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DOMAINS OF PARACOMPACTNESS AND REGULARITY

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Given a class $\mathscr{B}$ of topological spaces and a class $\mathscr{F}$ of maps of topological spaces, our interest is in characterization of the class $\mathscr{R}_{\mathscr{F}}(\mathscr{B})$ of topological spaces whose every $\mathscr{F}$-image lies in $\mathscr{B}$. The class $\mathscr{R}_{\mathscr{F}}(\mathscr{B})$ is referred to as the *$\mathscr{F}$-resolvent* of $\mathscr{B}$, and is the largest class of spaces smaller than $\mathscr{B}$ closed under $\mathscr{F}$-images (provided $\mathscr{F}$ is closed under composition and includes identity maps).

In the present paper, $\mathscr{B}$ will be either the class of paracompact spaces or the class of regular spaces, and the conditions determining $\mathscr{F}$ will always include separation axioms on the range, some of which can be dispensed with now by agreeing that *all spaces are Hausdorff*.

In what follows, the set of non-isolated points of a space X will frequently be singled out for attention; this set will be called the *accumulation set* of X or, more often, acc$X$.

1. Domains of regularity: continuous maps. Our purpose in this section is to discover conditions under which every Hausdorff continuous image of X will be regular. We need a few preliminary results.

LEMMA 1.1. *If every continuous Hausdorff image of X is regular, X is normal and countably compact.*

Proof. Normality is clearly necessary, for if A and B are closed sets in X which cannot be separated, the quotient of X obtained by identifying the points of A will be Hausdorff (since X must be regular) but not regular.

Now if X is not countably compact, let (x_n) be a sequence in X having no cluster point. If infinitely many of the points x_n are isolated in X, then X contains a closed copy D of the integers, and then the disjoint union of $X - D$ with any denumerable non-regular Hausdorff space will be a one-one continuous non-regular image of X.

Hence we assume each x_n is an accumulation point of X. Let $C = \{x_1, x_2, \ldots\}$ and pick $x_0 \notin C$. Let τ^* be the topology on X in which basic neighbourhoods of $x \neq x_0$ are unchanged, while basic neighbourhoods of x_0 have the form $V \cup (O - C)$ where V is an old-style neighbourhood of x_0 and O is any old-style open set containing all but finitely many points of C. In this (Hausdorff) topology on X, C is closed but cannot be separated from x_0. Since (X, τ^*) is a (one-one) continuous image of X, we are done.

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LEMMA 1.2 (Katětov [5]). *Every nowhere dense closed subset of X is compact if and only if* acc$X$ *is compact.*

For use in the proof of the following result, recall that a space X is *metacompact* if and only if every open cover has an open point-finite refinement. Arens and Dugundji [2] have shown that X is compact if and only if it is countably compact and metacompact.

With this we can prove the main result of the present section, characterizing the $\mathscr{F}$-resolvant of the class of regular spaces when $\mathscr{F}$ is the class of continuous maps with Hausdorff range.

THEOREM 1.3. *X is compact if and only if every continuous Hausdorff image of X is regular.*

Proof. Necessity is immediate.

Conversely, by Lemma 1.1 we may assume that X is normal and countably compact. Suppose first that acc$X$ is not compact. Then by Lemma 1.2, X contains a non-compact nowhere dense closed set A. Let βX be the Stone-Čech compactification of X and set $B = \mathrm{Cl}_{\beta X} A$. Now $B - A \neq \emptyset$ since A is non-compact, so we may choose $p \in B - A$. Let $\mathscr{U}$ be the neighbourhood system of p in βX and let

$$\mathscr{V} = \{(X \cap U) - A \mid U \in \mathscr{U}\}.$$

Each $U \in \mathscr{U}$ meets X and moreover, since $X \cap U$ is open in X and A has empty interior, each $(X \cap U) - A$ must be non-empty. It is then clear that $\mathscr{V}$ is an open filter base on X. Now choose any $q \in X - A$ and define a new topology on X by leaving neighbourhoods of points $r \neq q$ unchanged and letting the basic neighbourhoods of q be sets of the form $S \cup V$ where S is an old-style neighbourhood of q and $V \in \mathscr{V}$. Call X with this new topology X^*. Then X^* is a one-one continuous Hausdorff image of X, but is not regular since (routinely) A can no longer be separated from q by open sets.

Thus if every continuous Hausdorff image of X is regular, then acc$X$ must be compact. But then X is metacompact, whence by the Arens-Dugundji result referred to above, X must be compact.

2. Domains of regularity: quotient maps. We turn now to the problem of characterizing the $\mathscr{F}$-resolvent for the class of regular spaces when $\mathscr{F}$ is the collection of quotient maps (with Hausdorff range). So far, only partial results are available; their depth can be judged by the fact that we cannot yet say whether the real line has non-regular Hausdorff quotients.

In [7], an A-*space* is defined to be a metrizable space X such that acc$X$ is compact. We find it convenient now to define an A'-*space* to be a Hausdorff space X such that acc$X$ is compact.

It is easily proved that every A'-space is regular and that the quotient of an A'-space is again an A'-space. These facts lead to an easy sufficient condition for every quotient of X to be regular:

THEOREM 2.1. *Every Hausdorff quotient of an A'-space is regular.*

The attractive conjecture that *only* A'-spaces have this property remains unproved. We have only the following result.

THEOREM 2.2. *If X is first countable and every Hausdorff quotient of X is regular, then* acc$X$ *is locally countably compact.*

Proof. If acc$X$ is not locally countably compact then there exists a point p in acc$X$ having no countably compact neighbourhood, and $\operatorname{acc}(X - \{p\})$ contains a sequence $a_1, a_2, \ldots$ having no cluster point. By the regularity of X there is a neighbourhood W of p that misses some open set containing $\{a_i | i \in N\}$. Now by first countability, for each $i \in N$ there is a sequence $x_{i1}, x_{i2}, \ldots$ in $X - W$ that is disjoint from all such sequences $\{x_{jn}\}$ for $j < i$ and such that $\{x_{in}\} \to a_i$. Let $\mathscr{U} = \{U_1, U_2, \ldots\}$ be a countable neighbourhood base at p, with $U_1 = W$ and $U_n \supset U_{n+1}$. Then U_1 contains a sequence $y_{11}, y_{12}, \ldots$ having no cluster point. Having found for any $k \in N$ a sequence $y_{k1}, y_{k2}, \ldots$ which has no cluster point and which is contained in the neighbourhood

$$U_k - \{y_{in} | i = 1, \ldots, k - 1; n \in N\}$$

of p, we see that

$$U_{k+1} - \{y_{in} | i = 1, \ldots, k; n \in N\}$$

is a neighbourhood of p and must contain a sequence $y_{k+1,1}, y_{k+1,2}, \ldots$ having no cluster point. Thus, by induction, we find in each neighbourhood U_m of p a sequence $y_{m1}, y_{m2}, \ldots$ having no cluster point and disjoint from the closed set

$$\{a_i | i \in N\} \cup \{x_{in} | i = 1, \ldots, m; n \in N\} \cup \{y_{in} | i = 1, \ldots, m - 1; n \in N\}.$$

Now the quotient obtained by identifying y_{kn} with x_{kn} for all $k, n \in N$ is Hausdorff but not regular, since the point p and the closed set $\{a_i | i \in N\}$ cannot be separated by open sets.

The following example shows, however, that the conditions of the last theorem are far from sufficient, even for first countable spaces.

Example 2.3. For $m = 4, 5, 6, \ldots$. Let

$$W_m = \{0\} \cup \bigcup_{n=1}^{\infty} \left[\frac{1}{n+1} + \frac{1}{n^m}, \frac{1}{n} - \frac{1}{n^m}\right]$$

with the usual topology as a subspace of the real line. Let X be the disjoint union of the spaces $W_m, m = 4, 5, \ldots$ and the intervals $[1/n, 1], n = 1, 2, \ldots$. Then X is locally compact, σ-compact and without isolated points. But the quotient topology obtained by projecting X onto $[0, 1]$ is non-regular (and Hausdorff), since the set $\{1/n | n \in N\}$ is closed in this topology but cannot be separated from the origin by open sets.

The question of open maps seems difficult. Local compactness of X is certainly sufficient for regularity of every Hausdorff open quotient of X, but, since there are non-locally compact A'-spaces, this condition cannot be necessary (cf. Theorem 2.1). On the other hand, local compactness of acc$X$ is not sufficient, as is shown by the following example, which is a modification of Example 2 of [**1**, p. 70].

Example 2.4. Let

$$X_1 = \{(0, 0)\} \cup \left\{\left(0, \frac{1}{n} + \frac{1}{n^m}\right) \middle| n = 1, 2, \ldots; m = 2, 3, \ldots\right\}$$

and let

$$X_2 = \left\{\left(1, \frac{1}{n}\right) \middle| n = 1, 2, \ldots\right\} \cup \left\{\left(1, \frac{1}{n} + \frac{1}{n^m}\right) \middle| n = 1, 2, \ldots; m = 2, 3, \ldots\right\},$$

both considered as subspaces of the plane. Let $X = X_1 \cup X_2$. Now since the only non-isolated points of X are $(0, 0)$ and the points $(1, 1/n)$ for $n = 1, 2, \ldots$, acc$X$ is clearly a locally compact subspace of X. However, the open quotient map obtained by identifying the points $(0, 1/n + 1/n^m)$ and $(1, 1/n + 1/n^m)$ for $n = 1, 2, \ldots$, and $m = 2, 3, \ldots$ has a non-regular Hausdorff image, since the closed set $\{(1, 1/n) | n = 1, 2, \ldots\}$ cannot be separated from the point $(0, 0)$ by open sets.

3. Domains of paracompactness: continuous maps. The question now considered is: for which spaces X is every continuous regular image of X paracompact? The reason for imposing the condition of regularity on the range is evident, the question otherwise degenerating into the question of Section 1.

We will need the following lemma about paracompact spaces, of some independent interest.

LEMMA 3.1. *If X is paracompact, the following conditions on X are equivalent:*

(a) *X is Lindelöf,*

(b) *every open cover of X has a countable subcollection whose closures cover,*

(c) *every open cover of X has a countable subcollection whose union is dense in X,*

(d) *every uncountable subset of X has an accumulation point.*

Proof. (a) ⇒ (b) ⇒ (c) is obvious.

(a) ⇒ (d). If X contains an uncountable set A having no accumulation point, then for each point $x \in X$ there is some open set U_x containing only finitely many points of A. Now $\{U_x | x \in X\}$ is an open cover of X having no countable subcover, whence X cannot be Lindelöf.

(c) ⇒ (a). If $\mathscr{U}$ is any open cover of X, let $\mathscr{V}$ be a locally finite open cover such that the closure of each $V \in \mathscr{V}$ is contained in some $U \in \mathscr{U}$. Then $\mathscr{V}$ has a countable subcollection $\{V_1, V_2, \ldots\}$ whose union is dense in X and

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each $\bar{V}_i$ is contained in some $U_i \in \mathscr{U}$. Then, by local finiteness of $\mathscr{V}$ $X = \overline{\bigcup V_1} = \bigcup \bar{V}_i \subset \bigcup U_i$, so $\mathscr{U}$ has a countable subcover.

(d) $\Rightarrow$ (a). Suppose X is not Lindelöf. Let $\mathscr{U}$ be an open cover with no countable subcover and let $\mathscr{V}$ be a locally finite refinement of $\mathscr{U}$. Then $\mathscr{V}$ is an uncountable cover and if we pick $x_U \in \mathscr{U}$, for each $U \in \mathscr{V}$, the set $\{x_U | U \in \mathscr{V}\}$ can have no accumulation point since $\mathscr{V}$ is locally finite.

Recall now that T is an *absolute retract* if and only if whenever X is normal and $f:A \rightarrow T$ is continuous on a closed subset A of X, then f has an extension F to all of X. It is a standard fact [**3**, p. 151] that $R^{\aleph_1}$ is an absolute retract. This will be used in the proof of the following theorem, which does the job of characterizing the $\mathscr{F}$-resolvant of the class of paracompact spaces, when $\mathscr{F}$ is the class of continuous maps with regular range.

THEOREM 3.2. *X is regular Lindelöf if and only if every continuous regular image of X is paracompact.*

Proof. Necessity is immediate.

For sufficiency, note that X must be paracompact, and hence normal. If X is not Lindelöf, then, by Lemma 3.1 there is some uncountable set A in X having no accumulation point. Now A is a closed and relatively discrete subspace of X. But $N^{\aleph_1}$ is a closed non-normal subspace of $R^{\aleph_1}$ (A. H. Stone [**6**]) and any map f of A onto $N^{\aleph_1}$ will be continuous. Then, $R^{\aleph_1}$ being an absolute retract, f has an extension to a continuous map $F:X \rightarrow R^{\aleph_1}$. Because $f(A)$ is a closed non-paracompact subspace of $F(X)$, $F(X)$ is not paracompact.

4. Domains of paracompactness: quotient maps. Once again, partial results are available which point to an attractive but as yet unproved conjecture.

THEOREM 4.1. *If X is regular and* acc$X$ *is Lindelöf, then every regular quotient of X is paracompact.*

Proof. Let f be a quotient map of X onto a regular space Y. Then $f(\text{acc}X)$ is Lindelöf and, for $y \in f(X) - f(\text{acc}X)$, $f^{-1}(y)$ is open, so y is isolated. Thus $\text{acc} f(X) \subset f(\text{acc}X)$ and $\text{acc} f(X)$ is Lindelöf. Thus if $\mathscr{U}$ is an open cover of Y, there is a countable subcollection $\mathscr{W}$ covering $f(\text{acc}X)$. Hence

$$\mathscr{W} \cup \{\{y\} | y \in Y - f(\text{acc}X)\}$$

is an open σ-locally finite refinement of $\mathscr{U}$.

The necessity of the condition of the last theorem remains unproved in general, although we can prove it for a large class of spaces assuming the continuum hypothesis. We make use of the space Ψ as described in [**4, p. 79**], which can be constructed as follows. Let N be the discrete space of positive integers, $\mathscr{P}$ an infinite collection of infinite subsets of N, maximal with respect

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to the property that if $P_1, P_2 \in \mathscr{P}$ and $P_1 \neq P_2$, then $P_1 \cap P_2$ is finite. For each $P \in \mathscr{P}$, adjoin a point x_p to N whose neighbourhoods are $\{x_p\} \cup F$, where F is a cofinite subset of P. The space Ψ is locally compact, separable, pseudocompact, but not paracompact.

To prove the theorem we are after (4.3), we require the following result.

LEMMA 4.2. *Let f be a quotient mapping of a normal space X onto a Hausdorff space Z such that, for some closed subspace D of X*

(i) *$f(D)$ is regular, and*

(ii) *$f^{-1}f(a) = a$, for each $a \notin D$.*

Then Z is regular.

Proof. Let A be closed in Z and $c \notin A$. Note that $E = f(D)$ is regular and closed in Z. There are two cases.

Case 1. $c \notin E$. Then $f^{-1}(c)$ and $f^{-}(A) \cup D$ are disjoint closed subsets of X. If U is an open subset of X such that $f^{-1}(c) \subset U$ and $\bar{U}$ does not meet $f^{-1}(A) \cup D$, then $f(U)$ is open, since $f^{-1}f(U) = U$, and contains c, and $\overline{f(U)}$ does not meet A (since $\overline{f(U)} \subset f(\bar{U})$, the latter being closed since $f^{-1}f(\bar{U}) = \bar{U}$). This dispenses with Case 1.

Case 2. $c \in E$. Pick sets U and V open in E such that $c \in U$, $A \cap E \subset V$ and $\bar{U} \cap \bar{V} = \emptyset$. (Since E is closed in Z, we need not concern ourselves with which space we are taking closure in here.)

Then $f^{-1}(\bar{U})$ and $f^{-1}(\bar{V})$ are disjoint neighbourhoods of $f^{-1}(c)$ and $f^{-1}(A) \cap D$ in D and are closed in X. Thus, by normality we can find disjoint open sets S and T in X such that

$$f^{-1}(\bar{U}) \subset S, \qquad f^{-1}(\bar{V}) \cup f^{-1}(A) \subset T.$$

Now, define S^* and T^* by

$$S^* = f^{-1}(U) \cup (S - D),$$
$$T^* = f^{-1}(V) \cup T - D).$$

Then it is easily verified that S^* and T^* are open in X and that $f^{-1}f(S^*) = S^*$, $f^{-1}f(T^*) = T^*$. It follows that $f(S^*)$ and $f(T^*)$ are disjoint open sets in Z containing c and A, respectively.

Hence Z is regular.

THEOREM 4.3. *With the continuum hypothesis, if X is paracompact and* acc$X$ *contains a dense first countable subspace, then every regular quotient of X is paracompact if and only if* acc$X$ *is Lindelöf.*

Proof. Sufficiency follows from Theorem 4.1.

Conversely, suppose that X is paracompact and that acc$X$ contains a dense first countable subspace, but that acc$X$ is not Lindelöf. Then by Lemma 3.1, acc$X$ has an uncountable locally finite open cover $\mathscr{U}$ with no countable subcollection whose union is dense.

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Pick $U_1 \in \mathscr{U}$ and, since acc$X$ has a dense first countable subspace, find a converging sequence $x_{1n} \to x_1$ of distinct points contained in U_1. Suppose that for each $\alpha < \beta$ $(< w_1)$ we have chosen $U_\alpha \in \mathscr{U}$ and a converging sequence $x_{\alpha n} \to x_\alpha$ of distinct points contained in $U_\alpha - [\bigcup_{\gamma<\alpha} U_\gamma]^-$. Then

$$\overline{\bigcup_{\alpha<\beta} U_\alpha} \neq \text{acc}X$$

and hence we may pick $U_\beta \in \mathscr{U}$ so that $U_\beta - [\bigcup_{\alpha<\beta} U_\alpha]^-$ is a non-empty open set and find $x_{\beta n} \to x_\beta$ in $U_\beta - [\bigcup_{\alpha<\beta} U_\alpha]^-$.

Let $A_\alpha = \{x_{\alpha n} | n = 1, 2, \ldots\} \cup \{x_\alpha\}$ and let $A = \bigcup_{\alpha<w_1} A_\alpha$. Then A is closed in acc$X$, and hence in X, by local finiteness of $\mathscr{U}$.

On the continuum hypothesis, there is a one-one map h of some subset of $\{x_\alpha | \alpha < w_1\}$ onto $\{x_P \in \Psi | P \in \mathscr{P}\}$. We assume, without loss, that the domain of h is all of $\{x_\alpha | \alpha < w_1\}$. Now h can be extended to all of A by mapping the sequence $(x_{\alpha n})$ which converges to x_α one-one onto the sequence P which converges in Ψ to x_P. The resulting map g will not be one-one, although it is one-one when restricted to each A_α, but it does exhibit Ψ as a quotient of A.

Now define an equivalence relation R on X by xRy if $x = y$ or $g(x) = g(y)$, letting the resulting quotient space be denoted Z. If $f:X \to Z$ is the quotient map, then clearly $f(A)$ is a closed subspace of Z which is homeomorphic to Ψ. Since Ψ is not paracompact, neither is Z. But by Lemma 4.2, with A here taking the place of D there, Z is regular.

This completes the proof of 4.3.

COROLLARY 4.4. *If X is a metrizable space, then every regular quotient of X is paracompact if and only if* acc$X$ *is separable.*

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