# REPRESENTATIONS OF SPACES AS FUNCTION SPACES

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**0. Introduction.** Given a topological space $X$, we can consider the group $G(X)$ of all autohomeomorphisms of $X$. Much is known about the relationship between $X$ and $G(X)$ for certain restricted classes of the space $X$; Whittaker **[7]** has shown that the existence of an isomorphism between any two sufficiently large subgroups of $G(X)$ and $G(Y)$ implies that $X$ and $Y$ are actually homeomorphic, whenever these are both compact, locally Euclidean manifolds, with or without boundary; Fine and Schweigert **[1]** give a detailed analysis of $G(\mathbb{R})$; recently, Neumann **[4]**, Mekler **[3]** and Truss **[6]** have considered in depth the group $G(\mathbb{Q})$.

A proven technique when studying arbitrary spaces is to embed them within other spaces about which more is known; thus the study of compact Hausdorff spaces allows for a greater understanding of Tychonov spaces (i.e. those spaces which occur as subspaces of compact Hausdorff spaces). Similarly, Shimrat **[5]** has shown that every space $X$ can be embedded in a homogeneous superspace.

We shall show that every space $X$ embeds as a retract within the space $C(G(X), X)$ of continuous functions from $G(X)$ into $X$ (with suitably defined topologies), and that this embedding has the additional property that every autohomeomorphism of $X$ extends to an autohomeomorphism of $C(G(X), X)$. Moreover, if $X$ is Tychonov, so is $C(G(X), X)$, and our retraction extends to a retraction of $\beta C(G(X), X)$ onto $\beta X$.

**1. A class of topologies for** $G(X)$**.** Let $X$ be some fixed but arbitrary topological space. Throughout this paper, we denote by $p$ any topological property satisfying

(i) $p(\{x\})\ \forall x \in X$,

(ii) $[p(A) \text{ and } p(B)] \Rightarrow p(A \cup B)\ \forall A, B \in X$.

For example, $p(A)$ might mean "$A$ is compact", "$A$ is finite", or even "$A \subseteq X$". We shall write $(p; B)$ to mean that $p(A)$ holds for every relatively closed subset $A \subseteq B \subseteq X$.

If $A \subseteq X$, we denote by $G_A$ the stabiliser of $A$ in $G(X)$, viz:

$$G_A = \{g \in G(X):\ g(a) = a\ \forall a \in A\}.$$

Given some such property $p$, we may take $\mathbb{F}_p = \{G_A : A \subseteq X, p(A)\}$ to be a fundamental system of open sets for $G(X)$. For we need only check that

(i) if $U, V \in \mathbb{F}_p$, then there exists $W \in \mathbb{F}_p$ such that $W \subseteq U \cap V$;

(ii) if $U \in \mathbb{F}_p$, $g \in G(X)$, then there exists $V \in \mathbb{F}_p$ such that $g^{-1}Vg \subseteq U$.

See e.g. **[1**, p. 28**]**.

Now (i) follows by taking $W = G_{A \cup B}$, where $U = G_A$ and $V = G_B$, recalling our initial

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hypothesis that $p(A)$ and $p(B)$ together imply $p(A \cup B)$. Condition (ii) is satisfied since every conjugate of a stabiliser is again a stabiliser, and because $p$ is preserved under maps in $G(X)$, since it is topological. This gives a topology on $G(X)$ with which it becomes a topological group, which we shall denote $G_p(X)$. When $p$ is the particular property of being compact, we write $G_K(X)$ instead of $G_p(X)$.

LEMMA 1. *$G_p(X)$ is a zero-dimensional Tychonov space.*

*Proof.* If $g \in \bigcap \mathbb{F}_p$, then $g \in G_A$ for every $A \subseteq X$ satisfying $p(A)$. In particular, $g \in G_{\{x\}}$ for every $x \in X$, whence $g = 1$. So $G_p(X)$ is $T_1$ and so Hausdorff. It is now sufficient to note that every element of $\mathbb{F}_p$ is clopen, since its complement is a union of its cosets, each of which is open. ■

When discussing a topological group $G$, it is often of interest to determine whether $G$ is locally compact, since we may then define Haar measure on $G$. Note that, by taking $p(A) \equiv$ "$A \subseteq X$", we can always ensure that $G_p(X)$ is discrete; but for other properties $p$, $G_p(X)$ need not be locally compact, as we now demonstrate.

If $G_p(X)$ is locally compact, for some property $p$, then Lemma 1 tells us that the group identity, 1, has a basis of compact clopen sets, each in $\mathbb{F}_p$.

We are grateful to the referee for greatly improving upon our original proof of the next result.

THEOREM 2. *Let $A \subseteq B \subseteq X$, where $p(A)$ and $(p; B)$. Suppose $G_A$ is compact in $G_p(X)$. Then there exists a finite $S \subseteq B \backslash A$ such that*

$$G_B = G_{\overline{A \cup S}^B}.$$

*Proof.* For each finite subset $F$ of $B \backslash A$ define $A_F = \overline{A \cup F}^B$ and $H_F = G_{A_F} \backslash G_B$. Then $H_F$ is a closed subset of the compact set $G_A \backslash G_B$. Since the union of the $A_F$'s is $B$, we have $\bigcap H_F = \varnothing$, and so the $H_F$'s fail to have the finite intersection property. Hence we can find finite sets $F_1, \ldots, F_n$ in $B \backslash A$ and

$$\varnothing = H_{F_1} \cap \ldots \cap H_{F_n} = H_{F_1 \cup \ldots \cup F_n}.$$

So, putting $S = F_1 \cup \ldots \cup F_n$ completes the proof. ■

COROLLARY 3. *$G_K(\mathbb{Q})$ is not locally compact.*

*Proof.* Suppose that $G_A$ were compact, for some compact $A \subseteq \mathbb{Q}$. Let $T$ be any infinite compact subset of $\mathbb{Q}$, disjoint from $A$. Taking $B = A \cup T$ satisfies the conditions of the theorem, whence there exists some finite $S \subseteq T$ such that $G_{A \cup S} = G_{A \cup T}$ (since $\overline{A \cup S}^{A \cup T} = A \cup S$). But this is clearly nonsense. For, choose any $x \in T \backslash S$, and any interval $(a, b)$ containing $x$ and disjoint from $A \cup S$. Then we can easily find an element of $G(\mathbb{Q})$ fixing $\mathbb{Q} \backslash (a, b)$ and moving $x$. ■

**2. Representing spaces as function spaces.** We conclude this paper by showing that every space $X$ may be regarded as a space of continuous functions from $G_p(X)$ into $X$. Our method parallels that of the embedding of a vector space in its second dual.

LEMMA 4. *Given $x \in X$, define $\phi_x : G_p(X) \to X$ by $\phi_x(g) = g(x)$. Then $\phi_x^{-1}(S)$ is open in $G_p(X)$ for every $S \subseteq X$.*

*Proof.* We simply note that $\phi_x^{-1}(S) = \phi_x^{-1}(S) \, . \, G_{\{x\}}$, and the latter is open since $G_{\{x\}} \in \mathbb{F}_p$. ■

Consequently, we may sensibly define a map $\Phi : X \to C(G_p(X), X)$ by $\Phi(x) = \phi_x$ for all $x \in X$. The map $\Phi$ is injective, since $\phi_x = \phi_y$ implies $x = \phi_x(1) = \phi_y(1) = y$.

Let $C(G_p(X), X)$ be given the finite-open topology. That is, we take as a subbase for the topology all sets of the form

$$(K, U) = \{f \in C(G_p(X), X) : f(K) \subseteq U\},$$

where $K$ ranges over all finite subsets of $G(X)$, and $U$ over all open subsets of $X$. We shall always assume that $C(G_p(X), X)$ is equipped with this topology.

THEOREM 5. *The map $\Phi : X \to C(G_p(X), X)$ is a topological embedding.*

*Proof.* We have already seen that $\Phi$ is injective, so we need only demonstrate bicontinuity. If $(K, U)$ is a typical subbasic set in $C(G_p(X), X)$, then

$$\begin{aligned}\Phi^{-1}(K, U) &= \{x \in X : \phi_x(K) \subseteq U\} \\ &= \{x \in X : g(x) \in U \ \forall g \in K\} \\ &= \bigcap \{g^{-1}(U) : g \in K\}.\end{aligned}$$

The latter is a finite intersection of open sets, and so is open in $X$.

Conversely, if $U$ is an open set in $X$, then

$$\begin{aligned}\Phi(U) &= \{\phi_x : x \in U\} \\ &= \{f \in C(G_p(X), X) : f(1) \in U\} \cap \operatorname{im} \Phi \\ &= (\{1\}, U) \cap \operatorname{im} \Phi,\end{aligned}$$

which is open in $\operatorname{im} \Phi$, since $(\{1\}, U)$ is open in $C(G_p(X), X)$. ■

THEOREM 6. *Let $g \in G(X)$. Then $g$ has an extension $\psi_g \in G(C(G_p(X), X))$, i.e. $g = \psi_g |_X$, where we identify $X$ and $\Phi(X)$.*

*Proof.* Define $\psi_g : C(G_p(X), X) \to C(G_p(X), X)$ by $\psi_g(f)(\tilde{g}) = f(\tilde{g}g)$ for all $g \in G(X)$, $f \in C(G_p(X), X)$. We shall show that $\psi_g \in G(C(G_p(X), X))$.

Note first that $\psi_g(f) \in C(G_p(X), X)$ whenever $g \in G(X)$ and $f \in C(G_p(X), X)$. For suppose $U \subseteq X$ is open. Then

$$\begin{aligned}[\psi_g(f)]^{-1}(U) &= \{\tilde{g} : \psi_g(f)(\tilde{g}) \in U\} \\ &= \{\tilde{g} : f(\tilde{g}g) \in U\} \\ &= \{\tilde{g} : \tilde{g}g \in f^{-1}(U)\} \\ &= f^{-1}(U) \, . \, g^{-1},\end{aligned}$$

which is open, since $f^{-1}(U)$ is open in $G_p(X)$ by the continuity of $f$.

Now suppose that $\psi_g(f_1) = \psi_g(f_2)$. Then given any $\tilde{g} \in G(X)$, we have $f_1(\tilde{g}g) = f_2(\tilde{g}g)$. Since $\tilde{g}g$ ranges over all of $G(X)$ as $\tilde{g}$ does, we see that $f_1 = f_2$. So $\psi_g$ is injective.

Moreover, if $(K, U)$ is a typical subbasic open set in $C(G_p(X), X)$, then

$$\begin{aligned}\psi_g^{-1}(K, U) &= \{f : \psi_g(f)(K) \subseteq U\}\\ &= \{f : f(\tilde{g}g) \in U\ \forall \tilde{g} \in K\}\\ &= (Kg, U),\end{aligned}$$

which is open. So $\psi_g$ is continuous.

Likewise, $\psi_{g^{-1}}$ is an injective continuous map, and since $\psi_g \circ \psi_{g^{-1}} = \psi_{g^{-1}} \circ \psi_g = \text{id}$, we see that $\psi_g$ is a homeomorphism, as claimed.

It remains only to show that $\psi_g$ extends $g$. But this is clear, since $\psi_g(\phi_x)(\tilde{g}) = \phi_x(\tilde{g}g) = \tilde{g}g(x) = \phi_{g(x)}(\tilde{g})$. Thus $\Phi \circ g = \psi_g \circ \Phi$, as required. ■

We now show that $X$ is a retract of $C(G_p(X), X)$, and that, if $X$ is Tychonov, then so is $C(G_p(X), X)$. We obtain, as a corollary, that $\beta X = \bar{X}^{\beta C(G_p(X), X)}$ whenever $X$ is Tychonov.

LEMMA 7. *$X$ is a retract of $C(G_p(X), X)$.*

*Proof.* Define $\theta : C(G_p(X), X) \to X$ by $\theta(f) = f(1)$. If $x = X$, then $\theta(\phi_x) = \phi_x(1) = x$, so that $X$ is fixed by $\theta$. To show that $\theta$ is continuous, we consider a typical open set $U$ of $X$. Now

$$\begin{aligned}\theta^{-1}(U) &= \{f \in C(G_p(X), X) : \theta(f) \in U\}\\ &= (\{1\}, U),\end{aligned}$$

which is open in $C(G_p(X), X)$. ■

LEMMA 8. *If $X$ is Tychonov, so is $C(G_p(X), X)$.*

*Proof.* Let $f \in (K, U)$, where $(K, U)$ is some typical subbasic open set in $C(G_p(X), X)$. We have to find a continuous $F : C(G_p(X), X) \to [0, 1]$, such that $F(f) = 0$ and $F \equiv 1$ outside $(K, U)$ (see e.g. [**8**, 14.8]).

The set $f(K)$ is a finite subset of $U$. Since $X$ is Tychonov, there exists a continuous $\hat{F} : X \to [0, 1]$ such that $\hat{F} \equiv 0$ on $f(K)$ and $\hat{F} \equiv 1$ outside $U$. Define $F : C(G_p(X), X) \to [0, 1]$ by

$$F(\tilde{f}) = \max\{\hat{F}(\tilde{f}(g)) : g \in K\}.$$

Then $F(f) = 0$, while if $\tilde{f} \notin (K, U)$, then $\tilde{f}(g) \notin U$ for some $g \in K$, whence $F(\tilde{f}) = 1$.

We now show that $F$ is continuous. Write $K = \{g_1, \ldots, g_n\}$ and for each $i = 1, \ldots, n$, define $\psi_i : C(G_p(X), X) \to X$ by $\psi_i(\tilde{f}) = \tilde{f}(g_i)$. Then $\psi_i = \theta \circ \psi_{g_i}$, so $\psi_i$ is continuous for each $i = 1, \ldots, n$, where $\theta$ is as in Lemma 7 and $\phi_{g_i}$ as in Theorem 6. It is now enough to note that

$$\begin{aligned}F(\tilde{f}) &= \max\{\hat{F}(\tilde{f}(g)) : g \in K\}\\ &= \max\{\hat{F} \circ \psi_i(\tilde{f}) : i = 1, \ldots, n\}.\end{aligned}$$

Since each $F \circ \psi_i$ is continuous, so is their maximum.

We conclude the proof by showing that $C(G_p(X), X)$ is Hausdorff. If $f_1, f_2$ are distinct elements of $C(G_p(X), X)$, there exists some $g \in G(X)$ such that $f_1(g) \neq f_2(g)$. Since $X$ is Hausdorff, there exist disjoint open sets $U$, $V$ with $f_1(g) \in U$ and $f_2(g) \in V$. Now $f_1 \in (\{g\}, U)$, $f_2 \in (\{g\}, V)$, and these two open sets are disjoint. ■

COROLLARY 9. *Let $X$ be Tychonov. Then*

$$\beta X = \bar{X}^{\beta C(G_p(X), X)}$$

*and, moreover, $\theta$ extends to a retraction $\theta^\beta : \beta(C(G_p(X), X) \to \beta X$.*

*Proof.* According to Lemma 8, $C(G_p(X), X)$ is Tychonov, so that $\beta C(G_p(X), X)$ exists. Now $X$ is a retract of, and so is $C^*$-embedded in, $C(G_p(X), X)$. Hence $X$ is $C^*$-embedded in its compactification, $\bar{X}^{\beta C(G_p(X), X)}$, whence $\beta X = \bar{X}^{\beta C(G_p(X), X)}$.

Now $\theta$ has a natural extension $\theta^\beta : \beta C(G_p(X), X) \to \beta X$; since $\theta$ acts as the identity on $X$, $\theta^\beta$ must act as the identity on $\bar{X}^{\beta C(G_p(X), X)} = \beta X$, as claimed.

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