ON SUBSETS WITH INTERSECTIONS OF EVEN CARDINALITY

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This paper solves a question posed by P. Erdös:

THEOREM. *If* $A_1, A_2, \ldots, A$ *are distinct subsets of* n *elements and if* $|A_i \cap A_j| \equiv 0 \bmod 2$ $(i \neq j)$, *then*

$$M \leq \begin{cases} n+1 & \text{if } n \leq 5 \\ 2^{\frac{n}{2}} & \text{if } n \text{ even and } n \geq 6 \\ 1 + 2^{\frac{n-1}{2}} & \text{if } n \text{ odd and } n \geq 7 \end{cases}$$

and for each n, *there exists a collection of subsets which achieves this bound with equality.*

Proof. Without loss of generality, we assume the sets are ordered so that for some k,

$$|A_i| \equiv 1 \bmod 2 \qquad \text{if } i = 1, 2, \ldots, k$$

and

$$|A_i| \equiv 0 \bmod 2 \qquad \text{if } i = k+1, \ldots, M.$$

Let $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_M$ be the corresponding n-dimensional binary (GF(2)) row vectors. We may further assume that the sets are ordered so that $\vec{A}_{k+1}, \vec{A}_{k+2}, \ldots, \vec{A}_{k+\ell}$ form a basis of the set $\vec{A}_{k+1}, \vec{A}_{k+2}, \ldots, \vec{A}_M$, so that

$$(1) \qquad M \leq k + 2^{\ell}$$

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Since the binary dot product $\vec{A}_i \cdot \vec{A}_j$ gives the parity of $|A_i \cap A_j|$, we know that if $1 \leq i, j \leq k + \ell$, then

$$(2) \qquad \vec{A}_i \cdot \vec{A}_j = \begin{cases} 1 & \text{if} \quad i = j \leq k \\ 0 & \text{otherwise} \end{cases}$$

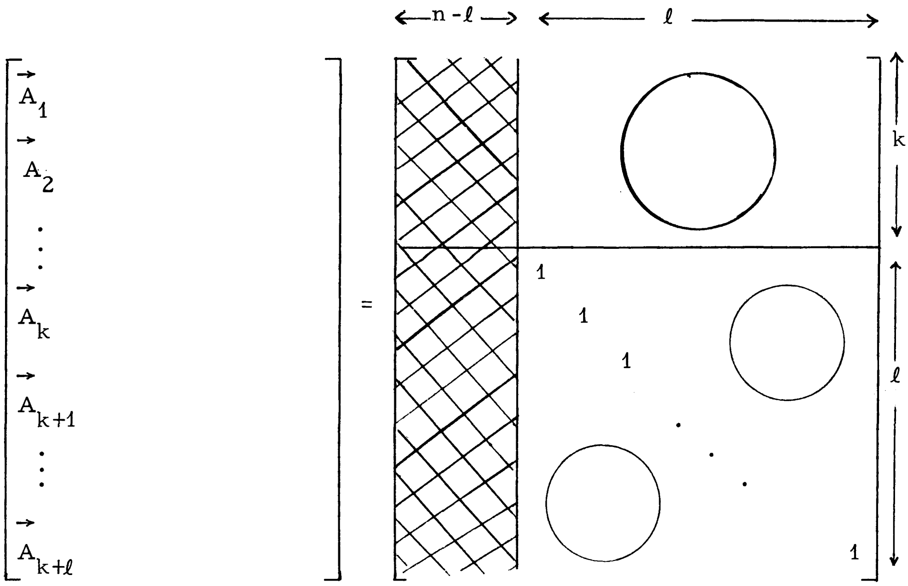
We also claim that the vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_{k+\ell}$ are linearly independent, for if there were an integer $i \leq k$ and binary elements $b_{i+1}, b_{i+2}, \ldots, b_{k+\ell}$ such that

$$(3) \qquad \vec{A}_i = \sum_{j=i+1}^{k+\ell} b_j \vec{A}_j$$

then we could take the dot product of $\vec{A}_i$ and each side of Equation (3) to obtain the contradiction

$$1 = \vec{A}_i \cdot \vec{A}_i = \sum_{j=i+1}^{k+\ell} b_j \vec{A}_j \cdot \vec{A}_i = \sum_{j=i+1}^{k+\ell} 0 = 0 .$$

If i is in the interval $k < i \leq k + \ell$ and if $j \neq i$, then we may replace $\vec{A}_j$ by $\vec{A}_j + \vec{A}_i$ and obtain a new set of $k + \ell$ linearly independent binary vectors which also satisfy Equation (2). If we select a column in which $\vec{A}_i$ has a 1 and then add $\vec{A}_i$ into each $\vec{A}_j$ which has a 1 in that column $(j \neq i)$, we obtain a set of vectors in which only $\vec{A}_i$ has a 1 in the selected column. If we repeat this procedure for all i in the interval $k < i \leq k + \ell$, and then permute columns appropriately, we obtain a set of vectors of the form



Let $B_1, B_2, \ldots, B_{k+\ell}$ be the $n - \ell$ dimensional binary vectors obtained by deleting the last ℓ columns of these A's. Since these A's satisfy Equation (2), we have

$$\vec{B}_i \cdot \vec{B}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

for all $1 \leq i, j < k + \ell$. Since the B's are orthonormal, they are linearly independent and we have that $k + \ell \leq n - \ell$ or $\ell \leq [\frac{1}{2}(n-k)]$, where $[x]$ denotes the greatest integer not exceeding x. Equation (1) now becomes

$$M \leq k + 2^{[\frac{1}{2}(n-k)]}$$

Since k is arbitrary, we write

$$M \leq \max_{0 \leq k \leq n} \left(k + 2^{[\frac{1}{2}(n-k)]} \right)$$

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If $n \leq 5$, a maximum is attained at $k = n$; if $n \geq 6$, a maximum is attained at $k = 0$ or 1, depending on the parity of n. This proves the bound stated in the theorem.

It is easy to construct collections of subsets satisfying the bound. For $n \leq 5$, the empty subset and the n one-element subsets suffice. For $n \geq 6$, we may select all $2^{[\frac{1}{2}n]}$ subsets in which each of $[\frac{1}{2}n]$ disjoint pairs of elements always occur together. If n is odd, we may also include the set consisting of the single unpaired element.

Remark. Professor J. E. Graver of Syracuse University also solved this problem, independently of the author.

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