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Zero-electron-mass limit of the compressible Navier–Stokes–Poisson equations with well/ill-prepared initial data

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In this study, we consider the viscous compressible Navier–Stokes–Poisson equations, which consist of the balance laws for electron density and moment, and a Poisson equation for the electrostatic potential. The limit of vanishing electron mass of this system with both well/ill-prepared initial data on the whole space is rigorously justified within the framework of local smooth solution. We first make use of the symmetric hyperbolic–parabolic structure of the compressible Navier–Stokes–Poisson equation to obtain uniform estimate in the short time, by which we show uniform existence of local classical solution to the compressible Navier–Stokes–Poisson equation in $\mathbb{R}^d (d \geqslant 1)$. Further, with uniform estimate of time derivatives, we show the zero-electron-mass limit of the solutions for the compressible Navier–Stokes–Poisson equation with well-prepared initial data in $\mathbb{R}^d (d \geqslant 1)$ by using Aubin's lemma. A detailed spectral analysis on the linearized system is done so that we are able to prove the zero-electron-mass limit of the solutions with ill-prepared initial data in $\mathbb{R}^d (d \geqslant 3)$, where the convergence occurs away from the time $t = 0$. Finally, note that the dissipation mechanism for the linearized compressible Navier–Stokes–Poisson system is different from that of the compressible Euler equations in Grenier (*Commun. Partial Diff. Eqns.* **21** (1996), 363–394); Grenier (*Commun. Pure Appl. Math.* **50** (1997), 821–865); Ukai (*J. Math. Kyoto Univ.* **26** (1986), 323–331), or that of the compressible Euler–Poisson equations in Ali and Chen (*Nonlinearity* **24** (2011), 2745–2761), since its eigenvalues are somehow similar to that of heat equation, and the fundamental solution contains a part behaving like the heat kernel, thus a big difficulty is the singularity of the heat kernel at $t = 0$.

Keywords: Navier–Stokes–Poisson equations; uniform local existence; zero-electron-mass limit; well/ill-prepared initial data

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1. Introduction

The compressible Navier–Stokes–Poisson equations are used to describe the motion of charged particles (i.e. the electrons and the holes or, the electrons and the ions) under the influence of the self-consistent electrostatic potential force arising from semiconductors and plasmas. In this model, the heavy holes and ions are assumed to be immobile and uniformly distributed in space, providing as a background of positive charge. The light electrons are modelled as a charged compressible fluid moving against the ionic forces. Neglecting magnetic effect and heat-conductive effect, the governing dynamics of the electron fluid are given by the following viscous isentropic compressible Navier–Stokes–Poisson equations (see [**7**, **21**, **35**]):

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ m_e[\partial_t(nu) + \operatorname{div}(nu \otimes u)] + \nabla p(n) = \operatorname{div} S(u) + \rho \nabla \phi, \\ \lambda^2 \Delta \phi = n - N, \end{cases} \tag{1.1}$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^d (d \geqslant 1)$. The unknown variables n, u, ϕ are the electron density, the velocity and the electrostatic potential, $p(n)$ is the pressure function, usually given by $p(\rho) = A\rho^\gamma$ with the constants $A > 0$ and $\gamma \geqslant 1$. In this work, we assume that $p(n)$ is smooth and strictly increasing. $S(u)$ denotes the viscous stress tensor with the form

$$\operatorname{div} S(u) = \mu' \Delta u + \nu' \nabla \operatorname{div} u,$$

here μ' and ν' are viscosity coefficients satisfying $\mu' > 0$ and $\mu' + \nu' > 0$ for the sake of simplicity. The constant m_e is the ratio of the electron/ions mass, and $\lambda > 0$ is the Debye length. N stands for a given (constant) density of positively charged ions. Moreover, we also mention that, without viscous terms, (1.1) becomes the compressible isentropic Euler–Poisson system, which is another model describing the motion of charged particles. Finally, if the electrostatic potential ϕ is neglected, the compressible Navier–Stokes–Poisson equations (1.1) is then reduced to the classical compressible Navier–Stokes equations.

Recently, some important progress has been made for the compressible Navier–Stokes–Poisson system. Here we only refer to some results about the isentropic compressible Navier–Stokes–Poisson system. The local and/or global existence of renormalized weak solutions to the Cauchy problem of the multi-dimensional compressible Navier–Stokes–Poisson system are proved in [**14**, **43**]. The existence of non-trivial stationary solutions with compact support and their stability related to a free-boundary value problem for the three-dimensional Navier–Stokes–Poisson system are discussed in [**15**]. Some nonexistence result of global weak solutions is obtained in [**5**]. Large-time behaviour of the spherically symmetric Navier–Stokes–Poisson system with degenerate viscosity coefficients and with vacuum in $\mathbb{R}^3$ is shown in [**42**]. The global existence of spherically symmetric weak solutions, and the regularity and long-time behaviour of global solution for free boundary value problem to three-dimensional spherically symmetric compressible Navier–Stokes–Poisson equations are shown in [**25**]. The linear and nonlinear dynamical stability for the Lane–Emden solutions to the compressible Navier–Stokes–Poisson system is studied in [**20**]. The global strong solutions of the initial value problem for the multi-dimensional compressible Navier–Stokes–Poisson system with the strictly positive background profile in Besov spaces are investigated in [**19**]. The global existence and L^2-decay rate of the smooth solution of the initial value problem for the compressible Navier–Stokes–Poisson system in $\mathbb{R}^3$ are proved in [**27**]. The pointwise estimates of the smooth solutions for the three-dimensional isentropic compressible Navier–Stokes–Poisson equation are obtained in [**38**]. The asymptotic stability of the nonlinear wave such as the rarefaction wave, the viscous shock wave and the stationary wave of the one-dimensional compressible Navier–Stokes–Poisson equation is studied in [**12**, **13**, **23**, **28**, **39**].

Moreover, the zero-electron-mass limit $m_e \to 0$ and the quasi-neutral limit $\lambda \to 0$ of the compressible Navier–Stokes–Poisson equations are the important problems in the theory of the compressible fluid of semiconductors and plasmas. Li and Liao [**26**] showed the existence and zero-electron-mass limit of strong solutions to the stationary compressible Navier–Stokes–Poisson equation with large external force. Donatelli *et al.* [**10**] discussed the vanishing electron-mass limit of weak solution for the plasma hydrodynamics in three-dimensional unbounded domain. Li *et al.* [**29**] investigated zero-electron-mass limit of the two-dimensional compressible Navier–Stokes–Poisson equations over bounded domain. The quasineutral limit of weak solution and smooth solutions of the compressible unipolar Navier–Stokes–Poisson system was studied in [**11**, **22**, **37**]. We also mention that many authors discussed the zero-electron-mass limit in the Euler–Poisson system for both well- and ill-prepared initial data, and we can refer to [**3**, **4**, **40**, **41**] and some references therein. To our knowledge, there were no results on the zero-electron-mass limit of the classical local solutions to compressible Navier–Stokes–Poisson equations (1.1). The goal of this work is to fill in the void and study the zero-electron-mass limit of the classical solutions to compressible Navier–Stokes–Poisson equations (1.1) with well- and ill-prepared initial data in $\mathbb{R}^d (d \geqslant 1)$.

To study the zero-electron-mass limit of the Navier–Stokes–Poisson equations (1.1), we denote $\varepsilon^2 = m_e$, and we assume $\mu' = \varepsilon^2\mu$, $\nu' = \varepsilon^2\nu$. Moreover, let us introduce the electrostatic field $E = \nabla\phi$ and define the enthalpy $h = h(n)$ by $h'(n) = p'(n)/n$ and $h(1) = 0$. Then the viscous isentropic compressible Navier–Stokes–Poisson system (1.1) can be written as:

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \varepsilon^2(\partial_t + u\cdot\nabla)u + \nabla h(n) = \dfrac{\varepsilon^2}{n}(\mu\Delta u + \nu\nabla\operatorname{div} u) + E, \\ \lambda^2 E = \nabla\Delta^{-1}(n-N) := K*(n-N), \quad K(x) = \dfrac{1}{\mathrm{d}\,\omega_{d-1}}\dfrac{x}{|x|^d}, \end{cases} \tag{1.2}$$

where ω_{d-1} is the volume of the unit ball in $\mathbb{R}^d$. From the analysis of [**3**], the operator Δ^{-1} represents symbolically the fundamental solution of the Poisson equation, and $\Delta^{-1}f$ might not be well defined for not good enough functions f when the problem is considered on the whole space. However, $\nabla\Delta^{-1}f$ is well defined for $f \in H^s(\mathbb{R}^d)$ as long as $s > 0$. Since we will prove the zero-electron-mass limit of the isentropic compressible Navier–Stokes–Poisson equations (1.2) with well- and ill-prepared initial data, we now supply the system (1.2) with the following initial data:

$$(n, u)(\cdot, 0) = (n_I^\varepsilon, u_I^\varepsilon). \tag{1.3}$$

Moreover, for smooth solutions, the initial value problem (1.2)–(1.3) is equivalent to the following hyperbolic–parabolic type system:

$$\begin{cases} (\partial_t + u \cdot \nabla)h + p'(n)\mathrm{div}\, u = 0, \\ \varepsilon^2(\partial_t + u \cdot \nabla)u + \nabla h = \dfrac{\varepsilon^2}{n(h)}(\mu \Delta u + \nu \nabla \mathrm{div}\, u) + E, \\ \lambda^2 E = K * (n - N), \end{cases} \tag{1.4}$$

with initial conditions

$$(h, u)(\cdot, 0) = (h_I^\varepsilon, u_I^\varepsilon), \quad h_I^\varepsilon = h(n_I^\varepsilon). \tag{1.5}$$

Here $n(h)$ is a smooth reversible function of $h(n)$, which can be assured by the assumption of $p(n)$.

The zero-electron-mass limit $\varepsilon \to 0$ in the problem (1.2)–(1.3) or (1.4)–(1.5) is reminiscent of the low Mach number (incompressible) limit of the compressible fluid equation, which has been investigated in a number of recent studies, see the monograph [**6**] and the survey papers [**2**, **16**, **30**, **32**], and the references cited therein. The objective of this paper is to perform the limit as $\varepsilon \to 0$ in (1.2). Then as in [**3**, **8**, **9**, **30**], we introduce the new variables:

$$(\tilde{n}, \tilde{E}) = \left(\frac{n - N}{\varepsilon}, \frac{E}{\varepsilon} \right), \quad \tilde{h} = \frac{h(n) - h^0}{\varepsilon}, \quad h^0 = h(N),$$

then systems (1.2) and (1.4) are rewritten as

$$\begin{cases} \partial_t \tilde{n} + \dfrac{1}{\varepsilon}\mathrm{div}((N + \varepsilon\tilde{n})u) = 0, \\ (\partial_t + u \cdot \nabla)u + \dfrac{1}{\varepsilon}h'(N + \varepsilon\tilde{n})\nabla\tilde{n} = \dfrac{1}{N + \varepsilon\tilde{n}}(\mu \Delta u + \nu \nabla \mathrm{div}\, u) + \dfrac{1}{\varepsilon}\tilde{E}, \\ \lambda^2 \tilde{E} = K * \tilde{n}, \end{cases} \tag{1.6}$$

and

$$\begin{cases} A(\varepsilon\tilde{h})(\partial_t + u \cdot \nabla)\tilde{h} + \dfrac{1}{\varepsilon}\mathrm{div}\, u = 0, \\ (\partial_t + u \cdot \nabla)u + \dfrac{1}{\varepsilon}\nabla\tilde{h} = \dfrac{1}{n(h^0 + \varepsilon\tilde{h})}(\mu \Delta u + \nu \nabla \mathrm{div}\, u) + \dfrac{1}{\varepsilon}\tilde{E}, \\ \lambda^2 \tilde{E} = K * \tilde{n}, \end{cases} \tag{1.7}$$

respectively, where $A(\varepsilon\tilde{h}) = 1/p'(n(h^0 + \varepsilon\tilde{h}))$. For smooth solutions, we use the variable $\tilde{h}$ instead of $\tilde{n}$, because the system (1.7) with some symmetric part is more convenient for standard energy estimates.

Notations. First, $L^p(\mathbb{R}^d)$ with $1 \leqslant p < \infty$ denotes the space of measurable functions whose p-powers are integrable on $\mathbb{R}^d$, with norm $\|\cdot\|_{L^p} = (\int_{\mathbb{R}^d} |\cdot|^p \mathrm{d}x)^{1/p}$,

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and $L^\infty(\mathbb{R}^d)$ is the space of bounded measurable functions on $\mathbb{R}^d$, with the norm $\|\cdot\|_{L^\infty} = \mathrm{esssup}_{x\in\mathbb{R}^d}|\cdot|$. Without confusion, we also denote the norm of $L^2(\mathbb{R}^d)$ by $\|\cdot\|$ for brevity. Next, for a nonnegative integer k, $H^k = H^k(\mathbb{R}^d)$ denotes the usual L^2-type Sobolev space of order k. We also write $\|\cdot\|_k$ for the standard norm of $H^k(\mathbb{R}^d)$. Moreover, we denote $|\!|\!|\cdot|\!|\!|_{s,T} = \sup_{0<t<T}\|\cdot\|_s$ for $s \geqslant 0$. In addition, we denote by $C([0, T], \mathbf{X})$ (resp. $L^2([0, T], \mathbf{X})$) the space of continuous (resp. square integrable) functions on $[0, T]$ with values in a Banach space $\mathbf{X}$. Finally, the symbols c_i $(i = 1, 2, \ldots)$ or C_j $(j = 0, 1, 2, \ldots)$ are always used to denote generic positive constants independent of ε, $c(\cdot)$ and $C(\cdot)$ denote some positive smooth functions which may vary from line to line.

The first result in this paper is the following uniform local existence.

Theorem 1.1 (Uniform-in-ε local-in-time existence). *Let $d \geqslant 1$, $s > d/2 + 2$ and $N > 0$, assume that the initial data $(n_I^\varepsilon, u_I^\varepsilon)$ satisfy*

$$n_I^\varepsilon - N \in L^1(\mathbb{R}^d), \qquad \left\| \left(\frac{n_I^\varepsilon - N}{\varepsilon}, u_I^\varepsilon \right) \right\|_s \leqslant M_0, \tag{1.8}$$

where M_0 a given constant independent of ε. Then there exist constants $T_0 > 0$ and $M_0' > 0$ independent of ε, and $\varepsilon_0(M_0) > 0$, such that, for all ε with $0 < \varepsilon < \varepsilon_0(M_0)$, the problem (1.2)–(1.3) has a classical solution $(n^\varepsilon, u^\varepsilon, E^\varepsilon)$ in $[0, T_0]$ satisfying

$$\left|\!\left|\!\left| \left(\frac{n^\varepsilon - N}{\varepsilon}, u^\varepsilon, \frac{E^\varepsilon}{\varepsilon} \right) \right|\!\right|\!\right|_{s,T_0} \leqslant M_0'. \tag{1.9}$$

Next, we have the following zero-electron-mass limit for the problem (1.2)–(1.3) with *well-prepared* initial data, that is, the initial data is prepared to make the 'initial time-derivatives' uniformly bounded. Here the 'initial time-derivative' is understood by compatibility condition using the differential equation that time-derivative can be replaced by spatial-derivatives. The term *ill-prepared* initial data refers to that is not well-prepared.

Theorem 1.2 (Limit for well-prepared initial data). *Let the assumption of theorem 1.1 be held. Assume that the initial data are well-prepared, that is, $u_I^\varepsilon = u_I^0 + \varepsilon u_I^1$ with $\nabla \cdot u_I^0 = 0$, and*

$$\left\| \frac{n_I^\varepsilon - N}{\varepsilon^2} \right\|_s \leqslant M_1, \tag{1.10}$$

where M_1 is a constant. Let $(n^\varepsilon, u^\varepsilon, E^\varepsilon)$ be a classical solution to (1.2)–(1.3) in $[0, T_0]$ with $T_0 > 0$ independent of ε as obtained in theorem 1.1. Then we have the limit as $\varepsilon \to 0$ that

$$\begin{aligned} &(n^\varepsilon, E^\varepsilon) \to (N, 0) \quad \text{strongly in} \quad L^\infty([0, T_0]; H^\alpha(\mathbb{R}^d)) \cap C^{0,1}([0, T_0]; L^2(\mathbb{R}^d)),\\ &u^\varepsilon \to u^0, \quad \text{strongly in} \quad C^0([0, T_0]; H^\alpha(\mathbb{R}^d)) \quad \text{for all} \quad \alpha < s, \end{aligned}$$

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where u^0 is the unique classical solution of the incompressible Navier–Stokes equations

$$\begin{aligned} &\nabla\cdot u^0=0, \quad (\partial_t+u^0\cdot\nabla)u^0=\frac{\mu}{N}\Delta u^0+\nabla\pi \quad \textit{for} \quad t>0,\\ &u^0(\cdot,0)=u_I^0, \end{aligned} \tag{1.11}$$

and π is the limit of

$$\frac{E^\varepsilon-\nabla h(n^\varepsilon)}{\varepsilon}\rightharpoonup\nabla\pi \quad \textit{weakly* in} \quad L^\infty([0,T_0];L^2(\mathbb{R}^d)).$$

Finally, for the ill-prepared initial data of the problem (1.2)–(1.3), we have

Theorem 1.3 (Limit for ill-prepared initial data). *Let the assumptions of theorem 1.1 hold and let $d\geqslant 3$, suppose that the initial data $u_I^\varepsilon\to u_I^0$ in $H^s(\mathbb{R}^d)$ as $\varepsilon\to 0$, and let $(n^\varepsilon,u^\varepsilon,E^\varepsilon)$ be a classical solution to (1.2)–(1.3) in $[0,T_0]$ with $T_0>0$ independent of ε. Then, as $\varepsilon\to 0$,*

$$\begin{aligned} &(n^\varepsilon,E^\varepsilon)\to(N,0) \quad \textit{strongly in} \quad L^\infty([0,T_0];H^s(\mathbb{R}^d)),\\ &u^\varepsilon\rightharpoonup u_*^0 \quad \textit{weakly* in} \quad L^\infty([0,T_0];H^s(\mathbb{R}^d)),\\ &u^\varepsilon\to u_*^0 \quad \textit{strongly in} \quad C^0_{loc}((0,T_0]\times\mathbb{R}^d), \end{aligned}$$

where $u_*^0\in L^\infty([0,T_0];H^s(\mathbb{R}^d))$ is the solution of the incompressible Navier–Stokes equations

$$\begin{aligned} &\nabla\cdot u_*^0=0, \quad (\partial_t+u_*^0\cdot\nabla)u_*^0=\frac{\mu}{N}\Delta u_*^0+\nabla\pi \quad \textit{for} \quad t>0,\\ &u_*^0(\cdot,0)=Pu_I^0, \end{aligned} \tag{1.12}$$

for some $\pi\in L^\infty([0,T_0];H^s(\mathbb{R}^d))$, and P is the orthogonal projection of H^s onto the subspace $\{v\in H^s:\nabla\cdot v=0\}$.

Remark 1.4. Theorem 1.3 holds only for $d\geqslant 3$, since the decay property of the fundamental solution is essentially used in the spectral analysis. This is compatible with the corresponding result in [**3**].

Remark 1.5. In the present work, we only discuss the zero-electron-mass limit for the unipolar isentropic Navier–Stokes–Poisson equation with well-prepared and ill-prepared initial data. It is also attractive for studying similar problems for the unipolar non-isentropic and bipolar compressible Navier–Stokes–Poisson system. These are expected to be done in the forthcoming papers.

The ideas and outlines of proving theorems 1.1, 1.2 and 1.3 are as follows. As a first step, we will get the uniform-in-ε estimate in short time using a similar idea as in [**1**, **3**, **4**]. Then applying the local existence of [**24**] and standard continuation argument, we can show theorem 1.1. To get the uniform estimate, we need to treat the singular terms with $\frac{1}{\varepsilon}$. Note that the terms $\frac{1}{\varepsilon}\mathrm{div}\,u$ and $\frac{1}{\varepsilon}\nabla\tilde{h}$ in (1.6) are symmetric, the key ingredient of the uniform local estimate is the control of the term $\frac{1}{\varepsilon}\tilde{E}$ in (1.6), which can be similarly estimated as in [**3**]. Then, after obtaining the estimate of time derivatives with a similar argument as the one used in [**3**, **4**],

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the zero-electron-mass limit for the well-prepared initial data on $\mathbb{R}^d (d \geqslant 1)$ can be achieved by using Aubin's lemma. We will prove theorem 1.2 in § 4.

For the proof of theorem 1.3, we still follow the approach of [**17**, **18**, **36**], which was used in [**3**]. That is, we first consider the linearized Navier–Stokes–Poisson equations, and next study the properties of the semigroup $\mathcal{L}^\varepsilon(t)$ generated by the linear operator. Then we can decompose the solution $(\tilde{n}, u) = (0, u_1) + (\tilde{n}, u_2)$ with u_1 being the divergence free part. Finally, by further use of the properties of $\mathcal{L}^\varepsilon(t)$, it is possible to get estimates for $\partial_t u_1$, which help the discussion of the convergence away from $t = 0$. Therefore, we can prove the limit for ill-prepared initial data in § 5. It is worth noting that the eigenvalues of the linear compressible Navier–Stokes–Poisson equations are different from that of the linear compressible Euler equations in [**17**, **18**, **36**] and that of the linear compressible Euler–Poisson equations in [**3**]. The solution of the linearized equation in [**17**, **18**, **36**] has an algebra decay rate, and the solution of the linearized equation in [**3**] has an exponential decay rate due to the damping. However, the dissipation mechanism for the linear compressible Navier–Stokes–Poisson system is different here, which can be seen from its eigenvalues

$$\lambda_* = \mu|\xi|^2, \qquad (d-1 \text{ multiple})$$
$$\lambda_\pm = \frac{\mu+\nu}{2}|\xi|^2 \pm \frac{i}{2\varepsilon}\sqrt{4(1+a|\xi|^2) - (\mu+\nu)^2|\xi|^4\varepsilon^2},$$

this is somehow similar to that of heat equation, and the fundamental solution contains some part like the heat kernel, and a big difficulty is the singularity of the heat kernel at $t = 0$. Next, we have the characteristic decomposition of the solution operator in the form (5.3), with each mode corresponding to an eigenvalue, and then analyse carefully the uniform-in-ε estimates of each mode. See lemma 5.1 for details. Due to the singularity of the heat kernel at $t = 0$ and good decay away from $t = 0$, we need to have more precise local in time estimate of the solution operator such that the result is integrable in time. Thus we need two sets of estimates: $t < \delta$ and $t \geqslant \delta$, for any given positive constant δ. Without loss of generality, we consider $\delta = 1$, that is, we derive the estimate (5.6) for $t < 1$, and, estimate (5.7) for $t \geqslant 1$. The solution of the full linearized problem (5.1) can be represented by Duhamel's principle using the solution operator of the linear part, which essentially relies on the time integrability of the estimate near $t = 0$. See also remark 5.2. Another difficulty in the analysis is due to the complicated structure of the eigenvalues $\lambda_\pm$. More specifically, the term inside the square-root in $\lambda_\pm$, $F(|\xi|^2) := 4(1 + a|\xi|^2) - (\mu + \nu)^2|\xi|^4\varepsilon^2$, is not monotone with respect to $|\xi|^2$, unlike $\tilde{F}(|\xi|^2) = 4(1 + a|\xi|^2) - \varepsilon^2$ for the Euler–Poisson case which was strictly increasing, will bring difficulty to our analysis. To deal with this problem, we properly decompose the solution operator with respect to frequencies, say, the low, medium and high frequency parts, and estimate each part respectively. See (5.13) and subsequent computations for details.

This paper is organized as follows. In the next section we state some useful lemmas which will be used later, then we give the uniform local estimate of the terms involving the term $\frac{1}{\varepsilon}\tilde{E}$. We will prove the uniform local existence of the compressible Navier–Stokes–Poisson equation in § 3. Finally, the limit for the solutions of

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the compressible Navier–Stokes–Poisson equation with well- and ill-prepared initial data will be considered in § 4 and § 5, respectively.

2. Preliminary

We make some preliminaries in this section, by first giving some useful lemmas which will be used later, then showing the uniform local estimate of the term involving the term $\frac{1}{\varepsilon}\tilde{E}$. To begin with, we list the following classical differential inequalities in Sobolev spaces [**31**].

Lemma 2.1. (i) *Let f, $g \in H^s(\mathbb{R}^d)$ for $s \geqslant \frac{d}{2}+1$. Then, for all multi-indices α with $|\alpha| \leqslant s$, it holds that $\partial_x^\alpha(fg) \in L^2(\mathbb{R}^d)$ and*

$$\|\partial_x^\alpha(fg)\| \leqslant C\|f\|_s\|g\|_s.$$

(ii) *Let $f \in H^s(\mathbb{R}^d)$ and $g \in H^{s-1}(\mathbb{R}^d)$ for $s \geqslant \frac{d}{2}+2$. Then for all multi-indices α with $|\alpha| \leqslant s$, it holds that the commutator $[\partial_x^\alpha,\, f]g \in L^2(\mathbb{R}^d)$ and*

$$\|[\partial_x^\alpha, f]g\| \leqslant C\|\nabla f\|_{s-1}\|g\|_{s-1}.$$

(iii) *Assume $g(u)$ is a smooth function on G, $u(x)$ is a continuous function with $u(x) \in G_1$, $\bar{G}_1 \subset\subset G$, and $u(x) \in L^\infty \cap H^s(\mathbb{R}^d)$. Then for $s \geqslant 1$,*

$$\|D^s g(u)\| \leqslant C|\frac{\partial g}{\partial u}|_{s-1,\bar{G}_1}\|u\|_{L^\infty}^{s-1}\|D^s u\|.$$

Here $|\cdot|_{r,\bar{G}_1}$ is the C^r-norm on the set $\bar{G}_1$ and C_s is a generic constant depending only on s.

Next, we recall the following Aubin's lemma in [**33**, **34**].

Lemma 2.2. *Assume $X \subset E \subset Y$ are Banach spaces and $X \hookrightarrow\hookrightarrow Y$. Then the following imbeddings are compact:*

(i) $\Big\{\varphi : \varphi \in L^q(0,\, T; X),\, \frac{\partial \varphi}{\partial t} \in L^1(0,\, T; Y)\Big\} \hookrightarrow\hookrightarrow L^q(0,\, T; E)$, *if* $1 \leqslant q \leqslant \infty$;

(ii) $\Big\{\varphi : \varphi \in L^\infty(0,\, T; X),\, \frac{\partial \varphi}{\partial t} \in L^\gamma(0,\, T; Y)\Big\} \hookrightarrow\hookrightarrow C(0,\, T; E)$, *if* $1 < \gamma \leqslant \infty$.

To end this section, we give the following estimates of the terms involving the term such as $\frac{1}{\varepsilon}\tilde{E}$.

Lemma 2.3. *Under the assumptions of theorem 1.1, let $(\tilde{h},\, u,\, \tilde{E})$ be the solution of the Cauchy problem for (1.7) with (1.5) in $[0,\, T^*]$ for some $T^* > 0$ (may depend*

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on ε), and (3.1) hold. Then we have

$$\int_{\mathbb{R}^d} \frac{1}{\varepsilon}\tilde{E}\cdot u\,\mathrm{d}x \leqslant -\frac{1}{2N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |\tilde{E}|^2\,\mathrm{d}x + c(M)(\|\tilde{E}\|^2 + \|u\|^2) + c(M), \tag{2.1}$$

and

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^d} \tilde{E}_t\cdot u_t\,\mathrm{d}x \leqslant -\frac{1}{2N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |\tilde{E}_t|^2\,\mathrm{d}x + c(M)(\|\tilde{E}_t\|^2 + \|\tilde{h}_t\|^2 + \|u_t\|^2). \tag{2.2}$$

Furthermore, for any multi-index α with $1 \leqslant |\alpha| \leqslant s$, denote

$$h_\alpha = \partial^\alpha \tilde{h}, \quad u_\alpha = \partial^\alpha u, \quad E_\alpha = \partial^\alpha \tilde{E},$$

we also have

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{1}{\varepsilon}E_\alpha\cdot u_\alpha\,\mathrm{d}x \leqslant& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \frac{A(\varepsilon\tilde{h})}{n'(h^0+\varepsilon\tilde{h})}|E_\alpha|^2\,\mathrm{d}x \\ &+ c(M)(\|h\|_s^2 + \|u\|_s^2 + \|\tilde{E}\|_s^2) + c(M).\end{aligned} \tag{2.3}$$

The proof is similar to that in [3], thus is omitted here.

3. Uniform local existence

In this section, we mainly show uniform-in-ε local existence of smooth solution for (1.2)–(1.3). That is, we prove theorem 1.1. Applying the local existence of [24] and standard continuation argument, we only need the following uniform a priori estimate.

Lemma 3.1 (Uniform-in-ε a priori estimates). *Under the assumptions of theorem 1.1, let $(\tilde{h}, u, \tilde{E})$ be the solution of the Cauchy problem for (1.7) with (1.5) in $[0, T^*]$ for some $T^* > 0$ (may depend on ε), and*

$$|||\tilde{h}|||_{s,T^*} + |||u|||_{s,T^*} \leqslant M \tag{3.1}$$

for some positive constant M independent of ε. Then there exist $\varepsilon_0 = \varepsilon_0(M)$ and $c(M) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, it holds that

$$\left|\left|\left|(\tilde{h}, u, \tilde{E})\right|\right|\right|_{s,T^*} + \|\nabla u\|_{L^2([0,T^*];H^s(\mathbb{R}^d))} \leqslant \mathrm{e}^{c(M)T^*}(M_0 + c(M)T^*). \tag{3.2}$$

Before the proof of lemma 3.1, we give some necessary preliminaries. First, from (3.1) and Sobolev inequalities, it is easy to see that, there exists a constant

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$C(M) > 0$, such that, for $|\alpha| \leqslant 2$,

$$\sup_{t\in[0,T*]} \|\partial_x^\alpha(\tilde{h}, u)\|_{L^\infty} \leqslant C(M) \tag{3.3}$$

and

$$0 < C_0(M) \leqslant A(\varepsilon\tilde{h}),\ n(h^0 + \varepsilon\tilde{h}) \leqslant C_1(M) \tag{3.4}$$

for sufficiently small ε. Furthermore, since

$$\nabla A(\varepsilon\tilde{h}) = \varepsilon A'(\varepsilon\tilde{h})\nabla\tilde{h}, \quad \nabla n(h^0 + \varepsilon\tilde{h}) = \varepsilon n'(h^0 + \varepsilon\tilde{h})\nabla\tilde{h},$$

we also have

$$\sup_{t\in[0,T*]} \|\nabla A(\varepsilon\tilde{h})\|_{L^\infty} \leqslant \varepsilon C(M), \qquad \sup_{t\in[0,T*]} \|\nabla A(\varepsilon\tilde{h})\|_{L^\infty} \leqslant \varepsilon C(M). \tag{3.5}$$

Moreover, using $(1.7)_1$, we have

$$\partial_t A(\varepsilon\tilde{h}) = -A'(\varepsilon\tilde{h})\left(\varepsilon u\cdot\nabla\tilde{h} + \frac{1}{A(\varepsilon\tilde{h})}\operatorname{div} u\right)$$

and

$$\partial_t A^{-1}(\varepsilon\tilde{h}) = \frac{A'(\varepsilon\tilde{h})}{A^2(\varepsilon\tilde{h})}\left(\varepsilon u\cdot\nabla\tilde{h} + \frac{1}{A(\varepsilon\tilde{h})}\operatorname{div} u\right),$$

which together with (3.3)–(3.4) imply

$$\sup_{t\in[0,T*]} \|\partial_t A(\varepsilon\tilde{h})\|_{L^\infty} \leqslant C(M) \quad \text{and} \quad \sup_{t\in[0,T*]} \|\partial_t A^{-1}(\varepsilon\tilde{h})\|_{L^\infty} \leqslant C(M). \tag{3.6}$$

Proof. Multiply $(1.7)_1$ and $(1.7)_2$ by $\tilde{h}$ and u, respectively, take the summation and integrate over $\mathbb{R}^d$, then we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}(A(\varepsilon\tilde{h})|\tilde{h}|^2 + |u|^2)\,\mathrm{d}x + \int_{\mathbb{R}^d}\frac{1}{n(h^0+\varepsilon\tilde{h})}(\mu|\nabla u|^2 + \nu|\operatorname{div} u|^2)\,\mathrm{d}x \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^d}\partial_t(A(\varepsilon\tilde{h})\tilde{h}^2 + \nabla\cdot(A(\varepsilon\tilde{h})u)\tilde{h}^2 + \operatorname{div} u\,|u|^2)\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^d}\frac{n'(h^0+\varepsilon\tilde{h})\varepsilon}{n^2(h^0+\varepsilon\tilde{h})}(\mu(\nabla\tilde{h}\cdot\nabla)u\cdot u + \nu(u\cdot\nabla\tilde{h})\operatorname{div} u)\,\mathrm{d}x + \int_{\mathbb{R}^d}\frac{1}{\varepsilon}\tilde{E}\cdot u\,\mathrm{d}x \\
&\quad =: H_1 + H_2 + H_3.
\end{aligned} \tag{3.7}$$

First, utilizing (3.3), (3.5) and (3.6), it is trivial that

$$H_1 \leqslant C(M)(\|\tilde{h}\|^2 + \|u\|^2). \tag{3.8}$$

Next, by using Cauchy's inequality and (3.3), we have

$$H_2 \leqslant \frac{1}{2}\int_{\mathbb{R}^d}\frac{1}{n(h^0+\varepsilon\tilde{h})}(\mu|\nabla u|^2 + \nu|\operatorname{div} u|^2)\,\mathrm{d}x + c(M)\|u\|^2. \tag{3.9}$$

Finally, from (2.1) in lemma 2.3, we have

$$H_3 \leqslant -\frac{1}{2N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |\tilde{E}|^2\,\mathrm{d}x + c(M)(\|\tilde{E}\|^2 + \|u\|^2) + c(M). \tag{3.10}$$

Hence, putting the estimates (3.8)–(3.10) into (3.7), using (3.3) and (3.4), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}(|\tilde{h}|^2 + |u|^2 + |\tilde{E}|^2)\,\mathrm{d}x + \|\nabla u\|^2 + \|\mathrm{div}\,u\|^2\\ &\quad\leqslant c(M)(\|\tilde{h}\|^2 + \|u\|^2 + \|\tilde{E}\|^2) + C(M).\end{aligned} \tag{3.11}$$

Next, we derive the estimates of derivatives of $(\tilde{h},\, u,\, \tilde{E})$. Let α be a multi-index with $1 \leqslant |\alpha| \leqslant s$, denote $h_\alpha = D^\alpha \tilde{h},\ u_\alpha = D^\alpha u,\ E_\alpha = D^\alpha \tilde{E}$, and define $|D^{|\alpha|}u| := \sup_\alpha |D^\alpha u|$. Then applying the operator D^α to (1.7), we have

$$\begin{cases} A(\varepsilon\tilde{h})(\partial_t + u\cdot\nabla)h_\alpha + \dfrac{1}{\varepsilon}\mathrm{div}\,u_\alpha = F_\alpha,\\ (\partial_t + u\cdot\nabla)u_\alpha + \dfrac{1}{\varepsilon}\nabla h_\alpha = \dfrac{1}{n(h^0+\varepsilon\tilde{h})}(\mu\Delta u_\alpha + \nu\nabla\mathrm{div}\,u_\alpha)\\ \quad + \displaystyle\sum_{\beta<\alpha,|\alpha-\beta|=1} D^{\alpha-\beta}\left(\frac{1}{n(h^0+\varepsilon\tilde{h})}\right)(\mu\Delta u_\beta + \nu\nabla\mathrm{div}\,u_\beta) + \frac{1}{\varepsilon}E_\alpha + G_\alpha, \end{cases} \tag{3.12}$$

where

$$F_\alpha = A(\varepsilon\tilde{h})[u\cdot,\, D^\alpha]\nabla\tilde{h} + \frac{1}{\varepsilon}A(\varepsilon\tilde{h})[A^{-1}(\varepsilon\tilde{h}),\, D^\alpha]\mathrm{div}\,u,$$

$$G_\alpha = [u\cdot,\, D^\alpha]\nabla u - \sum_{\beta<\alpha,|\alpha-\beta|=1}\left[D^\beta, D^{\alpha-\beta}\left(\frac{1}{n(h^0+\varepsilon\tilde{h})}\right)\right](\mu\Delta u + \nu\nabla\mathrm{div}\,u).$$

In fact $|\beta| = |\alpha| - 1$ here. Now multiply $(3.12)_1$ and $(3.12)_2$ by h_α and u_α, respectively, and integrate the resultant equations over $\mathbb{R}^d$, then the summation of the two equations yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}(A(\varepsilon\tilde{h})|h_\alpha|^2 + |u_\alpha|^2)\,\mathrm{d}x + \int_{\mathbb{R}^d}\frac{1}{n(h^0+\varepsilon\tilde{h})}(\mu|\nabla u_\alpha|^2 + \nu|\mathrm{div}\,u_\alpha|^2)\,\mathrm{d}x\\ &\quad = \frac{1}{2}\int_{\mathbb{R}^d}\left(\partial_t A(\varepsilon\tilde{h})h_\alpha^2 + \nabla\cdot(A(\varepsilon\tilde{h})u)h_\alpha^2 + \mathrm{div}\,u\,|u_\alpha|^2\right)\,\mathrm{d}x\\ &\qquad + \int_{\mathbb{R}^d}\frac{n'(h^0+\varepsilon\tilde{h})\varepsilon}{n^2(h^0+\varepsilon\tilde{h})}\left(\mu(\nabla\tilde{h}\cdot\nabla)u_\alpha\cdot u_\alpha + \nu(u_\alpha\cdot\nabla\tilde{h})\,\mathrm{div}\,u_\alpha\right)\,\mathrm{d}x\\ &\qquad + \sum_{\beta<\alpha,|\alpha-\beta|=1}\int_{\mathbb{R}^d} D^{\alpha-\beta}\left(\frac{1}{n(h^0+\varepsilon\tilde{h})}\right)(\mu\Delta u_\beta + \nu\nabla\mathrm{div}\,u_\beta)\cdot u_\alpha\,\mathrm{d}x\\ &\qquad + \int_{\mathbb{R}^d}\frac{1}{\varepsilon}E_\alpha\cdot u_\alpha\,\mathrm{d}x + \int_{\mathbb{R}^d}F_\alpha\cdot h_\alpha\,\mathrm{d}x + \int_{\mathbb{R}^d}G_\alpha\cdot u_\alpha\,\mathrm{d}x\\ &\quad =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.\end{aligned} \tag{3.13}$$

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First, using (3.3) and (3.6), it is easy to find that

$$I_1 \leqslant C(M)(\|\tilde{h}_\alpha\|^2 + \|u_\alpha\|^2). \tag{3.14}$$

Second, by using Cauchy's inequality and (3.3), we have

$$I_2 \leqslant \frac{1}{4}\int_{\mathbb{R}^d} \frac{1}{n(h^0+\varepsilon\tilde{h})}(\mu|\nabla u_\alpha|^2 + \nu|\mathrm{div}\, u_\alpha|^2)\,\mathrm{d}x + \ c(M)\|u_\alpha\|^2. \tag{3.15}$$

and similarly,

$$\begin{aligned} I_3 &\leqslant \frac{1}{4}\int_{\mathbb{R}^d} \frac{1}{n(h^0+\varepsilon\tilde{h})}(\mu|\Delta u_\beta|^2 + \nu|\nabla\mathrm{div}\, u_\beta|^2)\,\mathrm{d}x + \ c(M)\|u_\alpha\|^2 \\ &\leqslant \frac{1}{4}\int_{\mathbb{R}^d} \frac{1}{n(h^0+\varepsilon\tilde{h})}(\mu|\nabla u_\alpha|^2 + \nu|\mathrm{div}\, u_\alpha|^2)\,\mathrm{d}x + \ c(M)\|u_\alpha\|^2. \end{aligned} \tag{3.16}$$

Next, from (2.3) in lemma 2.3, we have

$$I_4 \leqslant -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \frac{A(\varepsilon\tilde{h})}{n'(h^0+\varepsilon\tilde{h})}|E_\alpha|^2\,\mathrm{d}x + c(M)(\|\tilde{h}\|_s^2 + \|u\|_s^2 + \|\tilde{E}\|_s^2) + c(M). \tag{3.17}$$

Finally, from lemma 2.1 and (3.4), we have

$$\begin{aligned} \|F_\alpha\| &\leqslant \|A(\varepsilon\tilde{h})\|_\infty \|\nabla u\|_{s-1}\|\tilde{h}\|_s + \frac{1}{\varepsilon}\|A(\varepsilon\tilde{h})\|_\infty\|\nabla A^{-1}(\varepsilon\tilde{h})\|_{s-1}\|\mathrm{div}\, u\|_s \\ &\leqslant C(M)(\|\tilde{h}\|_s^2 + \|u\|_s^2 + 1), \end{aligned}$$

and similarly, we have

$$\|G_\alpha\| \leqslant C(M)(\|\tilde{h}\|_s^2 + \|u\|_s^2 + 1),$$

thus, by using Cauchy–Schwarz's inequality, we obtain

$$I_5 + I_6 \leqslant C(M)(\|\tilde{h}\|_s^2 + \|u\|_s^2 + 1), \tag{3.18}$$

Putting the estimates (3.14)–(3.18) into (3.13), using again (3.3) and (3.4), we readily have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|h_\alpha\|_s^2 + \|u_\alpha\|_s^2 + \|E_\alpha\|_s^2) + \|\nabla u_\alpha\|_s^2 + \|\mathrm{div}\, u_\alpha\|_s^2 \\ &\quad \leqslant C(M)(\|\tilde{h}\|_s^2 + \|u\|_s^2 + \|\tilde{E}\|_s^2) + C(M), \end{aligned} \tag{3.19}$$

Finally, taking summation (3.11) and (3.19) for $1 \leqslant \alpha| \leqslant s$, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\tilde{h}\|_s^2 + \|u\|_s^2 + \|\tilde{E}\|_s^2) + \|\nabla u\|_s^2 + \|\mathrm{div}\, u\|_s^2 \\ &\quad \leqslant C(M)(\|\tilde{h}\|_s^2 + \|u\|_s^2 + \|\tilde{E}\|_s^2) + C(M), \end{aligned} \tag{3.20}$$

which together with Gronwall's inequality imply (3.2), this completes the proof of lemma 3.1. □

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4. Limit for well-prepared initial data

In this section, we consider the zero-electron-mass limit of the isentropic compressible Navier–Stokes–Poisson equation with well-prepared initial data, that is, we shall prove theorem 1.2. Similar to that in [**3**, **4**], we need the following uniform estimates of the time derivatives.

LEMMA 4.1. *Under the assumptions of theorem 1.2 and lemma 3.1, there exists* $\varepsilon_1 \in (0, \varepsilon_0)$ *such that for all* $0 < \varepsilon < \varepsilon_1$, *it holds that*

$$\sup_{t\in[0,T^*]} \|(\tilde{h}_t, u_t, \tilde{E}_t)\|^2 + \int_0^{T^*} \|\nabla u_t\|^2 + \|\operatorname{div} u_t\|^2 \leqslant c(M, M_0, M_1, T^*). \tag{4.1}$$

Proof. Take time derivative to (1.7), we have

$$\begin{cases} A(\varepsilon\tilde{h})(\partial_t + u\cdot\nabla)\tilde{h}_t + \dfrac{1}{\varepsilon}\operatorname{div} u_t = F_t, \\ (\partial_t + u\cdot\nabla)u_t + \dfrac{1}{\varepsilon}\nabla\tilde{h}_t = \dfrac{1}{n(h^0+\varepsilon\tilde{h})}(\mu\Delta u_t + \nu\nabla\operatorname{div} u_t) + \dfrac{1}{\varepsilon}\tilde{E}_t + G_t, \\ \lambda^2\tilde{E}_t = K * \tilde{n}_t, \end{cases} \tag{4.2}$$

where

$$F_t = A(\varepsilon\tilde{h})(u\cdot\nabla\tilde{h}_t - \partial_t(u\cdot\nabla\tilde{h})) + \frac{1}{\varepsilon}(\nabla\cdot u_t - A(\varepsilon\tilde{h})\partial_t(A^{-1}(\varepsilon\tilde{h})\nabla\cdot u)),$$

$$G_t = -u_t\cdot\nabla u + \varepsilon\tilde{h}_t\frac{n'(h^0+\varepsilon\tilde{h})}{n^2(h^0+\varepsilon\tilde{h})}(\mu\Delta u + \nu\nabla\operatorname{div} u).$$

Multiply $(4.2)_1$ and $(4.2)_2$ by $\tilde{h}_t$ and u_t, respectively, then integrate over $\mathbb{R}^d$ and take the summation, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} A(\varepsilon\tilde{h})|\tilde{h}_t|^2 + |u_t|^2\,\mathrm{d}x + \int_{\mathbb{R}^d}\frac{1}{n(h^0+\varepsilon\tilde{h})}(\mu|\nabla u_t|^2 + \nu|\operatorname{div} u_t|^2)\,\mathrm{d}x \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^d}\Big(\partial_t A(\varepsilon\tilde{h})\tilde{h}_t^2 + \nabla\cdot(A(\varepsilon\tilde{h})u)\tilde{h}_t^2 + \operatorname{div} u\,|u_t|^2\Big)\,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^d}\frac{n'(h^0+\varepsilon\tilde{h})\varepsilon}{n^2(h^0+\varepsilon\tilde{h})}\Big(\mu(\nabla\tilde{h}\cdot\nabla)u_t\cdot u_t + \nu(u_t\cdot\nabla\tilde{h})\operatorname{div} u_t\Big)\,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^d}\frac{1}{\varepsilon}\tilde{E}_t\cdot u_t\,\mathrm{d}x + \int_{\mathbb{R}^d} F_t\,\tilde{h}_t + G_t\cdot u_t\,\mathrm{d}x \\ &\quad =: J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{4.3}$$

Similar to (3.8) and (3.14), one has

$$J_1 \leqslant C(M)(\|\tilde{h}_t\|^2 + \|u_t\|^2). \tag{4.4}$$

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Next, similar to (3.9) and (3.15), one has

$$J_2 \leqslant \frac{1}{2}\int_{\mathbb{R}^d} \frac{1}{n(h^0+\varepsilon\tilde{h})}(\mu|\nabla u_t|^2 + \nu|\mathrm{div}\, u_t|^2)\,\mathrm{d}x + \ c(M)\|u_t\|^2. \tag{4.5}$$

For J_3, by using lemma 2.3, we have

$$J_3 \leqslant -\frac{1}{2N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |\tilde{E}_t|^2\,\mathrm{d}x + c(M)(\|\tilde{E}_t\|^2 + \|\tilde{h}_t\|^2 + \|u_t\|^2) + c(M). \tag{4.6}$$

Finally, by using Cauchy inequality, (3.3) and (3.4), we have

$$J_4 \leqslant c(M)(\|\tilde{h}_t\|^2 + \|u_t\|^2). \tag{4.7}$$

Now, put (4.4)–(4.7) into (4.3), and noting (3.4), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\tilde{h}_t\|^2 + \|u_t\|^2 + \|\tilde{E}_t\|^2) + \|\nabla u_t\|^2 + \|\mathrm{div}\, u_t\|^2 \\ &\quad \leqslant C(M)(\|\tilde{h}_t\|^2 + \|u_t\|^2 + \|\tilde{E}_t\|^2). \end{aligned} \tag{4.8}$$

Moreover, from $(1.7)_1$ and $u_I^\varepsilon = u_I^0 + \varepsilon u_I^1$ with $\nabla\cdot u_I^0 = 0$, we know

$$\|\tilde{h}_t(\cdot,0)\| \leqslant C(M_0)(\|\nabla(h_I^\varepsilon - h^0)\| + \|\nabla\cdot u_I^1\|) \leqslant C(M_0). \tag{4.9}$$

Similarly, with the condition (1.10), we also have

$$\|u_t(\cdot,0)\|,\ \|\tilde{E}_t(\cdot,0)\| \leqslant C(M_0). \tag{4.10}$$

Then, apply Gronwall's inequality to (4.8), by noting (4.9)–(4.10), we readily have (4.1). This completes the proof. □

Now we show the proof of theorem 1.2.

Proof of theorem 1.2. From (3.2) and (4.1), we have

$$\left|\!\left|\!\left|\left(\frac{h^\varepsilon - h^0}{\varepsilon}, u^\varepsilon, \frac{E^\varepsilon}{\varepsilon}\right)\right|\!\right|\!\right|_{s,T_0} \leqslant M, \quad \left|\!\left|\!\left|\left(\frac{h_t^\varepsilon}{\varepsilon}, u_t, \frac{E^\varepsilon}{\varepsilon}\right)\right|\!\right|\!\right|_{0,T^*} \leqslant M,$$

thus, as $\varepsilon \to 0$,

$$(h^\varepsilon, E^\varepsilon) \to (h^0, 0) \quad \text{strongly in} \quad L^\infty([0,T_0]; H^s(\mathbb{R}^d)) \cap C^{0,1}([0,T_0]; L^2(\mathbb{R}^d)),$$

and further by lemma 2.2, there exist a subsequence, still denoted by u^ε, such that as $\varepsilon \to 0$,

$$u^\varepsilon \to u^0, \quad \text{strongly in} \quad C^0([0,T_0]; H^\alpha(\mathbb{R}^d)) \quad \text{for all} \quad \alpha < s.$$

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It remains to show (1.11). In fact, it holds for all $\chi(t) \in C^\infty[0, T_0]$ and $\psi(x) \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$ such that $\nabla \cdot \psi = 0$,

$$\begin{aligned}
&\int_0^{T_0} \int_{\mathbb{R}^d} \left((\partial_t + u \cdot \nabla) u - \frac{1}{n(h^0 + \varepsilon \tilde{h})} (\mu \Delta u + \nu \nabla \mathrm{div}\, u) \right) \chi \psi \mathrm{d}x \mathrm{d}t \\
&\quad = \int_0^{T_0} \int_{\mathbb{R}^d} \frac{1}{\varepsilon} (\tilde{E} - \nabla \tilde{h}) \chi \psi \mathrm{d}x \mathrm{d}t \\
&\quad = \int_0^{T_0} \int_{\mathbb{R}^d} \frac{1}{\varepsilon} \big(\nabla (-\Delta)^{-1} \tilde{n} - \nabla \tilde{h} \big) \chi \psi \mathrm{d}x \mathrm{d}t \\
&\quad = \int_0^{T_0} \int_{\mathbb{R}^d} \frac{1}{\varepsilon} \big((-\Delta)^{-1} \tilde{n} - \tilde{h} \big) \chi \nabla \cdot \psi \mathrm{d}x \mathrm{d}t = 0.
\end{aligned}$$

Then passing the limit as $\varepsilon \to 0$ in above equation to get

$$\begin{aligned}
&\int_0^{T_0} \int_{\mathbb{R}^d} u^0 \psi \partial_t \chi \mathrm{d}x \mathrm{d}t \\
&\quad = \int_0^{T_0} \int_{\mathbb{R}^d} \left((\partial_t + u^0 \cdot \nabla) u^0 - \frac{1}{N} (\mu \Delta u^0 + \nu \nabla \mathrm{div}\, u^0) \right) \chi \psi \mathrm{d}x \mathrm{d}t.
\end{aligned}$$

By the definition of weak time derivative of u^0, we conclude that

$$\partial_t u^0 = -P \left(u^0 \cdot \nabla u^0 - \frac{1}{N} (\mu \Delta u^0 + \nu \nabla \mathrm{div}\, u^0) \right).$$

where P is the standard projection on the set of divergence-free vector fields. Since we already have

$$u^0 \in C^0([0, T_0]; C^1(\mathbb{R}^d)) \cap L^\infty([0, T_0]; H^s(\mathbb{R}^d)),$$

which implies that

$$u^0 \cdot \nabla u^0 - \frac{1}{N} (\mu \Delta u^0 + \nu \nabla \mathrm{div}\, u^0) \in C^0([0, T_0]; C^0(\mathbb{R}^d)) \cap L^\infty([0, T_0]; H^{s-2}(\mathbb{R}^d)),$$

we infer

$$u_t^0 \in C^0([0, T_0] \times (\mathbb{R}^d)) \cap L^\infty([0, T_0]; H^{s-2}(\mathbb{R}^d)).$$

Thus, $u^0 \in C^1([0, T_0] \times (\mathbb{R}^d))$ is a classical solution to

$$\nabla \cdot u^0 = 0, \quad P(\partial_t + u^0 \cdot \nabla) u^0 = P \frac{\mu}{N} \Delta u^0, \; u^0(\cdot, 0) = u_I^0, \quad \text{for} \quad x \in \mathbb{R}^d, \; t > 0,$$

The second equation and the regularity of $\partial_t u^0 + u^0 \cdot \nabla u^0 - (\mu/N) \Delta u^0$ show that there exists a function $\pi \in L^\infty([0, T_0]; H^s(\mathbb{R}^d))$ such that

$$\nabla \cdot u^0 = 0, \quad \partial_t u^0 + u^0 \cdot \nabla u^0 = \frac{\mu}{N} \Delta u^0 + \nabla \pi.$$

Further, taking into account the equation satisfied by u^ε and

$$u_t^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \frac{\mu}{N} \Delta u^\varepsilon \rightharpoonup u_t^0 + u^0 \cdot \nabla u^0 - \frac{\mu}{N} \Delta u^0 = \nabla \pi$$

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weakly* in $L^\infty([0,T];L^2(\mathbb{R}^d))$, we get

$$\frac{E^\varepsilon - \nabla h(n^\varepsilon)}{\varepsilon} \rightharpoonup \nabla\pi \quad \text{weakly* in} \quad L^\infty([0,T_0];L^2(\mathbb{R}^d)).$$

Finally, the uniqueness of smooth solutions to the incompressible Navier–Stokes equation implies the convergence of the whole sequence. This completes the proof. □

5. Limit for ill-prepared initial data

In this section, we study the zero-electron-mass limit of the initial value problem (1.2)–(1.3) with ill-prepared initial data. To begin with, we rewrite (1.6) into the following linearized form:

$$\begin{cases} \partial_t\tilde{n} + \dfrac{N}{\varepsilon}\operatorname{div} u = G_1^\varepsilon, \\ \partial_t u + h'(N)\dfrac{\nabla\tilde{n}}{\varepsilon} - \dfrac{1}{\varepsilon}\tilde{E} - \dfrac{1}{N}(\mu\Delta u + \nu\nabla\operatorname{div} u) = G_2^\varepsilon, \\ \lambda^2\tilde{E} = K * \tilde{n}, \end{cases} \tag{5.1}$$

where

$$\begin{cases} G_1^\varepsilon := -\operatorname{div}(\tilde{n}u), \\ G_2^\varepsilon := -u\cdot\nabla u + \dfrac{h'(N) - h'(N+\varepsilon\tilde{n})}{\varepsilon}\nabla\tilde{n} + \left(\dfrac{1}{N+\varepsilon\tilde{n}} - \dfrac{1}{N}\right)(\mu\Delta u + \nu\nabla\operatorname{div} u). \end{cases}$$

Further, we set $N=1$ without loss of generality, and denote $a = h'(N)$. Let us consider the Cauchy problem of the linear part of (5.1):

$$U_t + LU = 0, \quad U(x,0) = U_I(x), \tag{5.2}$$

where the linear operator

$$L = \begin{pmatrix} 0 & \frac{1}{\varepsilon}\nabla\cdot \\ \frac{1}{\varepsilon}(a + (-\Delta)^{-1})\nabla & \mu\Delta\cdot I_d + \nu\nabla\nabla\cdot \end{pmatrix}.$$

Let $\mathcal{L}^\varepsilon(t)$ be the semigroup generated by L, then $U(x,t) = \mathcal{L}^\varepsilon(t)U_I(x)$ solves Cauchy problem (5.2). We have

Lemma 5.1. *Assume $U_I \in L^1\cap H^s$. The solution of (5.2) can be decomposed as*

$$U(x,t) = \mathcal{L}^\varepsilon(t)U_I := \mathcal{L}_*^\varepsilon(t)U_I + \mathcal{L}_+^\varepsilon(t)U_I + \mathcal{L}_-^\varepsilon(t)U_I, \tag{5.3}$$

in which (i)

$$\mathcal{L}_*^\varepsilon(t)U_I = \begin{pmatrix} 0 \\ \mathcal{G} * Pu_I \end{pmatrix} \text{ for } U_I = \begin{pmatrix} n_I \\ u_I \end{pmatrix}, \tag{5.4}$$

where $\mathcal{G}(x,t) = (1/(4\pi\mu t))^{d/2}\exp(-(|x|^2/4\mu t))$ is the heat kernel, P is the orthogonal projection of H^s onto the subspace $\{v\in H^s : \nabla\cdot v = 0\}$, and

$$\|\mathcal{G} * Pu_I\|_s \leqslant \|u_I\|_s. \tag{5.5}$$

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(ii) *For* $t < 1$ *and* $\theta \in (0, \frac{1}{d})$,

$$|\mathcal{L}^{\varepsilon}_{\pm}(t)U_I| \leqslant \left[\frac{\varepsilon}{t}C\left(\mathrm{e}^{-((\mu+\nu)/2)(t/\varepsilon)^{2\theta}t}\left(\frac{t}{\varepsilon}\right)^{\theta(d-1)} + \left(\frac{t}{\varepsilon}\right)^{\theta d}\right)\right.$$
$$\left. + \mathrm{e}^{-((\mu+\nu)/2)(t/\varepsilon)^{2\theta}t} + \mathrm{e}^{-(at/((\mu+\nu)\varepsilon^2))}\right]\|U_I\|_{L^1}. \tag{5.6}$$

(iii) *For* $t \geqslant 1$,

$$|\mathcal{L}^{\varepsilon}_{\pm}(t)U_I| \leqslant \left[\frac{\varepsilon}{t}C\left(\mathrm{e}^{-(at/(2(\mu+\nu)\varepsilon^2))}\frac{1}{\varepsilon^{d-1}} + \frac{1}{t^{d/2}}\right) + \mathrm{e}^{-(at/(2(\mu+\nu)\varepsilon^2))}\right]\|U_I\|_{L^1}. \tag{5.7}$$

(iv) *Furthermore, for any fixed* $\tau > 0$,

$$\sup_{t\geqslant\tau}|\mathcal{L}^{\varepsilon}_{\pm}(t)U_I| \to 0 \quad as \quad \varepsilon \to 0. \tag{5.8}$$

REMARK 5.2. Note that (5.6), the local-in-time behaviour of the solution, is essential for the estimate of the full nonlinear system when we apply Duhamel's principle, since the singularity at $t = 0$ in (5.6) is integrable. The three terms on the right hand side of (5.6) correspond to low, medium and high frequency part, respectively. On the other hand, if the solution exist for longer time, the behaviour of the solution is controlled by (5.7).

Proof. (i) Taking the Fourier transform of system (5.2), we get

$$\hat{U}_t + B(\varepsilon,\xi)\hat{U} = 0, \quad \hat{U}(0) = \hat{U}_I, \tag{5.9}$$

where $\hat{U}(t) = \hat{U}(\xi, t) = \mathcal{F}U(\xi, t)$, $B(\varepsilon, \xi)$ is defined as

$$B(\varepsilon,\xi) = \begin{pmatrix} 0 & \dfrac{i}{\varepsilon}\xi^T \\ \dfrac{i}{\varepsilon}(a + \dfrac{1}{|\xi|^2})\xi & \mu|\xi|^2 I_d + \nu\xi\xi^T \end{pmatrix}.$$

Further, the eigenvalues of the matrix B are computed from the determinant

$$\det(B(\varepsilon,\xi) - \lambda I) = (\mu|\xi|^2 - \lambda)^{d-1}\left(\lambda^2 - |\xi|^2(\mu+\nu)\lambda + \frac{1 + a|\xi|^2}{\varepsilon^2}\right) = 0.$$

That is,

$$\begin{aligned} &\lambda_* = \mu|\xi|^2, \quad d-1(\text{multiple}) \\ &\lambda_\pm = \frac{\mu+\nu}{2}|\xi|^2 \pm \frac{i}{2\varepsilon}\sqrt{4(1 + a|\xi|^2) - (\mu+\nu)^2|\xi|^4\varepsilon^2}. \end{aligned} \tag{5.10}$$

The corresponding eigenvectors of B are

$$e_i(\xi) = \begin{pmatrix} 0 \\ \tilde{e}_i(\xi) \end{pmatrix}, \quad i = 1, \ldots, d-1, \qquad e_\pm(\xi) = \frac{1}{e(|\xi|)}\begin{pmatrix} \theta_\pm(|\xi|) \\ \xi/|\xi| \end{pmatrix},$$

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where

$$\tilde{e}_i(\xi)\ (i=1,\ldots,d-1) \text{ is the orthonormal basis vector of } \{\eta\in\mathbf{R}^d|\,\eta\cdot\xi=0\},$$
$$\theta_\pm(|\xi|)=\frac{i|\xi|}{\varepsilon\lambda_\pm(|\xi|)},$$
$$e(|\xi|)=\sqrt{|\theta_\pm(|\xi|)|^2+1}=\sqrt{\frac{|\xi|^2}{1+a|\xi|^2}+1} \text{ such that } |e_\pm(\xi)|=1.$$

Then the solution of (5.9) can be represented by

$$\hat{U}(\xi,t)=\mathrm{e}^{-B(\varepsilon,\xi)t}\hat{U}_I(\xi)=\sum_j \mathrm{e}^{-\lambda_j t}(e_j(\xi),\hat{U}_I(\xi))e_j(\xi),$$

and the inverse Fourier transform

$$\begin{aligned} U(x,t)=\mathcal{L}^\varepsilon(t)U_I&=\mathcal{F}^{-1}(\mathrm{e}^{-tB(\varepsilon,\xi)}\hat{U}_I(\xi))\\ &:=\mathcal{L}^\varepsilon_*(t)U_I+\mathcal{L}^\varepsilon_+(t)U_I+\mathcal{L}^\varepsilon_-(t)U_I \end{aligned} \tag{5.11}$$

gives the solution to (5.2), where

$$\begin{aligned} \mathcal{L}^\varepsilon_*(t)U_I&=\mathcal{F}^{-1}\left(\sum_{j=1}^{d-1}\mathrm{e}^{-\lambda_* t}(e_j(\xi),\hat{U}_I(\xi))e_j(\xi)\right),\\ \mathcal{L}^\varepsilon_\pm(t)U_I&=\mathcal{F}^{-1}\left(\mathrm{e}^{-\lambda_\pm t}(e_\pm(\xi),\hat{U}_I(\xi))e_\pm(\xi)\right). \end{aligned} \tag{5.12}$$

Now let us estimate each part of the above decomposition. First, by definition, we compute

$$\begin{aligned} \mathcal{L}^\varepsilon_*(t)U_I&=\mathcal{F}^{-1}\left(\mathrm{e}^{-\mu|\xi|^2 t}\sum_{j=1}^{d-1}(e_j(\xi),\hat{U}_I(\xi))e_j(\xi)\right)\\ &=\mathcal{G}*\mathcal{F}^{-1}\left(\sum_{j=1}^{d-1}(e_j(\xi),\hat{U}_I(\xi))e_j(\xi)\right), \end{aligned}$$

where $\mathcal{G}(x,\,t)=(1/4\pi\mu t)^{d/2}\exp(-(|x|^2/4\mu t))$ is the heat kernel. Since $e_j(\xi)=(0,\,\tilde{e}_j(\xi)^T)^T$, we get

$$\mathcal{L}^\varepsilon_*(t)U_I=\begin{pmatrix}0\\ \mathcal{G}*Pu_I\end{pmatrix}\quad\text{for}\quad U_I=\begin{pmatrix}n_0\\ u_I\end{pmatrix},$$

in which

$$Pu_I=\mathcal{F}^{-1}\left(\sum_{j=1}^{d-1}(\tilde{e}_j(\xi),\hat{u}_I(\xi))\tilde{e}_j(\xi)\right),$$

that is, P is the orthogonal projection of H^s onto the subspace $\{v\in H^s:\ \nabla\cdot v=0\}$. Furthermore, (5.5) is proved by using Hausdorff–Young inequality for convolution.

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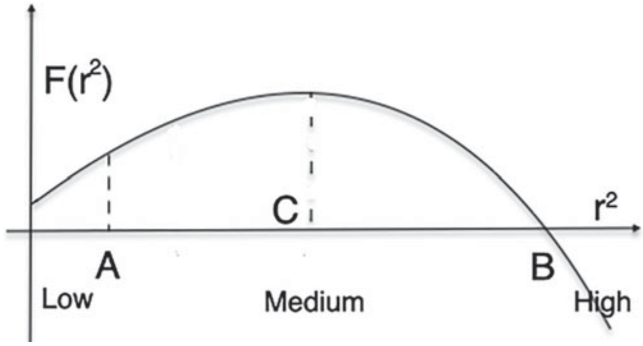

Figure 1. Frequency decomposition.

Next, to estimate $\mathcal{L}^{\varepsilon}_{\pm}$ for given ε, we recall that

$$\lambda_{\pm}=\frac{\mu+\nu}{2}|\xi|^2 \pm \frac{i}{2\varepsilon}\sqrt{F(|\xi|^2)}, \quad F(|\xi|^2) = 4(1+a|\xi|^2) - (\mu+\nu)^2|\xi|^4\varepsilon^2.$$

Due to the singularity of the heat kernel at $t = 0$ and good decay away from $t = 0$, we need to have more precise local in time estimate of the solution operator such that the result is integrable in time. Thus we need two sets of estimates: $t < \delta$ and $t \geqslant \delta$, for any given positive constant δ. Without loss of generality, we take $\delta = 1$, then derive the estimate for $t < 1$ and $t \geqslant 1$, respectively. (ii) For $t < 1$, let

$$A = \left(\frac{t}{\varepsilon}\right)^{2\theta}, \quad \theta \in \left(0, \frac{1}{d}\right), \quad B = \frac{2a + 2\sqrt{a^2 + (\mu+\nu)^2\varepsilon^2}}{(\mu+\nu)^2\varepsilon^2},$$

the frequency is decomposed into three parts: $|\xi|^2 \in [0, A]$, $[A, B]$, $[B, \infty)$. Note that $t < 1$, the above selection of A implies $A < C := 2a/((\mu+\nu)^2\varepsilon^2)$ for ε small, thus F is strictly increasing in the low frequency part. See figure 1.

Then we have

$$\mathcal{L}^{\varepsilon}_{\pm}(t)U_I = I_{\pm}+J_{\pm}+K_{\pm}, \tag{5.13}$$

where

$$I_{\pm} := \int_{|\xi|^2\leqslant A} \mathrm{e}^{-\lambda_{\pm}t}\mathrm{e}^{ix\cdot\xi}(e_{\pm}(\xi), \hat{U}_I(\xi))e_{\pm}(\xi)\,\mathrm{d}\xi,$$

$$J_{\pm} := \int_{A\leqslant|\xi|^2\leqslant B} \mathrm{e}^{-\lambda_{\pm}t}\mathrm{e}^{ix\cdot\xi}(e_{\pm}(\xi), \hat{U}_I(\xi))e_{\pm}(\xi)\,\mathrm{d}\xi,$$

$$K_{\pm} := \int_{|\xi|^2\geqslant B} \mathrm{e}^{-\lambda_{\pm}t}\mathrm{e}^{ix\cdot\xi}(e_{\pm}(\xi), \hat{U}_I(\xi))e_{\pm}(\xi)\,\mathrm{d}\xi.$$

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In the following we will give the estimates of $I_\pm$, $J_\pm$ and $K_\pm$, respectively. First, it is easy to compute

$$\begin{aligned} I_\pm(x,t) &= \int_{|\xi|^2\leqslant A} \mathrm{e}^{-((\mu+\nu)/2)|\xi|^2 t}\mathrm{e}^{\mp\frac{it}{2\varepsilon}\sqrt{F(r^2)}}\mathrm{e}^{ix\cdot\xi}(e_\pm(\xi),\hat{U}_I(\xi))e_\pm(\xi)\,\mathrm{d}\xi \\ &= \int_0^{\sqrt{A}} \mathrm{e}^{-((\mu+\nu)/2)r^2 t}\mathrm{e}^{\mp\frac{it}{2\varepsilon}\sqrt{F(r^2)}}\frac{r^{d-1}}{\mathrm{e}^2(r)}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d_y}\mathrm{e}^{ir(x-y)\cdot\omega} \\ &\quad\cdot(\theta_\pm(r)h_0(y)+\omega\cdot v_0(y))\begin{pmatrix}\theta_\pm(r)\\ \omega\end{pmatrix}\mathrm{d}y\mathrm{d}\omega\mathrm{d}r := \begin{pmatrix} I^1_\pm(x,t)\\ I^2_\pm(x,t)\end{pmatrix}. \end{aligned}$$

Setting

$$\begin{aligned} \zeta(r) &= \mathrm{e}^{-((\mu+\nu)/2)r^2 t}\frac{r^{d-1}}{\mathrm{e}^2(r)\big(\sqrt{F(r^2)}\big)'} \\ &= \mathrm{e}^{-((\mu+\nu)/2)r^2 t}r^{d-2}\frac{\sqrt{4(1+ar^2)-(\mu+\nu)^2r^4\varepsilon^2}}{2a-\varepsilon^2(\mu+\nu)^2r^2}\frac{1+ar^2}{1+(1+a)r^2}, \\ \alpha(r,x-y) &= \int_{\mathbb{S}^{d-1}}\mathrm{e}^{ir(x-y)\cdot\omega}\mathrm{d}\omega, \\ \beta(r,x-y) &= \int_{\mathbb{S}^{d-1}}\omega\,\mathrm{e}^{ir(x-y)\cdot\omega}\mathrm{d}\omega, \\ g_{\pm,1}(r,x) &= \zeta(r)\left(\theta_\pm(r)^2\int_{\mathbb{R}^d_y}\alpha(r,x-y)h_0(y)\mathrm{d}y+\theta_\pm(r)\int_{\mathbb{R}^d_y}\beta(r,x-y)v_0(y)\mathrm{d}y\right), \\ g_{\pm,2}(r,x) &= \zeta(r)\left(\theta_\pm(r)\int_{\mathbb{R}^d_y}\beta(r,x-y)h_0(y)\mathrm{d}y+\omega_{d-1}\int_{\mathbb{R}^d_y}\alpha(r,x-y)v_0(y)\mathrm{d}y\right), \end{aligned}$$

then for $k=1,\,2$, we have

$$\begin{aligned} I^k_\pm(x,t) &= \int_0^{\sqrt{A}}\mathrm{e}^{\mp\frac{it}{2\varepsilon}\sqrt{F(r^2)}}\big(\sqrt{F(r^2)}\big)'g_{\pm,k}(r,x)\mathrm{d}r \\ &= \mp\frac{2\varepsilon}{it}\left(\mathrm{e}^{\mp\frac{it}{2\varepsilon}\sqrt{F(r^2)}}g_{\pm,k}(r,x)\Big|_0^{\sqrt{A}}-\int_0^{\sqrt{A}}\mathrm{e}^{\mp\frac{it}{2\varepsilon}\sqrt{F(r^2)}}\frac{\partial g_{\pm,k}(r,x)}{\partial r}\mathrm{d}r\right). \end{aligned}$$

Note $g_{\pm,k}(0,\,x)=0$, we have

$$I^k_\pm(x,t) = \mp\frac{2\varepsilon}{it}\left(\mathrm{e}^{\mp\frac{it}{2\varepsilon}\sqrt{F(A)}}g_{\pm,k}(\sqrt{A},x)-\int_0^{\sqrt{A}}\mathrm{e}^{\mp\frac{it}{2\varepsilon}\sqrt{F(r^2)}}\frac{\partial g_{\pm,k}(r,x)}{\partial r}\mathrm{d}r\right),$$

thus

$$|I^k_\pm(x,t)| \leqslant \frac{2\varepsilon}{t}\left(|g_{\pm,k}(\sqrt{A},x)|+\int_0^{\sqrt{A}}\left|\frac{\partial g_{\pm,k}(r,x)}{\partial r}\right|\mathrm{d}r\right). \tag{5.14}$$

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Recall $A = (\frac{t}{\varepsilon})^{2\theta}$. Direct computation shows

$$\begin{cases} |\zeta(\sqrt{A})| \leqslant C\mathrm{e}^{-((\mu+\nu)/2)(t/\varepsilon)^{2\theta}t} \left(\dfrac{t}{\varepsilon}\right)^{\theta(d-1)}, \\ \theta_{\pm}(|\xi|) = \dfrac{i|\xi|}{\varepsilon\lambda_{\pm}(|\xi|)}, \qquad |\theta_{\pm}(|\xi|)| = \dfrac{|\xi|^2}{1+a|\xi|^2} \leqslant \dfrac{1}{a}, \\ |\alpha(r,x-y)|, |\beta(r,x-y)| \leqslant Cr^{-l}|x-y|^{-l} \ \text{ for } \ 0 \leqslant l \leqslant (d-1)/2, \end{cases} \tag{5.15}$$

thus the first term in the right hand side of (5.14) can be controlled by

$$|g_{\pm,k}(\sqrt{A},x)| \leqslant C\mathrm{e}^{-((\mu+\nu)/2)(t/\varepsilon)^{2\theta}t} \left(\frac{t}{\varepsilon}\right)^{\theta(d-1)} \|U_I\|_{L^1}. \tag{5.16}$$

Moreover, we compute

$$\begin{aligned} &\frac{\partial g_{\pm,1}(r,x)}{\partial r} \\ &\quad = \zeta'(r)\left(\theta_{\pm}(r)^2 \int_{\mathbb{R}^d_y} \alpha(r,x-y)h_0(y)\mathrm{d}y + \theta_{\pm}(r)\int_{\mathbb{R}^d_y} \beta(r,x-y)v_0(y)\mathrm{d}y\right) \\ &\qquad + \zeta(r)\theta'_{\pm}(r)\left(2\theta_{\pm}(r)\int_{\mathbb{R}^d_y} \alpha(r,x-y)h_0(y)\mathrm{d}y + \int_{\mathbb{R}^d_y} \beta(r,x-y)v_0(y)\mathrm{d}y\right) \\ &\qquad + \zeta(r)\left(\theta_{\pm}(r)^2 \int_{\mathbb{R}^d_y} \alpha'(r,x-y)h_0(y)\mathrm{d}y + \theta_{\pm}(r)\int_{\mathbb{R}^d_y} \beta'(r,x-y)v_0(y)\mathrm{d}y\right). \end{aligned}$$

Recall that for $d \geqslant 3$, one has (see [**3**, P2756]),

$$|\alpha'(r,x-y)|, \ |\beta'(r,x-y)| \leqslant \frac{C}{r},$$

and noting also that $\zeta(r)$ and $\zeta'(r)$ will both have a factor $\mathrm{e}^{-((\mu+\nu)/2)r^2t}$, after tedious calculation, with the estimates in (5.15), we find

$$\left|\frac{\partial g_{\pm,1}(r,x)}{\partial r}\right| \leqslant C\mathrm{e}^{-((\mu+\nu)/2)r^2t} r^{d-1}\|U_I\|_{L^1}.$$

Similar estimate holds for $g_{\pm,2}(r,\, x)$. Thus for $k = 1,\, 2$,

$$\int_0^{\sqrt{A}} \left|\frac{\partial g_{\pm,k}(r,x)}{\partial r}\right| \mathrm{d}r \leqslant C\int_0^{\sqrt{A}} r^{d-1}\mathrm{d}r\, \|U_I\|_{L^1} \leqslant C\left(\frac{t}{\varepsilon}\right)^{\theta d} \|U_I\|_{L^1}. \tag{5.17}$$

Plugging (5.16) and (5.17) into (5.14), we have

$$|I^k_{\pm}(x,t)| \leqslant \frac{\varepsilon}{t}C\left(\mathrm{e}^{-((\mu+\nu)/2)(t/\varepsilon)^{2\theta}t}\left(\frac{t}{\varepsilon}\right)^{\theta(d-1)} + \left(\frac{t}{\varepsilon}\right)^{\theta d}\right)\|U_I\|_{L^1}. \tag{5.18}$$

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Next, since

$$J_\pm = \int_{A\leqslant|\xi|^2\leqslant B} \mathrm{e}^{-((\mu+\nu)/2)|\xi|^2 t}\mathrm{e}^{\mp(i/2\varepsilon)\sqrt{4(1+a|\xi|^2)-(\mu+\nu)^2|\xi|^4\varepsilon^2}\,t} \times \mathrm{e}^{ix\cdot\xi}(e_\pm(\xi),\hat{U}_I(\xi))e_\pm(\xi)\,\mathrm{d}\xi,$$

we have

$$\begin{aligned}|J_\pm| &\leqslant \int_{A\leqslant|\xi|^2\leqslant B} \mathrm{e}^{-((\mu+\nu)/2)|\xi|^2 t}|(e_\pm(\xi),\hat{U}_I(\xi))e_\pm(\xi)|\,\mathrm{d}\xi \\ &\leqslant \mathrm{e}^{-((\mu+\nu)/2)At}\|U_I\|_{L^1} = \mathrm{e}^{-((\mu+\nu)/2)(t/\varepsilon)^{2\theta}t}\|U_I\|_{L^1}.\end{aligned} \tag{5.19}$$

Finally, noting that F is negative for $|\xi|^2 \geqslant B$, then $\lambda_\pm$ are real numbers:

$$\lambda_\pm = \frac{\mu+\nu}{2}|\xi|^2 \pm \frac{1}{2}\sqrt{(\mu+\nu)^2|\xi|^4 - \frac{4(1+a|\xi|^2)}{\varepsilon^2}} = \frac{\mu+\nu}{2}|\xi|^2(1\pm\sqrt{1-\delta}),$$

where $0<\delta = ((4(1+a|\xi|^2))/((\mu+\nu)^2|\xi|^4\varepsilon^2)) \leqslant 1$ in the case of high frequency. Moreover, since $\delta\in(0,\,1]$,

$$1+\sqrt{1-\delta}\geqslant 1 > \frac{1}{2}\delta, \quad 1-\sqrt{1-\delta}\geqslant \frac{1}{2}\delta,$$

then

$$\lambda_\pm \geqslant \frac{\mu+\nu}{2}|\xi|^2\cdot\frac{1}{2}\delta = \frac{1+a|\xi|^2}{(\mu+\nu)|\xi|^2\varepsilon^2} \geqslant \frac{a}{(\mu+\nu)\varepsilon^2}. \tag{5.20}$$

Hence we have

$$\begin{aligned}|K_\pm| &= \left|\int_{|\xi|^2\geqslant B} \mathrm{e}^{-\lambda_\pm t}\mathrm{e}^{ix\cdot\xi}(e_-(\xi),\hat{U}_I(\xi))e_-(\xi)\,\mathrm{d}\xi\right| \\ &\leqslant \mathrm{e}^{-(a\,t/((\mu+\nu)\varepsilon^2))}\int_{|\xi|^2\geqslant B}|(e_+(\xi),\hat{U}_I(\xi))e_+(\xi)|\,\mathrm{d}\xi \\ &\leqslant \mathrm{e}^{-(a\,t/((\mu+\nu)\varepsilon^2))}\|U_I\|_{L^1}.\end{aligned} \tag{5.21}$$

Combine (5.18), (5.19), (5.21) and (5.13), we conclude (5.6). (iii) Next, we prove (5.7) for the case $t\geqslant 1$. Indeed, we will use different frequency decomposition. That is, taking $\tilde{A} = C/2 = a/((\mu+\nu)^2\varepsilon^2)$, the three parts of the frequency are now $|\xi|^2\in[0,\,\tilde{A}]$, $[\tilde{A},\,B]$, $[B,\,\infty)$. The estimate (5.16) is modified to

$$|g_{\pm,k}(\sqrt{\tilde{A}},x)| \leqslant C\mathrm{e}^{-\frac{at}{2(\mu+\nu)\varepsilon^2}}\frac{1}{\varepsilon^{d-1}}\|U_I\|_{L^1},$$

the estimate (5.17) is modified to

$$\int_0^{\sqrt{\tilde{A}}}\left|\frac{\partial g_{\pm,k}(r,x)}{\partial r}\right|\mathrm{d}r \leqslant C\int_0^{\sqrt{\tilde{A}}}\mathrm{e}^{-((\mu+\nu)/2)r^2 t}r^{d-1}\mathrm{d}r\|U_I\|_{L^1} \leqslant C\frac{1}{t^{d/2}}\|U_I\|_{L^1},$$

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thus, instead of (5.18), we have

$$|I^k_\pm(x,t)| \leqslant \frac{\varepsilon}{t} C \left(\mathrm{e}^{-((\mu+\nu)/2)(\frac{t}{\varepsilon})^{2\theta} t} \left(\frac{t}{\varepsilon}\right)^{\theta(d-1)} + \left(\frac{t}{\varepsilon}\right)^{\theta d} \right) \|U_I\|_{L^1}. \tag{5.22}$$

Next, for the medium frequency part, the estimate (5.19) is modified to

$$\begin{aligned} |J_\pm| &\leqslant \int_{\tilde{A}\leqslant|\xi|^2\leqslant B} \mathrm{e}^{-((\mu+\nu)/2)|\xi|^2 t} |(e_\pm(\xi), \hat{U}_I(\xi)) e_\pm(\xi)| \, \mathrm{d}\xi. \\ &\leqslant \mathrm{e}^{-((\mu+\nu)/2)\tilde{A}t} \|U_I\|_{L^1} = \mathrm{e}^{-(at/(2(\mu+\nu)\varepsilon^2))} \|U_I\|_{L^1}. \end{aligned} \tag{5.23}$$

The high frequency part estimate is unchanged. Combining (5.21)–(5.23), we have the estimate (5.7) for $t > 1$. (iv) The estimate (5.8) is directly derived from (5.5)–(5.7). The proof of the lemma is finished. □

Moreover, from Hölder's inequality, lemma 2.1 and the uniform estimates in theorem 1.1, we also have

Lemma 5.3. *Under the assumptions of theorem 1.1, there exists a constant $C > 0$ independent of ε such that for any $\tau \in [0, T_0]$, $k = 1, 2$,*

$$\|G^\varepsilon_k\|_{L^1} + \|G^\varepsilon_k\|_{s-2} \leqslant C.$$

The proof of lemma 5.3 is same as that of lemma 7 in [**3**], and we can omit the details here.

Now let us consider the zero-electron-mass limit of the initial value problem (1.2)–(1.3) in the ill-prepared initial data case. First, by the Duhamel's principle, the solution of system (5.1) can be given as

$$\mathcal{U}^\varepsilon(t) := \begin{pmatrix} \tilde{n}(t) \\ u(t) \end{pmatrix} = \mathcal{L}^\varepsilon(t) U_I + \int_0^t \mathcal{L}^\varepsilon(t-\tau) G^\varepsilon(\tau) \mathrm{d}\tau, \tag{5.24}$$

where $\mathcal{L}^\varepsilon(t)$ is the solution operator studied in previous subsection, and

$$U_I = \begin{pmatrix} \tilde{n}_I \\ u_I \end{pmatrix}, \quad G^\varepsilon(\tau) = \begin{pmatrix} G^\varepsilon_1(\tau) \\ G^\varepsilon_2(\tau) \end{pmatrix}.$$

Note the decomposition (5.3), we rewrite

$$\mathcal{U}^\varepsilon(t) = \mathcal{U}^\varepsilon_*(t) + \mathcal{U}^\varepsilon_+(t) + \mathcal{U}^\varepsilon_-(t), \tag{5.25}$$

where

$$\begin{aligned} \mathcal{U}^\varepsilon_*(t) &= \mathcal{L}^\varepsilon_*(t) U_I + \int_0^t \mathcal{L}^\varepsilon_*(t-\tau) G^\varepsilon(\tau) \mathrm{d}\tau, \\ \mathcal{U}^\varepsilon_\pm(t) &= \mathcal{L}^\varepsilon_\pm(t) U_I + \int_0^t \mathcal{L}^\varepsilon_\pm(t-\tau) G^\varepsilon(\tau) \mathrm{d}\tau. \end{aligned}$$

By the definition of $\mathcal{L}^\varepsilon_*(t)$ and (5.4)–(5.5), we have

$$\mathcal{U}^\varepsilon_* = \begin{pmatrix} 0 \\ u^\varepsilon_* \end{pmatrix}, \qquad u^\varepsilon_* = \mathcal{G} * P u_I + \int_0^t \mathcal{G}(t-\tau) * P G^\varepsilon_2(\tau) d\tau. \tag{5.26}$$

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Note (5.5) and the heat kernel $\mathcal{G}$ is smooth, then, with use of lemma 5.1 and lemma 5.3, we have

LEMMA 5.4. *Under the assumptions of theorem 1.3, there exists a constant C independent of ε such that*

$$\|u_*^\varepsilon\|_s + \|\partial_t u_*^\varepsilon\|_{s-2} \leqslant C.$$

Proof. By the orthogonal projection, we firstly have

$$\|u_*^\varepsilon\|_s \leqslant \|u^\varepsilon\|_s \leqslant C$$

from the uniform estimate (1.9). Secondly, the solution u_*^ε defined in (5.26) satisfies

$$\begin{cases} \partial_t u_*^\varepsilon - \Delta u_*^\varepsilon = PG_2^\varepsilon, \\ u_*^\varepsilon(0) = Pu_I, \end{cases}$$

thus

$$\|\partial_t u_*^\varepsilon\|_{s-2} \leqslant \|\Delta u_*^\varepsilon\|_{s-2} + \|PG_2^\varepsilon\| \leqslant C$$

by using (1.9), lemma 5.1 and lemma 5.3. The proof is completed. □

LEMMA 5.5. *Assume $U_I \in L^1 \cap H^s$. For any fixed $s > 0$, it holds that*

$$\sup_{t \geqslant s} \|\mathcal{U}_\pm^\varepsilon(t)\|_\infty \to 0 \quad as \quad \varepsilon \to 0. \tag{5.27}$$

Proof. Without loss of generality, we assume $t > 1$ and decompose the integral

$$\mathcal{U}_\pm^\varepsilon(t) = \mathcal{L}_\pm^\varepsilon(t)U_I + \left(\int_0^{t-1} + \int_{t-1}^t\right) \mathcal{L}_\pm^\varepsilon(t-\tau)G^\varepsilon(\tau)\mathrm{d}\tau.$$

By using (5.8), we find

$$|\mathcal{L}_\pm^\varepsilon(t)U_I| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{5.28}$$

For $\tau \in [0, t-1]$, that is, $t - \tau \geqslant 1$, then using the estimate of $\mathcal{L}_\pm^\varepsilon$ in (5.7) and lemma 5.1, we compute

$$\begin{aligned} &\left|\int_0^{t-1} \mathcal{L}_\pm^\varepsilon(t-\tau)G^\varepsilon(\tau)\mathrm{d}\tau\right| \\ &\quad \leqslant \int_0^{t-1} \left[\frac{\varepsilon}{t-\tau} C\left(\mathrm{e}^{-((a(t-\tau))/(2(\mu+\nu)\varepsilon^2))} \frac{1}{\varepsilon^{d-1}} + \frac{1}{(t-\tau)^{d/2}}\right)\right. \\ &\qquad \left. + \mathrm{e}^{-((a(t-\tau))/(2(\mu+\nu)\varepsilon^2))}\right] \|G^\varepsilon(\tau)\|_{L^1}\mathrm{d}\tau \end{aligned}$$

thus we have

$$\left|\int_0^{t-1} \mathcal{L}_\pm^\varepsilon(t-\tau)G^\varepsilon(\tau)\mathrm{d}\tau\right| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{5.29}$$

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Similarly, for $\tau \in (t-1,\, t]$, that is, $0 \leqslant t-\tau < 1$, then use the estimate of $\mathcal{L}^{\varepsilon}_{\pm}$ in (5.6) and lemma 5.1, we compute

$$\begin{aligned}
&\left|\int_{t-1}^{t} \mathcal{L}^{\varepsilon}_{\pm}(t-\tau)G^{\varepsilon}(\tau)\mathrm{d}\tau\right| \\
&\quad \leqslant \int_{t-1}^{t} \left[\frac{\varepsilon}{t-\tau} C\left(\mathrm{e}^{-((\mu+\nu)/2)((t-\tau)/\varepsilon)^{2\theta}(t-\tau)}\left(\frac{t-\tau}{\varepsilon}\right)^{\theta(d-1)} + \left(\frac{t-\tau}{\varepsilon}\right)^{\theta d}\right)\right. \\
&\qquad \left. + \mathrm{e}^{-((\mu+\nu)/2)((t-\tau)/\varepsilon)^{2\theta}(t-\tau)} + \mathrm{e}^{-((a(t-\tau))/((\mu+\nu)\varepsilon^2))}\right] \|G^{\varepsilon}(\tau)\|_{L^1}\mathrm{d}\tau,
\end{aligned}$$

thus

$$\left|\int_{t-1}^{t} \mathcal{L}^{\varepsilon}_{\pm}(t-\tau)G^{\varepsilon}(\tau)\mathrm{d}\tau\right| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{5.30}$$

Combining (5.28)–(5.30), we have (5.27). This completes the proof. □

Now we are ready to prove the zero-electron-mass limit of (1.2)–(1.3) for ill-prepared initial data.

Proof of theorem 1.3. Let $(n^{\varepsilon},\, u^{\varepsilon},\, E^{\varepsilon})$ be a classical solution defined in theorem 1.1 to (1.2)–(1.3) in $[0,\, T_0]$, with $T_0 > 0$ independent of ε. From (1.9), we have

$$\begin{aligned}
&(n^{\varepsilon}, E^{\varepsilon}) \to (N, 0) \quad \text{strongly in} \quad L^{\infty}([0, T_0]; H^s(\mathbb{R}^d)), \\
&u^{\varepsilon} \to u^0_* \quad \text{weakly* in} \quad L^{\infty}([0, T_0]; H^s(\mathbb{R}^d)).
\end{aligned}$$

Recall (5.24)–(5.25), we write

$$(\tilde{n}^{\varepsilon}, u^{\varepsilon})^T = \mathcal{U}^{\varepsilon} = \mathcal{U}^{\varepsilon}_* + \mathcal{U}^{\varepsilon}_+ + \mathcal{U}^{\varepsilon}_-.$$

For the first part $\mathcal{U}^{\varepsilon}_* = (0,\, u^{\varepsilon}_*)^T$, by lemma 5.4, there exists a subsequence of u^{ε}_* (still denoted by itself), and $u^0_* \in C^0([0,\, T_0] \times \mathbb{R}^d) \cap L^{\infty}([0,\, T_0]; H^s(\mathbb{R}^d))$, such that

$$u^{\varepsilon}_* \to u^0_* \quad \text{strongly in} \quad C^0_{loc}([0, T_0] \times \mathbb{R}^d).$$

Further use lemma 5.5, we have

$$(\tilde{n}^{\varepsilon}, u^{\varepsilon})^T = \mathcal{U}^{\varepsilon}_* + \mathcal{U}^{\varepsilon}_+ + \mathcal{U}^{\varepsilon}_- \to (0, u^0_*)^T \quad \text{strongly in} \quad C^0_{loc}((0, T_0] \times \mathbb{R}^d).$$

By the orthogonal decomposition of $\mathcal{U}^{\varepsilon}$, we have $Pu^{\varepsilon}_* = u^{\varepsilon}_*$. Passing to the limit as $\varepsilon \to 0$ gives

$$Pu^0_* = u^0_*, \quad \text{that is} \quad \nabla \cdot u^0_* = 0.$$

Next, recall the uniform estimates in theorem 1.1, we have the weak-$*$ convergence of $(\tilde{n}^{\varepsilon},\, u^{\varepsilon})$ in $L^{\infty}([0,\, T_0]; H^s(\mathbb{R}^d))$, thus

$$G^{\varepsilon}_2 \to -u^0_* \cdot \nabla u^0_* \quad \text{weakly* in} \quad L^{\infty}([0, T_0]; H^{s-1}(\mathbb{R}^d)).$$

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Now pass to the limit as $\varepsilon \to 0$ in (5.26), we get

$$u_*^0 = \mathcal{G} * Pu_I - \int_0^t \mathcal{G}(t-\tau) * P[u_*^0 \cdot \nabla u_*^0(\tau)]\mathrm{d}\tau,$$

thus u_*^0 satisfies the equation

$$\partial_t u_*^0 - \mu\Delta u_*^0 = -P[u_*^0 \cdot \nabla u_*^0], \quad u_*^0(x,0) = Pu_I^0,$$

thus the proof of theorem 1.3 is complete. □

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