INEQUALITIES INVOLVING THE GENUS OF A GRAPH AND ITS THICKNESSES†

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Let G be a graph with p points and q lines, and genus γ. The *thickness* $t(G)$ has been introduced as the minimum number of planar subgraphs whose union is G. This topological invariant of a graph has been studied by Battle, Harary and Kodama [**1**], Tutte [**7**], Beineke, Harary and Moon [**3**], and Beineke and Harary [**2**].

It is natural to generalise this concept of the thickness of a graph to the union of graphs with a specified genus. We say that the *n-thickness* of G is the minimum number of subgraphs of genus at most n whose union is G. Denoting the n-thickness of G by t_n, we write in particular t_0 for the 0-thickness, i.e., the thickness.

1. Thickness and girth. We already know [**5**] that any graph G in which $q > 3p-6$ is not planar. The proof of this assertion uses the Euler formula for spherical polyhedra, $V-E+F = 2$, with substitution into this formula of $V = p$, $E = q$, and $F = 2q/3$. The reason for the last of these substitutions is that a plane graph which has as many lines as possible is triangulated, so that each line lies on two faces, and every face contains three lines. This substitution gives $q = 3p-6$. As an immediate consequence we obtain the inequality

$$q \leqq (3p-6)t_0. \tag{1}$$

We now generalise this inequality to the n-thickness. The Euler formula for a polyhedron of genus n is $V-E+F = 2-2n$. On substituting p and q into this equation, we find that

$$q = 3p-6+6n.$$

Consequently, for any non-negative integer n, whenever $q > 3p-6+6n$, we have $\gamma > n$.

Analogously to the inequality (1), it follows at once from this assertion that

$$q \leqq (3p+6n-6)t_n. \tag{2}$$

The preceding inequality can be rewritten in the following form which stresses the n-thickness:

$$t_n \geqq \frac{q}{3p+6n-6}. \tag{2a}$$

Since t_n is an integer, it follows that

$$t_n \geqq \left[\frac{q}{3p+6n-6}\right], \tag{2b}$$

where $[y]$ is the smallest integer not less than y.

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An *even graph* is one in which every cycle has even length. In particular every even graph has no triangles. Therefore, by the same reasoning as for arbitrary graphs, it is easy to verify, as in Errera [4], that if G is any graph with no triangles and $q > 2p-4$, then G is nonplanar.

The *girth* of G is the minimum length among all cycles of G. By a routine algebraic manipulation, one can verify the following generalisation of the inequality (2) to a graph with girth h:

$$t_n \geqq \frac{q(1-2/h)}{p+2n-2}. \tag{3}$$

Similarly, the genus of such a graph satisfies the inequality

$$\gamma \geqq \frac{1}{2}\left[\left(1-\frac{2}{h}\right)q-(p-2)\right]. \tag{4}$$

This may be rewritten in terms of the number $m = q-p+1$ of independent cycles in the case of a connected graph (cf. König [6]):

$$\gamma \geqq \frac{1}{2}\left(m+1-\frac{2q}{h}\right). \tag{4a}$$

2. The genus of the cube. Let Q_r be the graph of the r-dimensional cube, or briefly the r-cube. Then Q_r has $p = 2^r$ points which are all the binary sequences of length r. Two points of Q_r are adjacent whenever their sequences differ in exactly one place. Since the degree of each point is r, it follows that $q = r2^{r-1}$. Clearly the girth of Q_r is 4, so that (4) becomes

$$\gamma(Q_r) \geqq \tfrac{1}{4}q-\tfrac{1}{2}(p-2) = 1+(r-4)2^{r-3}. \tag{5}$$

In a later paper, it will be shown by a construction that the genus of Q_r is exactly given by the right hand member of (5).

3. The thicknesses of the complete graph. We have shown in [2] that the 0-thickness of the complete graph K_p is given by

$$t_0(K_p) = \left[\frac{p+7}{6}\right] \qquad (p \equiv -1, 0, 1, 2 \pmod 6), \tag{6}$$

but when $p \equiv 3$ or $4 \pmod 6$, the problem is in general unsolved. Thus, in a sense, this problem is two-thirds solved. In the same sense, the 1-thickness of the complete graph is five-sixths solved by

$$t_1(K_p) = \left[\frac{p+4}{6}\right] \qquad (p \equiv -1, 0, 1, 2, 3 \pmod 6). \tag{7}$$

However, the 2-thickness of K_p is solved for all p:

$$t_2(K_p) = \left[\frac{p+3}{6}\right]. \tag{8}$$

The proofs of equations (7) and (8) will be given elsewhere.

Added in proof. G. Ringel has just established equation (7) for all p, and his result will appear in the *Mathematische Zeitschrift*.

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