

## STILL MORE GENERALIZATIONS OF HARDY'S INEQUALITY

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### Abstract

These generalizations of Hardy's Integral Inequality are generalizations of some inequalities of B. G. Pachpatte.

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In [5] B. G. Pachpatte added one more to the multitude of papers generalizing Hardy's Inequality; and the present paper, inspired by [5], is yet another. The proof of our leading theorem is modelled on that of Pachpatte, and our theorem generalizes his. We have added iterated versions of our theorems. Our reference list mentions a few related papers.

**THEOREM 1.** *Let  $0 \leq a < b < \infty$ ,  $c > 0$ ,  $p > 0$  and  $q > 0$  be constants. Let  $r(x)$  be positive and locally absolutely continuous in  $[a, b)$ , and  $f(x)$  be almost everywhere non-negative and measurable on  $(a, b)$ . Let*

$$(1) \quad F(x) = \frac{1}{r(x)} \int_a^x \frac{r(t)f(t)}{t \log(b/t)} dt < \infty \quad \text{for all } x \text{ in } [a, b),$$

and

$$(2) \quad F(x) = o((b-x)^{-q/p}) \quad \text{as } x \rightarrow b-.$$

If  $p > 1$  and

$$(3) \quad 1 + \frac{p}{q} \frac{x r'(x)}{r(x)} \log \frac{b}{x} \geq \frac{1}{c}$$

for almost all  $x$  in  $(a, b)$ , then

$$(4) \quad \left( \int_a^b F(x)^p \frac{(\log(b/x))^{q-1}}{x} dx \right)^{1/p} \leq \frac{cp}{q} \left( \int_a^b f(x)^p \frac{(\log(b/x))^{q-1}}{x} dx \right)^{1/p}.$$

If  $0 < p < 1$  and the reverse inequality (3) holds, then the reverse inequality (4) holds also.

PROOF. We may suppose that  $f$  is not null.

(i) Suppose also that  $a > 0$ . The  $p$ th power of the left side of (4) is equal to

$$\begin{aligned}
 (5) \quad & \left[ -F(x)^p \frac{(\log(b/x))^q}{q} \right]_a^b \\
 & + \int_a^b \frac{(\log(b/x))^q}{q} p F(x)^{p-1} \left( \frac{f(x)}{x \log(b/x)} - \frac{r'(x)}{r(x)} F(x) \right) dx \\
 & = \frac{p}{q} \int_a^b \frac{(\log(b/x))^{q-1}}{x} F(x)^{p-1} f(x) dx \\
 & \quad - \frac{p}{q} \int_a^b \frac{r'(x)}{r(x)} (\log(b/x))^q F(x)^p dx;
 \end{aligned}$$

the integrated term at  $x = a$  has vanished because  $F(a) = 0$  and  $\log(b/a) < \infty$ , while that as  $x \rightarrow b^-$  has modulus, for  $x \geq b/2$ ,

$$\frac{F(x)^p}{q} \left( \log \left( 1 + \frac{b-x}{x} \right) \right)^q \leq \frac{F(x)^p}{q} \left( \frac{b-x}{x} \right)^q \leq \frac{1}{q(\frac{1}{2})^q} F(x)^p (b-x)^q,$$

which tends to zero by (2).

Now (5) can be re-written

$$\begin{aligned}
 (6) \quad & \int_a^b \left( 1 + \frac{p x r'(x)}{q r(x)} \log \frac{b}{x} \right) \frac{(\log(b/x))^{q-1}}{x} F(x)^p dx \\
 & = \frac{p}{q} \int_a^b x^{(p-1)/p} \frac{(\log(b/x))^{q-(q-1)(p-1)/p}}{x \log(b/x)} f(x) \\
 & \quad \times x^{-(p-1)/p} (\log(b/x))^{(q-1)(p-1)/p} F(x)^{p-1} dx.
 \end{aligned}$$

(ii) If  $p > 1$  and (3) holds, (6) and Hölder's Inequality give

$$\begin{aligned}
 \frac{1}{c} \int_a^b \frac{(\log(b/x))^{q-1}}{x} F(x)^p dx & \leq \frac{p}{q} \left( \int_a^b x^{p-1} \frac{(\log(b/x))^{q+p-1}}{x^p (\log(b/x))^p} f(x)^p dx \right)^{1/p} \\
 & \quad \times \left( \int_a^b x^{-1} (\log(b/x))^{q-1} F(x)^p dx \right)^{1-1/p}.
 \end{aligned}$$

The last integral factor is non-zero since  $F$  is not null. Dividing by it gives the required inequality (4). Similarly if  $0 < p < 1$  and (3) is reversed, the reverse inequality (4) is obtained.

(iii) Suppose instead that  $a = 0$ . If  $0 < a' < b$ , all the hypotheses hold with  $a$  replaced by  $a'$ , under their respective conditions on  $p$  and in (3). Call these inequalities (4'). As  $a' \searrow a+$ , the modified  $F(x)$  increases towards the value given in (1); and both

sides of (4') tend to the corresponding sides of (4), using the monotonic convergence theorem for the left sides. This limit process thus produces the inequality (4) as required.

REMARK. Pachpatte's Theorem 2 in [5] is the case of Theorem 1 (above) in which  $p > 1$ ,  $q = 1$ ,  $c = \beta$ ,  $[a, b]$  is replaced by  $[0, 1]$  and  $f(x)$  by  $f(x) \log(1/x)$ . (The assumed integrability of  $f$  in [5] should have been integrability of  $r(t)f(t)/t$ , in order that  $F$  should exist.)

COROLLARY 1. Let  $a, b, c, p, q, r$  be as in Theorem 1. Let  $f(x)$  be almost everywhere non-negative and locally integrable in  $[a, b)$ . Let

$$(7) \quad F(x) = \int_a^x f(t) dt \quad \text{for all } x \text{ in } [a, b), \quad \text{and}$$

$$(8) \quad F(x) = o\left((b-x)^{-q/p} r(x)\right) \quad \text{as } x \rightarrow b-.$$

If  $p > 1$  and (3) holds, then

$$(9) \quad \left\{ \int_a^b \left( \frac{F(x)}{r(x)} \right)^p x^{-1} (\log(b/x))^{q-1} dx \right\}^{1/p} \\ \leq \frac{cp}{q} \left\{ \int_a^b \left( \frac{f(x)}{r(x)} \right)^p x^{p-1} (\log(b/x))^{p+q-1} dx \right\}^{1/p}.$$

If  $0 < p < 1$  and the reverse inequality (3) holds, then the reverse inequality (9) holds also.

PROOF. In Theorem 1 replace  $f(x)$  by  $x \log(b/x) f(x)/r(x)$  and  $F(x)$  by  $F(x)/r(x)$ .

REMARK. In fact Corollary 1 is equivalent to Theorem 1; for, replacing  $f(x)$  by  $r(x) f(x)/x \log(b/x)$  and  $F(x)$  by  $F(x)r(x)$ , Corollary 1 becomes Theorem 1.

COROLLARY 2. Let  $0 \leq a < b < \infty$ ,  $p > 0$ ,  $q > 0$  and  $k < q/p$ . Let  $f(x)$  be almost everywhere non-negative and locally integrable in  $[a, b)$ . Let (7) hold, and

$$(10) \quad F(x) = o\left((b-x)^{k-q/p}\right) \quad \text{as } x \rightarrow b-.$$

If  $p > 1$  then

$$(11) \quad \left( \int_a^b F(x)^p x^{-1} (\log(b/x))^{q-kp-1} dx \right)^{1/p} \\ \leq \frac{p}{q-kp} \left( \int_a^b f(x)^p x^{p-1} (\log(b/x))^{p+q-kp-1} dx \right)^{1/p},$$

while if  $0 < p < 1$  the reverse inequality holds.

PROOF. In Corollary 1 let  $r(x) = (\log(b/x))^k$  and  $c = q/(q - kp)$ . Inequality (3) and its reverse are then satisfied simultaneously, because they are equalities.

COROLLARY 3. Let  $0 \leq a < b < \infty$ ,  $p > 0$ ,  $q > 0$ . Let  $r(x)$  be positive and locally absolutely continuous in  $[a, b)$ . Let  $f(x)$  be almost everywhere non-negative and locally integrable in  $[a, b)$ . Let  $F(x)$  satisfy (7) and (8).

If  $p > 1$  and  $r(x)$  is also increasing, then (9) holds with  $c = 1$ , that is,

$$(12) \quad \left\{ \int_a^b \left( \frac{F(x)}{r(x)} \right)^p x^{-1} (\log(b/x))^{q-1} dx \right\}^{1/p} \leq \frac{p}{q} \left\{ \int_a^b \left( \frac{f(x)}{r(x)} \right)^p x^{p-1} (\log(b/x))^{p+q-1} dx \right\}^{1/p}.$$

If  $0 < p < 1$  and  $r(x)$  is also decreasing, then the reverse inequality (12) holds.

PROOF. Immediate from Corollary 1 and (3).

REMARKS. Theorem 5.1 in [4], with  $\mu = 1 - q$ ,  $R = b$  and  $\omega(x) = 1/r(x)^p$ , gives inequality (12) under almost the same conditions as Corollary 3 with  $p > 1$ . The only significant differences are that  $r$  is not required to be locally absolutely continuous and  $F$  is not required to satisfy (8). More generally, the same can be said of Theorem 5.1 with  $\mu = 1 - q$ ,  $R \geq b$  and

$$\omega(x) = \left( \frac{\log(b/x)}{\log(R/x)} \right)^{q-1} \frac{1}{r(x)^p}$$

provided this last expression is decreasing in  $(a, b)$ ; differentiation shows that this is certainly so if  $q \geq 1$  and  $r(x)$  is increasing.

Theorem 4.2 in [4], with  $\mu = 1 - q$ ,  $(\lambda - 1)k' = q \geq 1$ ,  $0 < R \leq a < b < \infty$ ,  $f(x) = 0$  for  $x > b$  and  $F(b) = 0$ , together with

$$\omega(x) = \left( \frac{\log(b/x)}{\log(x/R)} \right)^{q-1} \frac{x^{\lambda-1}}{r(x)^p}$$

supposed increasing in  $(a, b)$ , gives the inequality (12) reversed, as in Corollary 3 with  $0 < p < 1$ . The first factor in this  $\omega(x)$  is clearly decreasing, but the second factor is certainly increasing if  $r(x)$  is decreasing; and so  $\omega(x)$  is increasing if  $r(x)$  decreases sufficiently rapidly.

THEOREM 2. Let  $0 \leq a < b < \infty$ ,  $c > 0$ ,  $p > 0$ ,  $q > 0$ ,  $n > 0$  be constants,  $n$  being an integer. Let  $r(x)$  be positive and locally absolutely continuous in  $[a, b)$ , and

$f(x)$  be almost everywhere non-negative and measurable on  $(a, b)$ . Let

$$F(x) = \frac{1}{(n-1)!} \int_a^x \frac{f(t)}{t} \left(\log \frac{x}{t}\right)^{n-1} dt < \infty \quad \text{for } a \leq x < b$$

and

$$F(x) = o\left((b-x)^{-q/p} r(x)\right) \quad \text{as } x \rightarrow b-.$$

If  $p > 1$ , together with

$$(13) \quad 1 + \frac{p}{q} \frac{xr'(x)}{r(x)} \log \frac{b}{x} \geq \frac{1}{c} \quad \text{and} \quad 1 + \frac{p}{q+(n-1)p} \frac{xr'(x)}{r(x)} \log \frac{b}{x} \geq \frac{1}{c}$$

for almost all  $x$  in  $(a, b)$ , then

$$(14) \quad \left\{ \int_a^b \left(\frac{F(x)}{r(x)}\right)^p \frac{(\log(b/x))^{q-1}}{x} dx \right\}^{1/p} \leq \frac{c^n}{(q/p)_n} \left\{ \int_a^b \left(\frac{f(x)}{r(x)}\right)^p \frac{(\log(b/x))^{np+q-1}}{x} dx \right\}^{1/p}.$$

Here  $(z)_n = z(z+1) \cdots (z+n-1)$ .

If  $0 < p < 1$  and the reverse inequalities (13) hold, the the reverse inequality (14) holds also.

PROOF. The case  $n = 1$  is Corollary 1 with  $f(x)$  replaced by  $f(x)/x$ .

First suppose the  $a > 0$ . Write  $F_n(x)$  instead of  $F(x)$ . We show that the theorem with  $a > 0$  results from iterating the case  $n = 1$ .

Suppose that the theorem holds for a particular positive integer  $n$ , and that its hypotheses with  $n$  replaced by  $n + 1$  are satisfied (the inductive hypothesis). We aim to show that the conclusion with  $n$  replaced by  $n + 1$  then holds.

For  $a < x < b$ ,

$$\begin{aligned} \int_a^x \frac{F_n(s)}{s} ds &= \frac{1}{(n-1)!} \int_a^x \frac{ds}{s} \int_a^s \frac{f(t)}{t} \left(\log \frac{s}{t}\right)^{n-1} dt \\ &= \frac{1}{(n-1)!} \int_a^x \frac{f(t)}{t} dt \int_t^x \left(\log \frac{s}{t}\right)^{n-1} \frac{ds}{s} \\ &= \frac{1}{n!} \int_a^x \frac{f(t)}{t} \left(\log \frac{x}{t}\right)^n dt; \quad \text{call this expression } F_{n+1}(x). \end{aligned}$$

By the inductive hypothesis  $F_{n+1}(x) < \infty$  for  $a < x < b$ , and so  $F_n(s)/s$  is locally integrable in  $[a, b)$ , and of course non-negative. Also since  $a > 0$

$$\begin{aligned} F_{n+1}(x) &\leq \frac{\log(b/a)}{n} \frac{1}{(n-1)!} \int_a^x \frac{f(t)}{t} \left(\log \frac{x}{t}\right)^{n-1} dt \\ &= O(F_n(x)) \\ &= o\left((b-x)^{-q/p} r(x)\right) \end{aligned}$$

as  $x \rightarrow b-$ . The case  $n = 1$ , with  $f$  replaced by  $F_n$  and consequently  $F$  by  $F_{n+1}$ , is thus applicable, giving

$$(15) \quad \left\{ \int_a^b \left( \frac{F_{n+1}(x)}{r(x)} \right)^p \frac{(\log(b/x))^{q-1}}{x} dx \right\}^{1/p} \leq \frac{c}{q/p} \left\{ \int_a^b \left( \frac{F_n(x)}{r(x)} \right)^p \frac{(\log(b/x))^{p+q-1}}{x} dx \right\}^{1/p}$$

if  $p > 1$ , and the reverse inequality if  $0 < p < 1$  and (13) is reversed.

We can now apply the assumed case of the theorem, with  $q$  replaced by  $q + p$ , giving

$$(16) \quad \left\{ \int_a^b \left( \frac{F_n(x)}{r(x)} \right)^p \frac{(\log(b/x))^{q+p-1}}{x} dx \right\}^{1/p} \leq \frac{c^n}{((q/p) + 1)_n} \left\{ \int_a^b \left( \frac{f(x)}{r(x)} \right)^p \frac{(\log(b/x))^{np+q+p-1}}{x} dx \right\}^{1/p}$$

if  $p > 1$ , and the reverse inequality if  $0 < p < 1$  and the inequalities (13) are reversed.

Together (15) and (16) give (14) with  $n$  replaced by  $n + 1$ . The case  $a > 0$  of the theorem follows by induction.

The extension to the case  $a = 0$  is made in the same way as the corresponding extension of Theorem 1 in part (iii) of its proof, replacing (3) and (4) by (13) and (14).

**THEOREM 3.** *Let  $0 < a < b \leq \infty$ ,  $c > 0$ ,  $p > 0$  and  $q > 0$  be constants. Let  $r(x)$  be positive and locally absolutely continuous in  $(a, b]$ , and let  $f(x)$  be almost everywhere non-negative and measurable on  $(a, b)$ . Let*

$$(17) \quad F(x) = \frac{1}{r(x)} \int_x^b \frac{r(t)f(t)}{t \log(t/a)} dt < \infty \quad \text{for all } x \text{ in } (a, b],$$

and

$$(18) \quad F(x) = o((x - a)^{-q/p}) \quad \text{as } x \rightarrow a +.$$

If  $p > 1$  and

$$(19) \quad 1 - \frac{p}{q} \frac{xr'(x)}{r(x)} \log \frac{x}{a} \geq \frac{1}{c}$$

for almost all  $x$  in  $(a, b)$ , then

$$(20) \quad \left( \int_a^b F(x)^p \frac{(\log(x/a))^{q-1}}{x} dx \right)^{1/p} \leq \frac{cp}{q} \left( \int_a^b f(x)^p \frac{(\log(x/a))^{q-1}}{x} dx \right)^{1/p}.$$

If  $0 < p < 1$  and the reverse inequality (19) holds, then the reverse inequality (20) also holds.

PROOF. On the same lines as the proof of Theorem 1, but supposing that  $b < \infty$ , we obtain, instead of (5),

$$(21) \quad \int_a^b F(x)^p \frac{(\log(x/a))^{q-1}}{x} dx = \frac{p}{q} \int_a^b F(x)^{p-1} f(x) \frac{(\log(x/a))^{q-1}}{x} dx + \frac{p}{q} \int_a^b F(x)^p \frac{r'(x)}{r(x)} (\log(x/a))^q dx;$$

the integrated term at  $x = b$  has vanished because  $F(b) = 0$  and  $\log(b/a) < \infty$ , while that as  $x \rightarrow a+$  tends to zero even more simply than before, by (18). Equation (6) is changed only in that  $\log(b/x)$  is replaced by  $\log(x/a)$  throughout, and  $+$  on the left is replaced by  $-$ .

Corresponding to part (iii) of the proof of Theorem 1, we let  $a < b' < \infty$ , apply the parts already proved with  $b$  replaced by  $b'$ , and make  $b' \nearrow \infty$  using monotony.

REMARK. Pachpatte's Theorem 1 in [5] is the case of Theorem 3 (above) in which  $p > 1, q = 1, c = \alpha$ ,  $\alpha$  is replaced by 1 and  $f(x)$  by  $f(x) \log(x)$ .

COROLLARY 4. Let  $a, b, c, p, q, r$  be as in Theorem 3. Let  $f(x)$  be almost everywhere non-negative and locally integrable in  $(a, b]$ ,

$$(22) \quad F(x) = \int_x^b f(t) dt \quad \text{for all } x \text{ in } (a, b]$$

and

$$(23) \quad F(x) = o((x - a)^{-q/p} r(x)) \quad \text{as } x \rightarrow a + .$$

If  $p > 1$  and (19) holds, then

$$(24) \quad \left\{ \int_a^b \left( \frac{F(x)}{r(x)} \right)^p x^{-1} (\log(x/a))^{q-1} dx \right\}^{1/p} \leq \frac{cp}{q} \left\{ \int_a^b \left( \frac{F(x)}{r(x)} \right)^p x^{p-1} (\log(x/a))^{p+q-1} dx \right\}^{1/p} .$$

If  $0 < p < 1$  and the reverse inequality (19) holds, then the reverse inequality (24) holds also.

PROOF. In Theorem 3 replace  $f(x)$  by  $x \log(x/a) f(x)/r(x)$  and  $F(x)$  by  $F(x)/r(x)$ .

REMARK. In fact Corollary 4 is equivalent to Theorem 3.

**COROLLARY 5.** *Let  $0 < a < b < \infty$ ,  $p > 0$ ,  $q > 0$  and  $k < q/p$ . Let  $f(x)$  be almost everywhere non-negative and locally integrable in  $(a, b]$ . Let (22) hold, and*

$$(25) \quad F(x) = o\left((x - a)^{k - q/p}\right) \quad \text{as } x \rightarrow a +.$$

*If  $p > 1$  then*

$$(26) \quad \left( \int_a^b F(x)^p \frac{(\log(x/a))^{q - kp - 1}}{x} dx \right)^{1/p} \leq \frac{p}{q - kp} \left( \int_a^b f(x)^p \frac{(\log(x/a))^{q - kp + p - 1}}{x^{1-p}} dx \right)^{1/p},$$

*while if  $0 < p < 1$  the reverse inequality holds.*

**PROOF.** In Corollary 4 let  $r(x) = (\log(x/a))^k$  and  $c = q/(q - kp)$ . Inequality (19) and its reverse are now satisfied simultaneously, because they are equalities.

**COROLLARY 6.** *Let  $0 < a < b \leq \infty$ ,  $p > 0$  and  $q > 0$ . Let  $r(x)$  be positive and locally absolutely continuous in  $(a, b]$ . Let  $f(x)$  be almost everywhere non-negative and locally integrable in  $(a, b]$ , and let (22) and (23) hold.*

*If  $p > 1$  and  $r(x)$  is also decreasing, then (24) holds with  $c = 1$ .*

*If  $0 < p < 1$  and  $r(x)$  is also increasing, then the reverse inequality (24) holds with  $c = 1$ .*

**PROOF.** Immediate from Corollary 4 and (19).

**REMARK.** Theorem 5.4 in [4], with  $\mu = 1 - q$  and  $\omega(x) = 1/r(x)^p$ , gives inequality (24) reversed with  $c = 1$  under almost the same conditions as Corollary 6 with  $0 < p < 1$ . However there does not seem to be in [4] any approximate counterpart of Corollary 6 with  $p > 1$ .

**THEOREM 4.** *Let  $0 < a < b \leq \infty$ ,  $c > 0$ ,  $p > 0$ ,  $q > 0$ ,  $n > 0$  be constants,  $n$  being an integer. Let  $r(x)$  be positive and locally absolutely continuous in  $(a, b]$ , and  $f(x)$  be almost everywhere non-negative and measurable on  $(a, b)$ . Let*

$$F(x) = \frac{1}{(n - 1)!} \int_x^b \frac{f(t)}{t} \left( \log \frac{t}{x} \right)^{n-1} dt < \infty \quad \text{for } a < x \leq b$$

*and*

$$F(x) = o\left((x - a)^{-q/p} r(x)\right) \quad \text{as } x \rightarrow a +.$$



If  $p > 1$ , together with

$$(27) \quad 1 - \frac{p}{q} \frac{xr'(x)}{r(x)} \log \frac{x}{a} \geq \frac{1}{c} \quad \text{and} \quad 1 - \frac{p}{q + (n-1)p} \frac{xr'(x)}{r(x)} \log \frac{x}{a} \geq \frac{1}{c}$$

for almost all  $x$  in  $(a, b)$ , then

$$(28) \quad \left\{ \int_a^b \left( \frac{F(x)}{r(x)} \right)^p \frac{(\log(x/a))^{q-1}}{x} dx \right\}^{1/p} \leq \frac{c^n}{(q/p)_n} \left\{ \int_a^b \left( \frac{f(x)}{r(x)} \right)^p \frac{(\log(x/a))^{np+q-1}}{x} dx \right\}^{1/p}.$$

If  $0 < p < 1$  and the reverse inequalities (27) hold, then the reverse inequality (28) holds also.

PROOF. The case  $n = 1$  is Corollary 4 with the function  $f(x)$  replaced by  $f(x)/x$ . First suppose that  $b < \infty$ . Write  $F_n(x)$  instead of  $F(x)$ .

Suppose that the theorem holds for a certain positive integer  $n$ , and that its hypotheses with  $n$  replaced by  $n + 1$  are satisfied.

For  $a < x < b$ ,

$$\begin{aligned} \int_x^b \frac{F_n(s)}{s} ds &= \frac{1}{(n-1)!} \int_x^b \frac{ds}{s} \int_s^b \frac{f(t)}{t} \left( \log \frac{t}{s} \right)^{n-1} dt \\ &= \frac{(-1)^{n-1}}{(n-1)!} \int_x^b \frac{f(t)}{t} dt \int_x^t \left( \log \frac{s}{t} \right)^{n-1} \frac{ds}{s} \\ &= \frac{(-1)^{n-1}}{n!} \int_x^b \frac{f(t)}{t} \left\{ - \left( - \log \frac{t}{x} \right)^n \right\} dt \\ &= \frac{1}{n!} \int_x^b \frac{f(t)}{t} \left( \log \frac{t}{x} \right)^n dt; \quad \text{call this expression } F_{n+1}(x). \end{aligned}$$

By the inductive hypothesis  $F_{n+1}(x) < \infty$  for  $a < x \leq b$ , and so  $F_n(s)/s$  is locally integrable in  $(a, b]$ , and of course non-negative. Also since  $b < \infty$ ,

$$\begin{aligned} F_{n+1}(x) &\leq \frac{\log(b/a)}{n} \frac{1}{(n-1)!} \int_x^b \frac{f(t)}{t} \left( \log \frac{t}{x} \right)^{n-1} dt \\ &= O(F_n(x)) \\ &= o((x-a)^{-q/p} r(x)) \end{aligned}$$

as  $x \rightarrow a+$ . The case  $n = 1$ , with  $f$  replaced by  $F_n$  and consequently  $F$  by  $F_{n+1}$ , is thus applicable, giving

$$(29) \quad \left\{ \int_a^b \left( \frac{F_{n+1}(x)}{r(x)} \right)^p \frac{(\log(x/a))^{q-1}}{x} dx \right\}^{1/p} \leq \frac{c}{q/p} \left\{ \int_a^b \left( \frac{F_n(x)}{r(x)} \right)^p \frac{(\log(x/a))^{p+q-1}}{x} dx \right\}^{1/p}$$

if  $p > 1$ , and the reverse inequality if  $0 < p < 1$  and (27) is reversed.

We can now apply the assumed case of the theorem with  $q$  replaced by  $q + p$ , because  $q + p > 0$  and the conditions corresponding to (27) are fulfilled. The latter of these conditions, namely

$$1 - \frac{p}{q + np} \frac{xr'(x)}{r(x)} \log \frac{x}{a} \geq \frac{1}{c},$$

is provided by the inductive hypothesis, while the former,

$$1 - \frac{p}{q + p} \frac{xr'(x)}{r(x)} \log \frac{x}{a} \geq \frac{1}{c},$$

holds because  $p/(q + p)$  lies between  $p/q$  and  $p/(q + np)$ . Thus

$$(30) \quad \left\{ \int_a^b \left( \frac{F_n(x)}{r(x)} \right)^p \frac{(\log(x/a))^{q+p-1}}{x} dx \right\}^{1/p} \\ \leq \frac{c^n}{((q/p) + 1)_n} \left\{ \int_a^b \left( \frac{f(x)}{r(x)} \right)^p \frac{(\log(x/a))^{np+q+p-1}}{x} dx \right\}^{1/p}$$

if  $p > 1$ , and the reverse inequality if  $0 < p < 1$  and the inequalities (27), with  $n$  replaced by  $n + 1$ , are reversed.

Together (29) and (30) give (28) with  $n$  replaced by  $n + 1$ . This completes the induction proof of the theorem in the case  $b < \infty$ .

If  $b = \infty$ , let  $a < b' < \infty$ . All the hypotheses hold with  $b$  replaced by  $b'$ , and consequently (28) and its reverse hold with  $b$  replaced by  $b'$ , under their respective conditions on  $p$  and conditions (27). This replacement affects  $F$ , but at  $b' \nearrow \infty$  the modified  $F$  tends increasingly to the original  $F$ . So, using the monotonic convergence theorem, this limit process gives (28) with  $b = \infty$ .

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