Canad. J. Math. Vol. **63** (6), 2011 pp. 1345–1363
doi:10.4153/CJM-2011-058-x

Pointed Torsors

J. F. Jardine

Abstract. This paper gives a characterization of homotopy fibres of inverse image maps on groupoids of torsors that are induced by geometric morphisms, in terms of both pointed torsors and pointed cocycles, suitably defined. Cocycle techniques are used to give a complete description of such fibres, when the underlying geometric morphism is the canonical stalk on the classifying topos of a profinite group G. If the torsors in question are defined with respect to a constant group H, then the path components of the fibre can be identified with the set of continuous maps from the profinite group G to the group H. More generally, when H is not constant, this set of path components is the set of continuous maps from a pro-object in sheaves of groupoids to H, which pro-object can be viewed as a "Grothendieck fundamental groupoid".

Introduction

This paper gives a characterization of homotopy fibres of the functors

$$p\colon B(H\text{-}\mathbf{tors}) \to B(p(H)\text{-}\mathbf{tors})$$

of torsor categories that are induced by exact functors $p\colon \operatorname{Shv}(\mathcal{C}) \to \operatorname{Shv}(\mathcal{D})$ between Grothendieck toposes. In this generality, H is a sheaf of groupoids, and H-**tors** stands for its associated category of torsors, which is the groupoid of global sections of the associated stack [8]. The inverse image part of a geometric morphism has the required exactness properties, and inverse image functors are common in examples.

These homotopy fibres are characterized as nerves of, equivalently, suitably defined categories of pointed torsors or pointed cocycles. The equivalence between pointed torsors and pointed cocycles (Lemma 1.3) is a refinement of the equivalence between categories of torsors and cocycles given in [8].

The first section of this paper consists of general results, culminating in Proposition 1.4, which identifies the homotopy fibres of the simplicial set map p with nerves of pointed cocycle categories.

Section 2 contains some first applications of this theory for simplicial sheaves on étale sites. Suppose that $f\colon T \to S$ is a scheme homomorphism, and that H is a sheaf of groupoids on the small étale site for S. Pick an object x of the groupoid $H(T)$. It is shown (Proposition 2.4) that the homotopy fibre of the map

$$f^*\colon B(H\text{-}\mathbf{tors}) \to B(f^*(H)\text{-}\mathbf{tors})$$

over x can be computed with pointed torsors on either the small or big étale sites for S and T. It is a consequence of a general result (Proposition 2.1) that the homotopy

Received by the editors February 2, 2010; revised February 27, 2011.
Published electronically August 31, 2011.
This research was supported by NSERC.
AMS subject classification: **18G50**, 14F35, 55B30.
Keywords: pointed torsors, pointed cocycles, homotopy fibres.

fibre of f^* (on the big site level) is the nerve of a category of cocycles in the slice category of simplicial sheaves under T (Proposition 2.1). The arguments for the results of this section are cocycle theoretic, and the interpretation for pointed torsors comes from Proposition 1.4.

In Section 3 we specialize to a discussion of the classifying topos $\mathcal{B}G$ of a profinite group $G = \{G_i\}$ (and thus sheaves on finite étale sites) and the canonical stalk $\pi^*\colon \mathcal{B}G \to \mathbf{Set}$. If H is a groupoid in $\mathcal{B}G$, then $\pi^*(H)$ is just a groupoid and the inverse image map has the form

$$\pi^*\colon B(H\text{-}\mathbf{tors}) \to B\pi^*(H).$$

A precise calculation of the homotopy fibre F_x of π^* over an object x of the groupoid $\pi^*(H)$ can be achieved in this case, in that F_x is a union of contractible spaces, indexed on morphisms $C(G_i) \to BH$ defined on the Čech resolutions $C(G_i)$ associated with the component groups G_i of G, up to refinement (Lemma 3.5 and Corollary 3.4, respectively).

In particular, if S is a connected Noetherian scheme, $\pi_1(S, y)$ is the Grothendieck fundamental group of S at some geometric point y, and A is a constant group, then set of path components $\pi_0(F)$ of the fibre F over the base point of the classifying space BA is the set of pro-group homomorphisms $\pi_1(S, y) \to A$.

More generally, if H is a sheaf of groupoids on the finite étale site of S and x is an object of the groupoid $y^*(H)$, then the corresponding set $\pi_0(F_x)$ for the homotopy fibre F_x of π^* over x is pro-represented in sheaves of groupoids by a *Grothendieck fundamental groupoid*, suitably defined as a pro-object in sheaves of groupoids. This object specializes to an *absolute Galois groupoid* if S is the spectrum $\mathrm{Sp}(k)$ of a field k.

These pro-objects are readily seen in practice. For example, the absolute Galois groupoid of a field k is the pro-object

$$L \mapsto E_{G(L/k)}\,\mathrm{Sp}(L),$$

which is indexed on finite Galois extensions L/k, where $E_{G(L/k)}\,\mathrm{Sp}(L)$ is the translation groupoid for the action of the Galois group $G(L/k)$ on $\mathrm{Sp}(L)$. Its associated pro-object of connected components is the absolute Galois group for the field k. For each Galois extension L/k, the nerve $B(E_{(G(L/k)}\,\mathrm{Sp}(L))$ of the groupoid $E_{G(L/k)}\,\mathrm{Sp}(L)$ is the Borel construction $EG(L/k) \times_{G(L/k)} \mathrm{Sp}(L)$ as well as the Čech resolution associated with the étale cover $\mathrm{Sp}(L) \to \mathrm{Sp}(k)$.

Historically, there has been some difficulty in finding a completely satisfying relation between étale homotopy theory [2] and the homotopy theory of simplicial sheaves [6]. In particular, the Grothendieck fundamental group is "the" fundamental group functor for étale homotopy theory, but it has been an interesting problem to interpret this object directly in simplicial sheaves. The results of this paper give a first representation of the Grothendieck fundamental group as an explicit invariant of the homotopy theory of simplicial sheaves, in the theory of pointed torsors. The generalization of the Grothendieck fundamental group construction to a pro-object in sheaves of groupoids is a further interesting outcome of the pointed torsor approach.

1 Pointed Torsors

Suppose that $\mathcal{C}$ is a small Grothendieck site.

We shall use Joyal's injective model structure [6] on the category $s\,\mathrm{Shv}(\mathcal{C})$ of simplicial sheaves on $\mathcal{C}$ throughout this paper. The cofibrations for this structure are the monomorphisms, the weak equivalences are the local weak equivalences, and the fibrations are the injective fibrations. The local weak equivalences are those simplicial sheaf maps $f\colon X \to Y$ that induce weak equivalences $X_x \to Y_x$ of simplicial sets in all stalks if stalks are available, but more generally are those maps that induce isomorphisms in all possible sheaves of homotopy groups. The injective fibrations are defined by a right lifting property with respect to all trivial cofibrations. Injective fibrations are also called global fibrations in the literature.

The injective model structure has a list of attributes: it is a cofibrantly generated, proper, closed, simplicial model structure such that weak equivalences are closed under finite products. The associated homotopy category $\mathrm{Ho}(s\,\mathrm{Shv}(\mathcal{C}))$ is a non-abelian derived category for the sheaf category on the site $\mathcal{C}$.

For simplicial sheaves X and Y, the *cocycle category* $h(X,Y)$ has objects consisting of simplicial sheaf maps

$$X \xleftarrow[\simeq]{g} Z \xrightarrow{f} Y \tag{1.1}$$

and morphisms consisting of commutative diagrams

$$\begin{array}{ccccc} & & Z & & \\ & \overset{g}{\swarrow} & \Big\downarrow & \overset{f}{\searrow} & \\ X & \simeq & & & Y. \\ & \underset{g'}{\nwarrow} & & \underset{f'}{\nearrow} & \\ & & Z' & & \end{array}$$

It is a basic property of cocycle categories [8] (since the injective model structure is proper and has the property that weak equivalences are closed under finite products) that the assignment that takes a cocycle (1.1) to the morphism fg^{-1} in the homotopy category of simplicial sheaves induces a bijection $\pi_0 h(X,Y) \cong [X,Y]$ between the set of path components of the category $h(X,Y)$ (equivalently the path component set $\pi_0 Bh(X,Y)$ of the nerve $Bh(X,Y)$) and the set $[X,Y]$ of morphisms in the homotopy category $\mathrm{Ho}(s\,\mathrm{Shv}(\mathcal{C}))$.

Suppose that H is a sheaf of groupoids on the site $\mathcal{C}$. Recall [8] that an *H-torsor* is an H-diagram $\pi\colon F \to \mathrm{Ob}(H)$ in sheaves such that the induced map $\varinjlim\mathrm{holim}_H F \to *$ is a weak equivalence.

This definition of torsor involves a partial description of an internally defined functor. In more prosaic terms, the functor F consists of ordinary functors $F(U)\colon H(U) \to \mathbf{Set}$ such that the collections

$$\bigsqcup_{x\in \mathrm{Ob}(H(U))} F(U)(x) \to \mathrm{Ob}(H(U)), \quad U \in \mathcal{C}$$

https://doi.org/10.4153/CJM-2011-058-x Published online by Cambridge University Press

form a sheaf map $\pi\colon F \to \mathrm{Ob}(H)$, and such that the diagrams

$$\begin{array}{ccc} F(U)(x) & \xrightarrow{\alpha_*} & F(U)(y) \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} \\ F(V)(\phi^*(x)) & \xrightarrow[\phi^*(\alpha)_*]{} & F(V)(\phi^*(y)) \end{array}$$

commute for all morphisms $\alpha\colon x \to y$ of the groupoid $H(U)$ and all morphisms $\phi\colon V \to U$ of the underlying site $\mathcal{C}$.

All constituent functors $F(U)\colon H(U) \to \mathbf{Set}$ have *translation groupoids* $E_{H(U)}F(U)$ whose objects are pairs (x, y) with $x \in \mathrm{Ob}(H(U))$ and $y \in F(U)(x)$, and then a morphism $\alpha\colon (x, y) \to (x', y')$ is a morphism $\alpha\colon x \to x'$ of $H(U)$ such that $\alpha_*(y) = y'$. The homotopy colimit $\underrightarrow{\operatorname{holim}}_{H(U)} F(U)$ for the functor $F(U)$ is the nerve $B(E_{H(U)}F(U))$; this is the usual construction [4, IV.1.8]. The homotopy colimit sheaf $\underrightarrow{\operatorname{holim}}_H F$ is the simplicial sheaf, which is defined by

$$U \mapsto \underrightarrow{\operatorname{holim}}_{H(U)} F(U).$$

The canonical functors $E_{H(U)}F(U) \to H(U)$ defined by $(x, y) \mapsto x$ assemble to define the simplicial sheaf map

$$\pi\colon \underrightarrow{\operatorname{holim}}_H F \to BH.$$

The diagram of simplicial sheaves

$$(1.2) \qquad \begin{array}{ccc} F & \longrightarrow & \underrightarrow{\operatorname{holim}}_H F \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Ob}(H) & \longrightarrow & BH \end{array}$$

is homotopy cartesian, since H is a sheaf of groupoids. In effect, the diagram is homotopy cartesian in each section by a standard result of Quillen [4, IV.5.7] concerning diagrams of weak equivalences.

A morphism of H-diagrams is a map of sheaves

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ & \searrow \quad \swarrow & \\ & \mathrm{Ob}(H) & \end{array}$$

over $\mathrm{Ob}(H)$ that defines a natural transformation in each section. The sheaf map $f\colon F \to F'$ is necessarily a local weak equivalence of simplicial sheaves, and hence a sheaf isomorphism: this is proved by comparing homotopy cartesian diagrams of the form (1.2). It follows that the category of H-torsors and the H-equivariant maps between them form a groupoid, which will be denoted by H-**tors**.

This groupoid H-**tors** is the groupoid of global sections of a presheaf of groupoids H-**Tors**, which is a model for the stack associated to H; see [8].

Example 1.1 Standard examples of torsors include the representable functors $H(x,\,\cdot\,)$ and $H(\,\cdot\,,x)$ associated with a global section x of the object sheaf $\mathrm{Ob}(H)$. There is an isomorphism

$$H(x,\,\cdot\,) \xrightarrow{\cong} H(\,\cdot\,,x)$$

of H-torsors that is defined in sections by taking the morphism $x \xrightarrow{\alpha} y$ to $y \xrightarrow{\alpha^{-1}} x$. These are the trivial torsors. In general, an H-torsor morphism (*i.e.*, a trivialization over x) $f\colon H(x,\,\cdot\,) \to F$ is completely determined by the global section $f(1_x)$ of F, which section maps to x under the structure map $F \to \mathrm{Ob}(H)$. Observe that the structure map $H(x,\,\cdot\,) \to \mathrm{Ob}(H)$ is defined in sections by sending the morphism $x \xrightarrow{\alpha} y$ to the target object y.

Every H-torsor $F \to \mathrm{Ob}(H)$ has a functorially assigned cocycle

$$* \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_{H}\, F \to BH.$$

It follows that there is a functor

$$\underrightarrow{\mathrm{holim}}_{H}\colon H\text{-}\mathbf{tors} \to h(*, BH).$$

The homotopy colimit functor $\underrightarrow{\mathrm{holim}}_{H}$ has a left adjoint

$$\mathrm{pb}\colon h(*, BH) \to H\text{-}\mathbf{tors}$$

that is defined by sheafifying a presheaf, which presheaf is defined in sections by taking path components $\pi_0(\mathrm{pb}_x)$ of spaces pb_x. These spaces pb_x are defined for a cocycle $* \xleftarrow{\simeq} Z \to BH$ by the pullback diagrams

$$\begin{array}{ccc} \mathrm{pb}_x & \longrightarrow & Z(U) \\ \downarrow & & \downarrow \\ B(H(U)/x) & \longrightarrow & BH(U), \end{array}$$

where x ranges through the set $\mathrm{Ob}(H(U))$ of objects of $H(U)$ and $U \in \mathcal{C}$.

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck sites, and that the functor

$$p\colon \operatorname{Shv}(\mathcal{C}) \to \operatorname{Shv}(\mathcal{D})$$

between sheaf categories is exact in the sense that it preserves finite limits and all colimits.

Examples of such functors p include

(i) inverse image functors f^* associated with geometric morphisms

$$f\colon \operatorname{Shv}(\mathcal{D}) \to \operatorname{Shv}(\mathcal{C}),$$

(ii) restriction functors $q_*\colon \operatorname{Shv}(\mathcal{C}) \to \operatorname{Shv}(\mathcal{C}/U)$, defined by composition with the canonical functors $q\colon \mathcal{C}/U \to \mathcal{C}$. Such a functor q takes an object $V \to U$ to V, and one variously writes $F|_U = q_*F = F \cdot q$ for sheaves F.

The restriction functors q are often inverse images in practice. For example, suppose that $f\colon T \to S$ is an object of the big étale site $(Sch|_S)_{et}$. Then there is an isomorphism of categories

$$(Sch|_S)/f \cong Sch|_T,$$

and the standard inverse image functor

$$f^*\colon \operatorname{Shv}((Sch|_S)_{et}) \to \operatorname{Shv}((Sch|_T)_{et})$$

is defined by precomposition with the scheme homomorphism f.

All functors

$$p\colon \operatorname{Shv}(\mathcal{C}) \to \operatorname{Shv}(\mathcal{D}),$$

which satisfy the indicated exactness conditions above, preserve local weak equivalences. This is a standard fact in the case where p is the inverse image functor for some geometric morphism; see for example [3, 2.6]. The quickest proof of this result (which is a straightforward generalization of the proof for inverse images) follows from the observations that such a functor p commutes with the formation of the sheaf-level Ex^∞-functor and preserves local trivial fibrations.

The functor p also preserves homotopy colimits, and it follows that if $F \to \operatorname{Ob}(H)$ is an H-torsor, then the induced object $p(F) \to \operatorname{Ob}(p(H))$ is a $p(H)$-torsor. The functor p therefore induces a functor

$$p\colon H\text{-}\mathbf{tors} \to p(H)\text{-}\mathbf{tors}.$$

Suppose that $x\colon * \to \operatorname{Ob}(p(H))$ is a global section of the sheaf of groupoids $p(H)$. A *pointed H-torsor over x* is an H-torsor $F \to \operatorname{Ob}(H)$ together with a fixed lifting

$$\begin{array}{ccc} & & p(F) \\ & \overset{z}{\nearrow} & \downarrow \\ * & \xrightarrow[x]{} & \operatorname{Ob}(p(H)). \end{array}$$

A morphism of pointed H-torsors over x is a morphism of H-torsors that respects the liftings. Write $H\text{-}\mathbf{tors}_x$ for the corresponding groupoid.

Lemma 1.2 *The groupoid H-$\textbf{tors}_x$ of pointed H-torsors over x is the homotopy fibre of the map*

$$p\colon H\text{-}\textbf{tors} \to p(H)\text{-}\textbf{tors}.$$

Proof In the category of groupoids, the homotopy fibre F_x of the functor

$$p\colon H\text{-}\textbf{tors} \to p(H)\text{-}\textbf{tors}$$

over the torsor $p(H)(x,\,\cdot\,)$ has objects consisting of all $p(H)$-torsor morphisms $p(H)(x,\,\cdot\,) \to p(F)$. The morphisms of this groupoid are the commutative diagrams

$$\begin{array}{ccc} & \nearrow & p(F) \\ p(H)(x,\,\cdot\,) & & \big\downarrow{\scriptstyle \theta_*} \\ & \searrow & p(F'), \end{array}$$

where $\theta\colon F \to F'$ is a morphism of H-torsors.

A global section $z\colon * \to p(F)$ of the torsor $p(F)$ extends to a unique H-torsor morphism

$$z_*\colon p(H)(x,\,\cdot\,) \to p(F).$$

It follows that the assignment which takes the pointed torsor $* \xrightarrow{z} p(F)$ to the torsor morphism $p(H)(x,\,\cdot\,) \xrightarrow{z_*} p(X)$ defines an isomorphism $F_x \cong H$-$\textbf{tors}_x$ of groupoids. ∎

A *pointed H-cocycle over x* is a cocycle $* \underset{\simeq}{\overset{g}{\leftarrow}} Z \xrightarrow{f} BH$ together with a morphism

$$\begin{array}{ccccc} & \swarrow & * & \searrow^{x} & \\ & & \big\downarrow{\scriptstyle v} & & \\ * & \underset{p(g)}{\overset{\simeq}{\longleftarrow}} & p(Z) & \underset{p(f)}{\longrightarrow} & Bp(H) \end{array}$$

of $p(H)$-cocycles. A morphism of pointed H-cocycles over x is a morphism of cocycles that respects choices of sections. Write $h(*, BH)_x$ for the corresponding category.

The unit of the adjunction

$$\mathrm{pb}\colon h(*, BH) \leftrightarrows H\text{-}\textbf{tors} : \underrightarrow{\mathrm{holim}}_H$$

is a natural cocycle morphism

$$\begin{array}{ccc} & * & \\ {\scriptstyle\simeq}\nearrow & & \nwarrow{\scriptstyle\simeq} \\ Z & \xrightarrow{\eta} & \underrightarrow{\mathrm{holim}}_H\ \mathrm{pb}(Z). \\ {\scriptstyle f}\searrow & & \swarrow \\ & BH & \end{array}$$

If the object $p(Z)$ is pointed by a morphism $v\colon * \to p(Z)$ over x, then the composite sheaf map

$$* \xrightarrow{v} p(Z)_0 \xrightarrow{p(\eta)} p(\underrightarrow{\mathrm{holim}}_H\ \mathrm{pb}(Z))_0 = p(\mathrm{pb}(Z))$$

gives $p(\mathrm{pb}(Z))$ a point over x. It follows that the pullback functor restricts to a functor

$$\mathrm{pb}\colon h(*, BH)_x \to H\text{-}\mathbf{tors}_x.$$

If the torsor $F \to \mathrm{Ob}(H)$ is pointed by a map

$$* \xrightarrow{z} p(F) = p(\underrightarrow{\mathrm{holim}}_H F)_0$$

over x, then the global section z defines a map $* \xrightarrow{z} p(\underrightarrow{\mathrm{holim}}_H F)$, which gives the canonical cocycle

$$* \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_H F \to BH$$

the structure of a pointed cocycle over x. Thus, the canonical cocycle construction restricts to a functor

$$\underrightarrow{\mathrm{holim}}_H\colon H\text{-}\mathbf{tors}_x \to h(*, BH)_x.$$

We have proved the following lemma.

Lemma 1.3 *The pullback and homotopy colimit functors induce an adjoint pair of functors*

$$\mathrm{pb}\colon h(*, BH)_x \leftrightarrows H\text{-}\boldsymbol{tors}_x : \underrightarrow{\mathrm{holim}}_H.$$

Proposition 1.4 *The nerve $Bh(*, BH)_x$ is weakly equivalent to the homotopy fibre of the simplicial set map*

$$\pi\colon B(H\text{-}\boldsymbol{tors}) \to B(p(H)\text{-}\boldsymbol{tors})$$

over the trivial torsor $p(H)(x, \cdot\,)$.

Proof This result is a consequence of Lemma 1.2 and Lemma 1.3. ■

https://doi.org/10.4153/CJM-2011-058-x Published online by Cambridge University Press

2 Restriction Functors

Suppose that U is an object of a site $\mathcal{C}$, and consider the corresponding restriction functor

$$q_*\colon \operatorname{Shv}(\mathcal{C}) \to \operatorname{Shv}(\mathcal{C}/U).$$

In general, for any sheaf F on $\mathcal{C}$, a global section of q_*F is a section of $F(U)$, and there is a natural bijection $\hom(*, q_*(F)) \cong F(U)$. It follows that the objects of the category $h(*, BH)_x$ of pointed cocycles over $x\,\mathrm{Ob}(H)(U)$ can be identified with commutative diagrams

$$\begin{array}{ccccc} & & U & & \\ & \swarrow & \downarrow{\scriptstyle v} & \searrow{\scriptstyle x} & \\ * & \xleftarrow[\simeq]{} & Z & \xrightarrow[f]{} & BH, \end{array}$$

where the bottom row forms a cocycle in simplicial sheaves on $\mathcal{C}$. Here, we have notationally identified the object U with the sheaf associated with the functor represented by U. From this point of view, a morphism of pointed cocycles is a cocycle morphism in simplicial sheaves on $\mathcal{C}$ that respects choices of U-sections.

The slice category $U/s\operatorname{Shv}(\mathcal{C})$ has a model structure, for which a morphism

$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a weak equivalence (respectively cofibration, fibration) if the simplicial sheaf map $f\colon X \to Y$ is a local weak equivalence (respectively cofibration, injective fibration) of simplicial sheaves. Compare [5, 1.1.8], but the existence of such model structures for slice categories is an easy exercise.

In this category, a cocycle $t \xleftarrow{\simeq} v \xrightarrow{f} x$ is a pointed cocycle

$$\begin{array}{ccccc} & & U & & \\ & \swarrow{\scriptstyle t} & \downarrow{\scriptstyle v} & \searrow{\scriptstyle x} & \\ * & \xleftarrow[\simeq]{} & Z & \xrightarrow[f]{} & BH \end{array}$$

as described above, and the category $h(*, BH)_x$ of pointed cocycles can be identified with the cocycle category $h(t, x)$ for the slice category.

Weak equivalences are closed under finite products in the slice category $U/s\,\mathbf{Shv}(\mathcal{C})$, and its model structure is right proper. Theorem 1 of [8] therefore says that there is a canonical bijection

$$\pi_0 h(*, BH)_x = \pi_0 h(t, x) \cong [t, x],$$

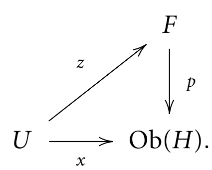
where $[t, x]$ is the set of morphisms from t to x in the homotopy category $\mathrm{Ho}(U/s\,\mathrm{Shv}(\mathcal{C}))$. We have proved the following proposition.

Proposition 2.1 *Suppose that H is a sheaf of groupoids on site $\mathcal{C}$, and let U be an object of $\mathcal{C}$. Suppose that $x \in \mathrm{Ob}(H)(U)$ is a U-section of H. Then there is a canonical bijection*

$$\pi_0 h(*, BH)_x = \pi_0 h(t, x) \cong [t, x],$$

where $[t, x]$ denotes morphisms from t to x in the homotopy category $\mathrm{Ho}(U/s\,\mathrm{Shv}(\mathcal{C}))$.

A pointed H-torsor over x can be identified with an H-torsor $p\colon F \to \mathrm{Ob}(H)$, together with a commutative diagram



A morphism of pointed torsors over x is then an H-torsor morphism that respects U-sections.

We have the following corollary of Proposition 2.1 and Lemma 1.3.

Corollary 2.2 *In the presence of the assumptions and notation of Proposition* 2.1*, there is an isomorphism*

$$\pi_0 B(H\text{-}\boldsymbol{tors}_x) \cong [t, x]$$

of isomorphism classes of pointed torsors with morphisms in the homotopy category for the category $U/s\,\mathrm{Shv}(\mathcal{C})$.

Suppose that S is scheme and let $(Sch|_S)_{et}$ be the big site of S-schemes $X \to S$, equipped with the étale topology. The inclusion $i\colon et|_S \subset (Sch|_S)_{et}$ of the standard étale site in the big site is a site morphism, and induces a geometric morphism

$$i\colon\ \mathrm{Shv}(Sch|_S)_{et} \to \mathrm{Shv}(et|_S).$$

The corresponding direct image functor

$$i_*\colon\ \mathrm{Shv}(Sch|_S)_{et} \to \mathrm{Shv}(et|_S)$$

is defined by composition with, or restriction along, the inclusion functor i. The inverse image functor i^* and restriction functors i_* are both exact (this is uncommon for direct images), and therefore preserve local weak equivalences of simplicial sheaves. The canonical map $\eta\colon F \to i_* i^* F$ is an isomorphism for all sheaves F on the étale site $et|_S$.

Lemma 2.3 (i) *The restriction functor i_* induces a weak equivalence*

$$i_*\colon Bh(*,X) \xrightarrow{\simeq} Bh(*,i_*X).$$

for all simplicial sheaves X on the big étale site $(Sch|_S)_{et}$.

(ii) *The inverse image functor i^* induces a weak equivalence*

$$i^*\colon Bh(*,Y) \xrightarrow{\simeq} Bh(*,i^*Y)$$

for all simplicial sheaves Y on the étale site $et|_Y$.

Proof To prove statement (i), suppose that X is a simplicial sheaf on the big site $(Sch|_S)_{et}$. Then restriction along i preserves weak equivalences and therefore defines a functor

$$i_*\colon h(*,X) \to h(*,i_*X)$$

that sends the cocycle $* \xleftarrow{\simeq} U \xrightarrow{f} X$ to the cocycle $* \xleftarrow{\simeq} i_*U \xrightarrow{i_*f} i_*X$. There is a functor

$$\tilde{i}\colon h(*,i_*X) \to h(*,X)$$

that sends the cocycle $* \xleftarrow{\simeq} Z \xrightarrow{f} i_*X$ to the cocycle $* \xleftarrow{\simeq} i^*Z \xrightarrow{f_*} X$, where f_* is the adjoint of f. The functor $\tilde{i}$ is left adjoint to the functor i_*, so that the simplicial set map

$$i_*\colon Bh(*,X) \to Bh(*,i_*X)$$

is a homotopy equivalence.

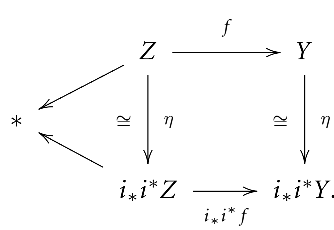
For statement (ii), the inverse image induces a functor

$$i^*\colon h(*,Y) \to h(*,i^*Y)$$

for simplicial sheaves Y on the étale site $et|_S$. The composite functor i_*i^* sends the cocycle

$$* \xleftarrow{\simeq} Z \xrightarrow{f} Y \quad \text{to the cocycle} \quad * \xleftarrow{\simeq} i_*i^*Z \xrightarrow{i_*i^*f} i_*i^*Y,$$

and there is a commutative diagram



It follows that the composite $i_*i^*\colon Bh(*,Y) \to Bh(*,i_*i^*Y)$ is homotopic to the isomorphism $\eta_*\colon Bh(*,Y) \xrightarrow{\cong} Bh(*,i_*i^*Y)$, which is defined by composition with η.

The composite map i_*i^* is therefore a weak equivalence, and the map i_* is a weak equivalence by statement (i). ∎

https://doi.org/10.4153/CJM-2011-058-x Published online by Cambridge University Press

Suppose that the scheme homomorphism $\phi\colon T \to S$ is an object of the big site $(Sch|_S)_{et}$. The map ϕ induces a geometric morphism

$$\phi\colon \operatorname{Shv}(Sch|_T)_{et} \to \operatorname{Shv}(Sch|_S)_{et},$$

in the standard way, for which the direct image functor ϕ_* is defined by composition with pullback along the scheme homomorphism ϕ. The diagram of direct image functors

$$\begin{array}{ccc}
\operatorname{Shv}(Sch|_T)_{et} & \xrightarrow{i_*} & \operatorname{Shv}(et|_T) \\
{\scriptstyle \phi_*}\downarrow & & \downarrow{\scriptstyle \phi_*} \\
\operatorname{Shv}(Sch|_S)_{et} & \xrightarrow[i_*]{} & \operatorname{Shv}(et|_S)
\end{array}$$

commutes, so that there is a canonical isomorphism $\gamma\colon i^*\phi^* \xrightarrow{\cong} \phi^* i^*$. It follows that there is a homotopy commutative diagram

$$(2.1)\qquad \begin{array}{ccccc}
Bh(*, X) & & \xrightarrow[\simeq]{i^*} & & Bh(*, i^*X) \\
{\scriptstyle \phi^*}\downarrow & & & & \downarrow{\scriptstyle \phi^*} \\
Bh(*, \phi^*X) & & & & Bh(*, \phi^* i^* X) \\
& \searrow^{\simeq}_{i^*} & & \swarrow^{\cong}_{\gamma_*} & \\
& & Bh(*, i^*\phi^*X) & &
\end{array}$$

for each simplicial sheaf X on the étale site $et|_S$, in which the instances of i^* are weak equivalences by Lemma 2.3.

Recall that the inverse image functor

$$\phi^*\colon \operatorname{Shv}(Sch|_S)_{et} \to \operatorname{Shv}(Sch|_T)_{et}$$

on the big site level is given by precomposition with the functor

$$\operatorname{Shv}(Sch|_T)_{et} \to \operatorname{Shv}(Sch|_S)_{et},$$

which is defined by composition with ϕ.

The existence of diagram (2.1) means that the small and big site versions of ϕ^* (the vertical maps in diagram (2.1)) have weakly equivalent homotopy fibres.

Proposition 2.4 *Suppose that H is a sheaf of groupoids on the étale site $et|_S$ and that $x \in \operatorname{Ob}(H)(T)$ is a global section of ϕ^*H. Then there is a weak equivalence*

$$B(H\text{-}\boldsymbol{tors}_x) \simeq B(i^*H\text{-}\boldsymbol{tors}_x)$$

relating the respective groupoids of pointed torsors for the small and big étale sites.

Proof The weak equivalences of diagram (2.1) (with $X = BH$) imply that the maps $\phi^* \colon Bh(*, BH) \to Bh(*, f^*BH)$ and $\phi^* \colon Bh(*, i^*BH) \to Bh(*, f^*i^*BH)$ have weakly equivalent homotopy fibres over x. These homotopy fibres can be identified with $B(H\text{-}\mathbf{tors}_x)$ and $B(i^*H\text{-}\mathbf{tors}_x)$, respectively, by Lemma 1.2 and Lemma 1.3. ■

3 Profinite Groups

Suppose that the group-valued functor $G \colon I \to \mathbf{Grp}$ is a profinite group. Then in particular the category I is left filtered (any two objects i, i' have a common lower bound, and any two morphisms $i \rightrightarrows j$ have a weak equalizer), and all constituent groups G_i, $i \in I$, are finite. We shall also assume that all transition homomorphisms $G_i \to G_j$ in the diagram are surjective.

Example 3.1 The standard example is the *absolute Galois group* $G(k)$ of a field k. One takes all finite Galois extensions L/k inside an algebraically closed field Ω containing k in the sense that one has a fixed imbedding $i \colon k \to \Omega$, and the Galois extensions are commutative diagrams of field homomorphisms

$$\begin{array}{ccc} k & \xrightarrow{\;i\;} & \Omega \\ & \searrow \quad \nearrow & \\ & L, & \end{array}$$

where L is a finite Galois extension of k. These are the objects of a right filtered category, for which the morphisms $L \to L'$ respect structure, and the contravariant functor $G(k)$ that associates the Galois group $G(L/k)$ with each of these pictures is the absolute Galois group.

More generally, suppose that S is a connected Noetherian scheme, and let $y \colon \mathrm{Sp}(\Omega) \to S$ be a geometric point of S. The *Grothendieck fundamental group* $\pi_1(S, y)$ is the pro-group defined on finite Galois extensions

$$\begin{array}{ccc} & & T \\ & \nearrow & \downarrow \\ \mathrm{Sp}(\Omega) & \xrightarrow[y]{} & S \end{array}$$

of S in y by the assignment $T \mapsto G(T/S)$ where the Galois group $G(T/S) = \mathrm{Aut}_S(T)$ is the group of S-scheme automorphisms of T that respect the geometric point y.

Suppose that G is a profinite group, and let $G\text{-}\mathbf{Set}_{df}$ be the category of finite discrete G-sets, as in [7].

Recall that a discrete G-set is a set F equipped with an action

$$G \times F \to G_i \times F \to F,$$

where $G = \varprojlim_i G_i$ (note the abuse of notation), and a morphism of discrete G-sets is a G-equivariant map.

The category G-$\mathbf{Set}_{df}$ has a topology for which the covering families are the G-equivariant surjections $V \to U$. A presheaf F on G-$\mathbf{Set}_{df}$ is a sheaf for this topology if and only if

(i) F takes disjoint unions to products, and
(ii) each canonical map $G_i \to G_i/H$ defined by a subgroup H of G induces a bijection
$$F(G_i/H) \xrightarrow{\cong} F(G_i)^H.$$

It follows in particular that the topology is subcanonical: every finite discrete G-set X represents a sheaf $\hom(\,\cdot\,, X)$. The resulting sheaf category $\mathcal{B}G := \mathrm{Shv}(G\text{-}\mathbf{Set}_{df})$ is the *classifying topos* for the profinite group G.

Suppose that the functor $\tilde{\pi}\colon G\text{-}\mathbf{Set}_{df} \to \mathbf{Set}$ takes a finite discrete G-set to its underlying set. Every set X represents a sheaf $\pi_*(X)$ on G-$\mathbf{Set}_{df}$ with

$$\pi_*(X)(U) = \hom(\tilde{\pi}(U), X).$$

The left adjoint π^* of the corresponding functor π_* has the form $\pi^*(F) = \varinjlim F(G_i)$, by a cofinality argument, and the functors π^* and π_* form a geometric morphism $\pi\colon \mathbf{Set} \to \mathcal{B}G$.

The list of points consisting of the geometric morphism π alone is an "adequate" collection of points for the classifying topos $\mathcal{B}G$ in the sense that the inverse image functor π^* is faithful. It follows that a simplicial sheaf morphism $f\colon X \to Y$ on G-$\mathbf{Set}_{df}$ is a local weak equivalence if and only if the induced map $\pi^*(X) \to \pi^*(Y)$ is a weak equivalence of simplicial sets.

Suppose that H is a sheaf of groupoids on G-$\mathbf{Set}_{df}$ and that x is an object of the groupoid $\pi^*(H)$. As above, let H-$\mathbf{tors}_x$ denote the category of H-torsors pointed by x, and let $h(*, BH)_x$ be the category of pointed H-cocycles over x. Recall from Lemma 1.3 that pullback and homotopy colimit define an adjunction

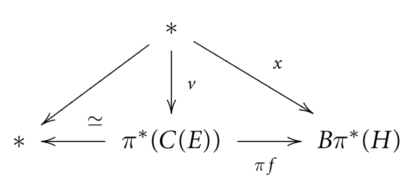
$$\mathrm{pb}\colon h(*, BH)_x \leftrightarrows H\text{-}\mathbf{tors}_x : \underrightarrow{\mathrm{holim}}\,_H,$$

and a corresponding homotopy equivalence

$$Bh(*, BH)_x \simeq B(H\text{-}\mathbf{tors}_x)$$

of the associated nerves.

A *pointed Čech cocycle* over x is a cocycle $* \xleftarrow{\simeq} C(E) \xrightarrow{f} BH$ together with a morphism



of $\pi^*(H)$-cocycles, where $C(E)$ is the Čech resolution associated with an epimorphism $E \to *$ in $\mathcal{B}G$.

A morphism of pointed Čech cocycles over x is a sheaf morphism $\theta\colon E \to E'$, which induces a morphism of cocycles

$$\begin{array}{ccccc} & \overset{\simeq}{\swarrow} & C(E) & \overset{f}{\searrow} & \\ * & & \big\downarrow \theta_* & & BH, \\ & \underset{\simeq}{\nwarrow} & C(E') & \underset{f'}{\nearrow} & \end{array}$$

which preserves base points in the sense that the diagram

$$\begin{array}{ccc} & \overset{v}{\nearrow} & \pi^*(E) \\ * & & \big\downarrow \pi\theta \\ & \underset{v'}{\searrow} & \pi^*(E') \end{array}$$

commutes. Write $h_{\mathrm{Cech}}(*, BH)_x$ for the category of Čech cocycles over x.

Lemma 3.2 *The inclusion functor*

$$i\colon h_{\mathrm{Cech}}(*, BH)_x \to h(*, BH)_x$$

is fully faithful, and induces a bijection $\pi_0 h_{\mathrm{Cech}}(*, BH)_x \xrightarrow{\cong} \pi_0 h(*, BH)_x$.

Proof The inclusion functor i has a left adjoint. The unit of the adjunction is a canonical cocycle morphism

$$\begin{array}{ccccc} & \overset{\simeq}{\swarrow} & Z & \overset{f}{\searrow} & \\ * & & \big\downarrow & & BH, \\ & \underset{\simeq}{\nwarrow} & C(Z_0) & \underset{f_*}{\nearrow} & \end{array}$$

which exists, since $C(Z_0)$ is the fundamental groupoid of the locally contractible simplicial sheaf Z. ■

Write $h_G(*, BH)_x$ for the subcategory of $h_{\mathrm{Cech}}(*, BH)_x$ whose objects are the pointed cocycles

$$* \xleftarrow{\simeq} C(G_i) \xrightarrow{f} BH$$

https://doi.org/10.4153/CJM-2011-058-x Published online by Cambridge University Press

such that $\pi^*(f)(e_i) = x$ in $\mathrm{Ob}(\pi^*(H))$. Here,

$$e_i \in \pi^*(G_i) = \varinjlim_j \hom(G_j, G_i)$$

is the element represented by the identity homomorphism $1\colon G_i \to G_i$.

If $\phi\colon G_j \to G_i$ is a structure homomorphism of the profinite group G, then $e_j \mapsto e_i$ under the function $\pi^*(\phi)\colon \pi^*(G_j) \to \pi^*(G_i)$. A morphism of $h_G(*, BH)_x$ is a structure homomorphism $\phi\colon G_j \to G_i$ for G, which respects cocycles.

Lemma 3.3 *The inclusion functor*

$$i\colon h_G(*, BH)_x \subset h_{\mathrm{Cech}}(*, BH)_x$$

induces a weak equivalence $i_*\colon Bh_G(*, BH)_x \xrightarrow{\simeq} Bh_{\mathrm{Cech}}(*, BH)_x$.

Proof Suppose that (f, v) is an object of $h_{\mathrm{Cech}}(*, BH)_x$, where $f\colon C(E) \to BH$ is a cocycle and $v \in \pi^*(E)$. The element v corresponds to a map $v_*\colon G_i \to E$ for some G_i, and $v_*(e_i) = v$. It follows that the category $i/(f, v)$ is non-empty. The category $i/(f, v)$ is also left filtered since $\pi^*(U)$ is defined by the filtered colimit

$$\pi^*(E) = \varinjlim_i E(G_i).$$

The category $i/(f, v)$ is therefore non-empty and left filtered for all objects (f, v) of the category $h_{\mathrm{Cech}}(*, BH)_x$. The desired result follows from Quillen's Theorem B [4, IV.5.6]. ■

Corollary 3.4 *Suppose that G is a profinite group, H is a sheaf of groupoids on G-$\mathbf{Set}_{df}$, and that x is an object of the stalk groupoid $\pi^*(H)$. Then there is an isomorphism*

$$\pi_0 Bh(*, BH)_x \cong \varinjlim_i \hom(C(G_i), BH)_x,$$

where $\hom(C(G_i), BH)_x$ *is the set of groupoid morphisms $f\colon C(G_i) \to H$ such that $\pi^*(f)(e_i) = x$.*

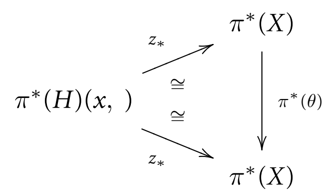
The weak equivalence

$$\mathrm{pb}\colon Bh(*, BH)_x \xrightarrow{\simeq} B(H\text{-}\mathbf{tors}_x)$$

is a fibrant model for the space $Bh(*, BH)_x$ in simplicial sets, since the pointed torsor category H-$\mathbf{tors}_x$ is a groupoid. Every automorphism θ of a pointed torsor

$$\begin{array}{ccc} & & \pi^*(X) \\ & \overset{z}{\nearrow} & \downarrow \\ * & \xrightarrow[x]{} & \mathrm{Ob}(\pi^*(H)) \end{array}$$

induces a diagram of torsor morphisms



so that $\pi^*(\theta)$ is the identity. But the functor π^* is faithful, so that θ is the identity as well.

We have therefore shown the following lemma.

Lemma 3.5 *The canonical simplicial set map*

$$Bh(*, BH)_x \to \pi_0 Bh(*, BH)_x$$

is a weak equivalence of simplicial sets for all $x \in \mathrm{Ob}(\pi^*(H))$ *and for all sheaves of groupoids* H *on the site* G-$\mathbf{Set}_{df}$.

Example 3.6 Suppose that $\Gamma^* A$ is the constant sheaf on a group A, and $*$ is the unique object of the group $A = \pi^*(\Gamma^* A)$ (thought of as a groupoid with one object). Then there is an isomorphism

$$\pi_0 Bh(*, B\Gamma^* A)_* \cong \varinjlim_i \hom(G_i, A),$$

where $\hom(G_i, A)$ is the set of group homomorphisms $G_i \to A$.

Suppose that H is a sheaf of groupoids on the finite étale site $fet|_S$ for a connected Noetherian scheme S. Let $y\colon \mathrm{Sp}(\Omega) \to S$ be a geometric point of S, and let $\pi_1(S, y)$ be the corresponding Grothendieck fundamental group. There is a well-known equivalence of categories

$$\mathrm{Shv}(fet|_S) \simeq \mathcal{B}\pi_1(S, y)$$

[1, V.7], [9, I.5], under which the inverse image $y^*\colon \mathrm{Shv}(fet|_S) \to \mathbf{Set}$ maps to the canonical stalk $\pi^*\colon \mathcal{B}\pi_1(S, y) \to \mathbf{Set}$.

Take an element $x \in \mathrm{Ob}(y^*(H))$. It is a consequence of Proposition 1.4, Lemma 3.2, and Lemma 3.3 that the homotopy fibre $Bh(*, BH)_x$ of the canonical stalk map

$$y^*\colon B(H\text{-}\mathbf{tors}) \to B(y^*(H))$$

over the object x has its path component set specified by

$$\pi_0 Bh(*, BH)_x \cong \varinjlim_T \hom(C(\mathrm{Sp}(T)), BH)_x,$$

https://doi.org/10.4153/CJM-2011-058-x Published online by Cambridge University Press

where the colimit is indexed over finite Galois extensions T/S in Ω, and the set $\hom(C(\mathrm{Sp}(T)), BH)_x$ consists of simplicial sheaf maps (or morphisms of sheaves of groupoids) $f\colon C(\mathrm{Sp}(T)) \to BH$ such that the degree 0 part $f\colon \mathrm{Sp}(T) \to \mathrm{Ob}(H)$ represents $x \in \mathrm{Ob}(y^*(H))$ (this is the "section condition"). There are isomorphisms

$$EG(T/S) \times_{G(T/S)} T \cong C(\mathrm{Sp}(T)),$$

so that the maps f above can be rewritten as simplicial sheaf maps

$$f\colon EG(T/S) \times_{G(T/S)} T \to BH$$

that satisfy the section condition in simplicial degree 0. The simplicial sheaf $EG(T/S) \times_{G(T/S)} T$ is the nerve of the translation groupoid $E_{G(T/S)}T$ for the action of the Galois group $G(T/S)$ on the sheaf T, and the maps f can be further identified with homomorphisms $f\colon E_{G(T/S)}T \to H$ of sheaves of groupoids that satisfy the section condition in objects.

If A is a group (see Example 3.6), then the homotopy fibre $Bh(*, B\Gamma^*A)_*$ of the map

$$y^*\colon B(\Gamma^*A\textbf{-tors}) \to BA$$

has

$$\pi_0 Bh(*, B\Gamma^*A)_* \cong \varinjlim_T \hom(G(T/S), A).$$

We have proved the following proposition.

Proposition 3.7 *Suppose that S is a connected Noetherian scheme, and let $y\colon \mathrm{Sp}(\Omega) \to S$ be a geometric point. Suppose that H is a sheaf of groupoids on the finite étale site $fet|_S$, and that A is a group.*

(i) *Suppose that x is an object of the groupoid $y^*(H) = \varinjlim_{T/S} H(T)$. Then there is an isomorphism*

$$\pi_0 Bh(*, BH)_x \cong \varinjlim_T \hom(E_{G(T/S)}T, H)_x.$$

(ii) *There is an isomorphism*

$$\pi_0 Bh(*, B\Gamma^*A)_* \cong \varinjlim_T \hom(G(T/S), A).$$

In all cases, the colimits are indexed over the finite Galois extensions T/S, and these extensions have Galois groups $G(T/S)$.

The pro-object in sheaves of groupoids that is defined on Galois extensions T/S by the assignment $T \mapsto E_{G(T/S)}T$ is the *Grothendieck fundamental groupoid $E\pi_1(S, y)$* of S at the geometric point y. Observe that applying the connected components functor to all objects $E_{G(T/S)}T$ gives the Grothendieck fundamental group $\pi_1(T, y)$.

Proposition 3.7 says that the Grothendieck fundamental groupoid $E\pi_1(S, y)$ pro-represents all pointed torsors, while the Grothendieck fundamental group $\pi_1(S, y)$ pro-represents pointed torsors with constant group coefficients.

In the special case that $S = \mathrm{Sp}(k)$ for a field k then the pro-object which is defined on finite Galois extensions L/k (inside some fixed algebraically closed extension) by

$$L \mapsto E_{G(L/k)} \mathrm{Sp}(L)$$

is the *absolute Galois groupoid* $EG(k)$ of the field k. The associated pro-object of path components is the absolute Galois group $G(k)$ of k.

References

[1] *Revêtements étales et groupe fondamental. Exposés I à XIII. Séminaire de géométrie algébrique du Bois Marie 1960/61 (SGA 1), dirigé par Alexander Grothendieck. Augmenté de deux exposés de M. Raynaud.* Lecture Notes in Mathematics, 224, Springer-Verlag, Berlin-New York, 1971.

[2] E. M. Friedlander, *Étale homotopy of simplicial schemes.* Annals of Mathematics Studies, 104, Princeton University Press, Princeton, NJ, 1982.

[3] P. G. Goerss and J. F. Jardine, *Localization theories for simplicial presheaves.* Canad. J. Math. **50**(1998), no. 5, 1048–1089. doi:10.4153/CJM-1998-051-1

[4] ———, *Simplicial homotopy theory.* Progress in Mathematics, 174, Birkhäuser Verlag, Basel, 1999.

[5] M. Hovey, *Model categories.* Mathematical Surveys and Monographs, 63, American Mathematical Society, Providence, RI, 1999.

[6] J. F. Jardine, *Simplicial presheaves.* J. Pure Appl. Algebra **47**(1987), no. 1, 35–87. doi:10.1016/0022-4049(87)90100-9

[7] ———, *Generalized étale cohomology theories.* Progress in Mathematics, 146, Birkhäuser Verlag, Basel, 1997.

[8] ———, *Cocycle categories.* In: Algebraic topology, Abel Symp., 4, Springer, Berlin, 2009, pp. 185–218.

[9] J. S. Milne, *Étale cohomology.* Princeton Mathematical Series, 33, Princeton University Press, Princeton, NJ, 1980.

Mathematics Department, University of Western Ontario, London, ON N6A 5B7
e-mail: jardine@uwo.ca