A BASIS FOR THE LAWS OF THE VARIETY $s\mathfrak{A}_{30}$

C. CHRISTENSEN

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To Bernhard Hermann Neumann on his 60th birthday

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A group is called an $s\mathfrak{A}$-group if and only if it is locally finite and all its Sylow subgroups are abelian. Kovács [1] has shown that for any integer e the class $s\mathfrak{A}_e$ of all $s\mathfrak{A}$-groups of exponents dividing e is a variety. Little is known about the laws of these varieties; in particular it is unknown whether they have finite bases. Whenever $s\mathfrak{A}_e$ is soluble it is an easy matter to establish explicitly a finite basis for its laws namely the exponent law, the appropriate solubility length law and all laws of the type $[x^m, y^m]^m$ where $e = p^\alpha m$, p is a prime and p does not divide m. (The significance of the last type of law is made clear by Proposition 2 below and the obvious fact that any group that satisfies a law of this type for given prime p has abelian Sylow p-subgroups.) For e less than thirty $s\mathfrak{A}_e$ is clearly soluble whilst PSL(2, 5), the non-abelian simple group of order 60, is contained in $s\mathfrak{A}_{30}$ so that the case $e = 30$ is, in a sense, the first non-trivial case to be considered.

The purpose of this note is to establish the following set of laws as a basis for the variety $s\mathfrak{A}_{30}$:

(i) x^{30}

(ii) $\{((x^6y^{12})^5(x^6y^{18})^5)^3[x^6, y^6]^6\}^6$

(iii) $((x^{10}y^{10})^6[x^{10}, y^{10}]^2)^{10}$

(iv) $[[u_{60}, u_{60}^y], [u_{60}, u_{60}^y]^z]$

Here, u_{60} is one of the chief centraliser laws of Kovács and Newman defined by:

$$u_2 = [x_1, x_2, (x_1^{-1}x_2)^{y_{1,2}}]$$

and inductively for $n \geqq 2$ by

$$u_n = [u_{n-1}, x_n^{y_n}, (x_1^{-1}x_n)^{y_{1,n}}, \cdots, (x_{n-1}^{-1}x_n)^{y_{n-1,n}}].$$

With the obvious exceptions of the two definitions made already, the notation and terminology used above and in what follows is that of Hanna Neumann's book [2].

The second law above is one of the laws in the basis for the variety generated by PSL(2, 5) obtained by Cossey and Macdonald [3] whilst the third law is the square of another of them. The fourth law was brought to my notice by Dr. M. F. Newmann and I am indebted to him for his permission to reproduce his proof of Proposition 6. This law is a considerable improvement on the law originally used to force local finiteness in the sense that it uses one quarter the number of variables approximately.

The proof of Theorem 4.2 of [3] yields a stronger theorem than that stated, namely:

THEOREM A (*Cossey and Macdonald*). *Let $\mathfrak{V}$ be a variety of $s\mathfrak{A}$-groups of exponent dividing* 30 *in which* $PSL(2, 5)$ *is the only non-abelian simple group, then every finite group in $\mathfrak{V}$ is a direct product of copies of* $PSL(2, 5)$ *and a soluble subgroup.*

Since it is a consequence of a result announced by J. H. Walter (see [4] p. 485) that PSL(2,5) is the only finite non-abelian simple $s\mathfrak{A}$-group of exponent dividing thirty, it follows from Theorem A that PSL(2,5) is the only insoluble finite monolithic group in $s\mathfrak{A}_{30}$. Moreover, Taunt has shown [5] that the solubility length of a finite soluble $s\mathfrak{A}$-group cannot exceed the number of primes dividing its order. This establishes the following result.

PROPOSITION 1. *Every finite monolithic group in $s\mathfrak{A}_{30}$ is soluble of length at most three or is isomorphic with* $PSL(2, 5)$.

The usefulness of this result stems from the fact that $s\mathfrak{A}_{30}$ is generated by its finite monolithic groups (see (51.32) and (51.41) of [2]). Therefore, in order to show that the variety $\mathfrak{U}$ defined by the laws (i) to (iv) contains $s\mathfrak{A}_{30}$ it is sufficient to show that PSL(2, 5) and the finite soluble monolithic groups (of length not more than three) in $s\mathfrak{A}_{30}$ satisfy these laws. At first sight this appears to be more than is necessary but Kovács and Newman have shown that a finite $s\mathfrak{A}$-group is critical if and only if it is monolithic.

The following result is well-known.

PROPOSITION 2. *Let G be a finite soluble group with abelian Sylow p-subgroups for some prime p and let G have exponent $p^{\alpha}m$ where p and m are coprime, then $[x^m, y^m]^m$ is a law in G.*

PROOF. Lemma (1.2.3) of [6] implies that G has p-length one so that $[x^m, y^m]$ is a p'-element of G.

For convenience we state the following result of Cossey [7].

PROPOSITION 3 (*Cossey*). *Let G be a finite soluble monolithic $s\mathfrak{A}$-group, then the last non-trivial term of the derived series of G is a normal Sylow p-subgroup of G for some prime p.*

In fact, a finite monolithic $s\mathfrak{A}$-group has a non-trivial normal Sylow

p-subgroup for some prime p if and only if it has a nontrivial soluble normal subgroup namely the last non-trivial term of the derived series of the soluble radical of G.

PROPOSITION 4. *The laws* (ii), (iii) *and* (iv) *are laws in* $s\mathfrak{A}_{30}$.

PROOF. It can be checked directly that (ii) and (iii) are laws of PSL(2.5). Since PSL(2,5) has order 60, it follows from (52.32) of [2] that it satisfies u_{60} and hence (iv).

Let G be a finite soluble monolithic group in $s\mathfrak{A}_{30}$. Then, by Proposition 3, G has a normal Sylow p-subgroup for $p = 2, 3$ or 5, and clearly we need consider only the case $P \neq G$ so that $P = G'$ or $P = G''$, and G'/P is abelian.

Consider first the law (ii). Let $x, y \in G$, then, by Proposition 2, $[x^6, y^6]^6 = 1$. If $p = 5$ then $x^6, y^6 \in P$ so that $(x^6 y^{12})^5 = (x^6 y^{18})^5 = 1$. If $p = 2$ or 3 then $(x^6 y^{12})^5 G' = (xy^2)^{30} G' = G'$ so that $(x^6 y^{12})^5 \in G'$ and similarly $(x^6 y^{18})^5 \in G'$. Let $g = (x^6 y^{12})^5 (x^6 y^{18})^5$, then since G'/P is abelian gP has order dividing 6. But the exponent of G'/P divides $5q$ where $6 = pq$ and it follows that g has order dividing pq. Hence (ii) is a law in G.

Consider now law (iii). Again let $x, y \in G$, then if $p = 3$, x^{10} and y^{10} are contained in P so that $[x^{10}, y^{10}] = 1 = (x^{10} y^{10})^3$. If $p = 2$ or 5 then we use the fact that $(x^{10} y^{10})^3 G' = G'$. Let $h = ((x^{10} y^{10})^6 [x^{10}, y^{10}]^2)^{10}$, then since G'/P is abelian $hP = (x^{10} y^{10})^{60} [x^{10}, y^{10}]^{20} P$ which is P by Proposition 2. Hence h is both a p-element for $p = 2$ or 5 and a 3-element; that is $h = 1$. It follows that (iii) is a law in G.

To show that $\mathfrak{U}$ and $s\mathfrak{A}_{30}$ are the same variety it remains only to show that the p-groups in $\mathfrak{U}$ are abelian and that $\mathfrak{U}$ is locally finite.

PROPOSITION 5. *Every p-group in $\mathfrak{U}$ is abelian.*

PROOF. Due to the exponent law, all the elements of a 2-group in $\mathfrak{U}$ are involutions which implies that such groups are abelian. Let G be a 5-group in $\mathfrak{U}$ and let $x, y \in G$. Substituting these elements of G in law (ii) yields the equation $[x, y] = 1$ as required. Similarly the law (iii) ensures that every 3-group in $\mathfrak{U}$ is abelian.

COROLLARY. *The finite groups in $\mathfrak{U}$ are contained in $s\mathfrak{A}_{30}$.*

PROPOSITION 6. *The only non-abelian simple group that satisfies* (iv) *is PSL*(2,5).

PROOF. Let H be a non-abelian simple group that satisfies (iv) and let v be a value of $[u_{60}, u_{60}^y]$ in H. Law (iv) implies that v commutes with all its conjugates in H so that its normal closure in H is abelian. It follows that $v = 1$. Using the same argument again we deduce that u_{60} is a law in H.

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This implies that the centraliser of every chief factor of H has index at most 60 in H (see (52.32) of [2]) and hence that H has order at most 60. It follows that H is isomorphic with PSL(2,5).

Now, in view of the corollary above, a finite k-generator group G in $\mathfrak{U}$ has order bounded by the order of the relatively free group of rank k of $s\mathfrak{A}_{30}$. It follows that G is finite for if this were not the case G would have a non-abelian simple composition factor in contradiction to Proposition 6. Thus $\mathfrak{U}$ is locally finite and we have proved:

THEOREM. *The set of laws* (i) *to* (iv) *given above is a basis for the laws of the variety* $s\mathfrak{A}_{30}$.

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Australian National University
Canberra