction as in Step 1 of Theorem 1.6 to conclude that $|N| \leqq |V|^{\alpha}/\lambda^n$.

Let $M/N$ be a chief factor of $G$, so that $M/N$ is an elementary abelian $p$-group for a prime $p$. Repetition of the argument in the first paragraph of Step 2 in Theorem 1.6 yields that $|M/N| = n$ and $\mathbf{C}_{G/N}(M/N) = M/N$. Thus $M/N$ is a faithful, irreducible $G/M$-module. Induction yields that $|G/M| \leqq n^{\alpha}/\lambda$, and hence that

$$|G| \leqq n^{\alpha+1}|V|^{\alpha}/\lambda^{n+1}.$$

We may assume that $n^{\alpha+1} > \lambda^n$. Since $\alpha \leqq 7/3$, we have that $n^{10} > 24^n$. This implies that $2 \leqq n \leqq 5$.

We have that $|G| = |G/N|\,|V^{\alpha}/\lambda^n$, that $n = |M:N|$, and that $\mathbf{C}_{G/N}(M/N) = M/N$. If $n = 4$, then

$$|G/N| \leqq 24 \text{ and } |G| \leqq 24\,|V^{\alpha}|/\lambda^4 = |V|^{\alpha}/\lambda.$$

The cases $n = 2, 3,$ and $5$ are handled similarly.

*Step* 3. Conclusion.

By Step 2, every normal abelian subgroup of $G$ is cyclic (see Theorem 3.2.3 of [**8**]). Corollary 2.4 applies to $G$ and we adopt the notation of that corollary. In particular, $A = \mathbf{C}_G(Z) \geqq \mathbf{F}(G)$. Since $Z$ is cyclic, $|G/A| \leqq |Z| \leqq |U|$. If $|T:U| = 2$, then $2||S|$ and a Sylow-2-subgroup of $Z$ is central in $G$ and $|G/A| \leqq |Z|/2$. In any case, we have that $|G/A|\,|T| \leqq |U|^2$.

If $p$ is prime and $P \in \mathrm{Syl}_p(E/Z)$, then Corollary 2.4 yields that $P$ is elementary abelian and a completely reducible $A/\mathbf{F}(G)$-module. By Lemma 2.5, $|E/Z| = e^2$ for an integer $e$. Since $\mathbf{C}_A(E/Z) = \mathbf{F}(G)$, induction and the method in Step 1 yield that $e = 1$ or that $|A/\mathbf{F}(G)| \leqq e^{2\alpha}/\lambda$. If $e > 1$, we then have that

$$|G| = |G/A|\,|A/\mathbf{F}(G)|\,|E/Z|\,|T| \leqq |U|^2 e^{2\alpha+2}/\lambda.$$

By Step 2, $V$ is the direct sum of isomorphic, irreducible faithful $EU$-modules. Let $0 \neq W$ be an irreducible $U$-submodule of $V$, so that

Step 2 implies that $W$ is a faithful $U$-submodule. Let $r = |W|$, so that $r$ is a prime power. Since $U$ is cyclic, $|U|\,|\,(r-1)$. By Lemma 2.5, there is an integer $t$ such that $\dim(V) = te \dim(W)$. In particular, $|V| = r^{te}$.

Assume that $e > 1$, so that $|G| \leqq |U|^2 e^{2\alpha+2}/\lambda$. Since $|U| < r$, since $|V| = r^{te}$, and since we may assume that $|G| > |V|^\alpha/\lambda$, it follows that $e^{2\alpha+2} > r^{\alpha e-2}$. Since $\alpha > 2$, we have that

(1) $\quad e^3 > r^{(e-1)}$ (for $e > 1$).

But $1 < Z \leqq U$ and $U|(r-1)$. Hence $r \geqq 3$ and $e^3 > 3^{e-1}$. The last inequality implies that $2 \leqq e \leqq 5$. Each prime divisor of $e$ divides $|Z|$, $|U|$, and $(r-1)$. If $e = 5$, then $r \geqq 11$ and inequality $(I)$ yields a contradiction. Thus $e \leqq 4$.

Suppose that $e = 4$. Since each prime divisor of $e$ divides $(r-1)$, inequality (1) yields that $r = 3$. Since $|U|\,|\,(r-1)$, we have that $|U| = 2 = |T|$ and $G = A$. Induction yields that $|G/\mathbf{F}(G)| \leqq 16^\alpha/\lambda$. Thus $|G| \leqq 16^\alpha \cdot 32/\lambda$. Since $|V| \geqq 3^4$, we may assume that $32 > (81/16)^\alpha$. This is impossible, since $\alpha > 11/5$. Thus $e \leqq 3$.

If $e = 3$, then $r = 4$ via Inequality (1). Then $|U| = 3 = |T|$ and $|G/A| \leqq 2$. Since $|A/\mathbf{F}(G)| \leqq |\mathrm{Aut}\,(E/Z)|$, we have that $|A/\mathbf{F}(G)| \leqq 48$ and that $|G| \leqq 2^5 \cdot 3^4$. Then

$$|G| \leqq 2^5 \cdot 3^4 \leqq (4^3)^{11/5}/\lambda \leqq 4^{3\alpha}/\lambda \leqq |V|^\alpha/\lambda.$$

We may assume that $e \leqq 2$.

If $e = 2$, then $|A/\mathbf{F}(G)| \leqq 6$ and $|A/T| \leqq 24$. Inequality (1) implies that $r$ is 3, 5, or 7. In any case, $|G/A|\,|T| \leqq 12$. Thus

$$|G| \leqq 2^5 \cdot 3^2 \leqq (25)^{11/5}/\lambda.$$

Since $|V| \geqq r^2$, we may assume that $r = 3$. But then $|U| = 2 = |T|$ and $G = A$. Thus

$$|G| \leqq 48 = 9^\alpha/\lambda \leqq |V|^\alpha/\lambda.$$

Without loss of generality, $e = 1$.

We may assume that $|G| > |V|^2/2$ and that $|V| \geqq 7$. Otherwise, we have that $|V| < 7$, since $x^2/2 \leqq x^{11/5}/\lambda \leqq x^\alpha/\lambda$ for $x \geqq 7$. But if $2 \leqq |V| \leqq 5$, then

$$|GL(V)| \leqq |V|^{11/5}/\lambda \leqq |V|^\alpha/\lambda.$$

By Corollary 2.4, we have that $T = \mathbf{F}(G) = A = \mathbf{C}_G(Z)$. Write $T = R \times S$ with $|R|$ odd and $S \in \mathrm{Syl}_2(T)$. Then $T = \mathbf{C}_G\,(R \cap Z)$, and thus $G/T$ is isomorphic to a subgroup of $\mathrm{Aut}(R \cap Z)$. In particular, $|G/T| \leqq |R \cap Z|$. We have that $|T:U| \leqq 2$, and that $|T| \geqq 8$ if $|T:U| = 2$. Since $|U| < r \leqq |V|$, since $|G:T| \leqq |R \cap Z|$, and since $|G| > |V|^2/2$, it follows that $S = 1$. Since $U$ permutes the non-identity elements of $V$ in orbits of size $|U|$, since $|G/U| \leqq |U|$, and since $|G| > |V|^2/2$, we must have that $|V| - 1 = |U|$. Since $|U|$ is odd, $|V|$ is a power of 2, say $2^f$. Since $|V| - 1 = |U|$, we have that $U$ and hence $G$ act transitively on the non-identity elements of $V$. Then $V$ may be identified with the additive group of the field $GF(2^f)$ in such a way that $G$ may be viewed as a subgroup of the semi-linear group

$$\Gamma L(1, 2^f) = \{x \to ax^\sigma | a \in GF(2^f), \sigma \text{ a field automorphism}\}$$

in its action on $V$ (see Theorem 19.9 of [**13**]). Since $|\Gamma L(1, 2^f)| = f \cdot 2^f$ and since $|G| > |V|^2/2$, we have that $f \cdot 2^f > 2^{2f-1}$, a contradiction. This completes the proof.

We next show that the exponent in the bound in Theorem 3.1(a) cannot be improved. In fact, the proof of Proposition 3.2 will show that the bound in Theorem 3.1(a) is obtained for infinitely many values of $|V|$ (namely, $\log(|V|) = 4^n \log(9)$ and $n \geqq 1$).

3.2. PROPOSITION. *Let $D$ and $\delta$ be constants. Assume that whenever $G$ and $V$ satisfy the hypotheses of Theorem* 3.1, *then* $|G| \leqq D|V|^\delta$. *Then* $\alpha \leqq \delta$. *Furthermore, if* $\delta = \alpha$, *then* $(24)^{-1/3} \leqq D$.

*Proof.* Given a group $H$ and irreducible $H$-module $W$, we form a group $H^*$ and an irreducible $H^*$-module as follows. We let $W^*$ be the direct sum of four copies of $W$. We let the symmetric group $S_4$ transitively permute four copies of $H$, and then we let $H^*$ be the semi-direct product $(H \times H \times H \times H)S_4$. Then $W^*$ is easily seen to be an irreducible $H^*$-module. Now let $W_0$ be the vector space of order 9 over a field of order 3, and let $W_0 = GL(V)$. We define $W_n$ and $H_n$ inductively for $n \geqq 1$ by $W_n = W^*_{n-1}$ and $H_n = H^*_{n-1}$. Thus $W_n$ is an irreducible $H_n$-module and $H_n$ is solvable for each $n$. Note that

$$\log(|V_n|) = 4^n \log(9) \text{ and}$$
$$\log(|H_n|) = 4^n \log(48) + (1 + 4 + 4^2 + \ldots + 4^{n-1}) \log(24).$$

Thus, for any $j$,

$$\log(|H_n|/|V_n|^j) = 4^n[\log(48) + (\log(24)/3) - j \log(9)] + (-1/3) \log(24).$$

Thus $\log(|H_n|/|V_n|^j) \to \infty$ as $n \to \infty$ if $j < \alpha$. Also, $|H_n|/|V_n|^\alpha = (24)^{-1/3}$ if $n \geqq 1$.

3.3 COROLLARY. *Let $G$ be a solvable primitive subgroup of the symmetric group $S_m$. Then* $|G| \leqq m^{\alpha+1}/(24)^{1/3}$.

*Proof.* We have that $G$ acts primitively and faithfully on a set $\Omega$ of $m$ elements. Let $\alpha \in \Omega$, so that $G_\alpha$ is a maximal subgroup of $G$ and $G_\alpha$ contains no non-trivial normal subgroups of $G$. Let $M$ be a minimal normal subgroup of $G$. Then $MG_\alpha = G$. Since $M$ is abelian; $M \cap G_\alpha \trianglelefteq G$ and thus $1 = M \cap G_\alpha$. In particular, $|M| = m$. Since $\mathbf{C}_G(M) \cap G_\alpha \trianglelefteq$

$G_\alpha M = G$; it follows that $M = \mathbf{C}_G(M)$ and that $M$ is a faithful, irreducible $G/M$-module. By Theorem 3.1, $|G/M| \leqq m^\alpha/(24)^{1/3}$ and thus $|G| \leqq m^{\alpha+1}/(24)^{1/3}$.

We note that Dixon [**4**] shows that a solvable subgroup of $S_m$ has order at most $(24)^{m/3}$ and that this bound is obtained for infinitely many values of $m$. Also, the bound in Corollary 3.3 is obtained for infinitely many values of $m$. This is evident by the proof of Proposition 3.2. For $n \geqq 1$, the semi-direct product $G_n = V_n H_n$ (same notation as Proposition 3.2) has faithful, primitive permutation representation on $|V_n|$ objects (namely the conjugates of $H_n$ in $G_n$); and $|G_n| = |V_n|^{\alpha+1}/(24)^{1/3}$.

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