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Laurent family of simple modules over quiver Hecke algebras

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Abstract

We introduce the notions of quasi-Laurent and Laurent families of simple modules over quiver Hecke algebras of arbitrary symmetrizable types. We prove that such a family plays a similar role of a cluster in quantum cluster algebra theory and exhibits a quantum Laurent positivity phenomenon similar to the basis of the quantum unipotent coordinate ring $\mathcal{A}_q(\mathfrak{n}(w))$, coming from the categorification. Then we show that the families of simple modules categorifying Geiß–Leclerc–Schröer (GLS) clusters are Laurent families by using the Poincaré–Birkhoff–Witt (PBW) decomposition vector of a simple module X and categorical interpretation of (co)degree of $[X]$. As applications of such $\mathbb{Z}$-vectors, we define several skew-symmetric pairings on arbitrary pairs of simple modules, and investigate the relationships among the pairings and Λ-invariants of R-matrices in the quiver Hecke algebra theory.

Contents

1. Introduction

A cluster algebra and its non-commutative version quantum cluster algebra, were introduced by Berenstein, Fomin and Zelevinsky [FZ02, BZ05] in an attempt to provide an algebraic and combinatorial framework for investigating the upper global basis of the quantum group.

The quantum cluster algebra $\mathscr{A}_q$ is a non-commutative $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra in the skew field $\mathbb{Q}(q^{1/2})(X_i)_{i \in \mathsf{K}}$ generated by the cluster variables, which are obtained from the initial cluster $\{X_i\}_{i \in \mathsf{K}}$ via the sequences of procedures, called *mutations*. Even though mutations involve non-trivial fractions, $\mathscr{A}_q$ is still contained in $\mathbb{Z}[q^{\pm 1/2}][X_i^{\pm 1}]_{i \in \mathsf{K}}$ with amazing reductions of fractions which is referred to as the *quantum Laurent phenomenon* [BZ05]. The famous conjecture, which

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is not completely proved yet at this moment, is the *quantum Laurent positivity* conjecture which asserts that every cluster variable is an element in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1/2}][\mathsf{X}_i^{\pm 1}]_{i\in\mathsf{K}}$ for any cluster $\{\mathsf{X}_i\}_{i\in\mathsf{K}}$. Note that the conjecture is proved in [Dav18] (see also [LS15, GHKK18]) when $\mathscr{A}_q$ is of skew-symmetric type and is widely open when it is of non-skew-symmetric type.

The notion of *monoidal categorification* of (quantum) cluster algebra was introduced by Hernandez and Leclerc in [HL10] (see also [KKKO18]) as the categorical framework for proving the conjecture as follows: a monoidal category $\mathcal{C}$ with an autofunctor q is a monoidal categorification of $\mathscr{A}_q$, if (a) $\mathbb{A}\otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$ ($\mathbb{A}:=\mathbb{Z}[q^{\pm 1/2}]$) is isomorphic to $\mathscr{A}_q$ and (b) the cluster monomials of $\mathscr{A}_q$ are the classes of real simple objects of $\mathcal{C}$. Once $\mathcal{C}$ is a monoidal categorification of $\mathscr{A}_q$, then the conjecture for $\mathscr{A}_q$ follows since it can be interpreted as the existence of a Jordan–Hölder series of an object. In [KKKO18], it is proved that the category $\mathscr{C}_w$ over *symmetric* quiver Hecke algebra R is a monoidal categorification of the quantum unipotent coordinate ring $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ associated with an element w of the Weyl group W by using the $\mathbb{Z}$-invariant $\Lambda(M,N)$ of a pair of simple objects $M,N\in\mathscr{C}_w$.

For *non-symmetric cases*, the monoidal categorification is still out of reach. We know that $\mathscr{C}_w$ categorifies $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ as an algebra [KL09, KL11, Rou08, Kim12] and $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ has a quantum cluster algebra structure [GLS13a, GY17] in *every symmetrizable case*. The quantum cluster algebra structure is skew-symmetric if the corresponding generalized Cartan matrix is symmetric. However, we cannot prove that $\mathscr{C}_w$ is a monoidal categorification of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ in *non-symmetric cases* due to the obstacle that we do not know whether every simple module $M\in\mathscr{C}_w$ admits an affinization [KP18] or not. Note that the existence of affinizations guarantees that one can define R-matrices and the $\mathbb{Z}$-invariant $\Lambda(M,N)$.

In this paper, we study the quantum Laurent positivity for $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ of not necessarily symmetric type in the view point of the monoidal categorification. More precisely, we show that the basis of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ corresponding to the simple modules in $\mathscr{C}_w$ exhibits a quantum Laurent positivity phenomenon with respect to any *quasi-Laurent family*, which is a central notion we introduce in this paper and plays the similar role of a cluster in the quantum cluster algebra theory.

The *quasi-Laurent family* (respectively, *Laurent family*) $\mathcal{M}=\{M_j\}_{j\in J}$ consists of mutually commuting affreal simple modules in $\mathscr{C}_w$ satisfying additional conditions (Definition 3.2). Among others, the most important condition is that if a simple module X commutes with all M_j, then there are monomials (i.e. convolution products) $\mathcal{M}(\mathbf{a})$ and $\mathcal{M}(\mathbf{b})$ in $\{M_j\}_{j\in J}$ such that $X\circ\mathcal{M}(\mathbf{a})$ is isomorphic to $\mathcal{M}(\mathbf{b})$. We say the family $\mathcal{M}$ is *Laurent* if $\mathcal{M}$ is *maximal* in the sense that, if a simple module X commutes with all M_j, then X is isomorphic to a monomial $\mathcal{M}(\mathbf{b})$ in $\{M_j\}_{j\in J}$.

The main results of this paper are the following.

(A) We show that if $\mathcal{M}$ is a quasi-Laurent family in $\mathscr{C}_w$, then the class $[X]$ in the Grothendieck ring $K(\mathscr{C}_w)$ of any simple object X in $\mathscr{C}_w$ can be written as a Laurent polynomial in $\{[M_j]\}_{j\in J}$ whose coefficients belong to $\mathbb{Z}_{\geq 0}[q^{\pm 1}]$ (Proposition 3.6).
(B) If $\mathcal{M}$ is a monoidal seed in $\mathscr{C}_w$, then $\mathcal{M}$ is a Laurent family.
(C) In particular, for any reduced sequence $\boldsymbol{i}$ of w, we show that the family $\mathcal{M}^{\boldsymbol{i}}:=\{M(w^{\boldsymbol{i}}_{\leqslant k}\varpi_{i_k},\varpi_{i_k})\}$ is a Laurent family and, hence, any class $[X]$ of a module X in $\mathscr{C}_w$ can be written as a Laurent polynomial in the unipotent quantum minors $D(w^{\boldsymbol{i}}_{\leqslant k}\varpi_{i_k},\varpi_{i_k})$ with coefficients in $\mathbb{Z}_{\geq 0}[q^{\pm 1}]$. Note that $D(w^{\boldsymbol{i}}_{\leqslant k}\varpi_{i_k},\varpi_{i_k})=[M(w^{\boldsymbol{i}}_{\leqslant k}\varpi_{i_k},\varpi_{i_k})]$ and we call $\{D(w^{\boldsymbol{i}}_{\leqslant k}\varpi_{i_k},\varpi_{i_k})\}$ the *GLS seed associated with* $\boldsymbol{i}$ (Proposition 4.5).
(D) We show that if $\mathcal{M}$ is a quasi-Laurent family, then the class $[X]$ of a simple module X in $\mathscr{C}_w$ is pointed and copointed with respect to the partial order $\preccurlyeq_{\mathcal{M}}$. That is, the set of

degrees of the monomials appearing in the Laurent expansion of $[X]$ with respect to $\mathcal{M}$ has a unique maximal element and a unique minimal element with respect to $\preccurlyeq_{\mathcal{M}}$. We define vectors $\mathbf{g}^R_{\mathcal{M}}(X)$ and $\mathbf{g}^L_{\mathcal{M}}(X) \in \mathbb{Z}^{\oplus J}$ as the maximal and the minimal element, respectively.

(E) Each quasi-Laurent family $\mathcal{M}$ also induces new $\mathbb{Z}$-values $\mathrm{G}^R_{\mathcal{M}}(X,Y)$ and $\mathrm{G}^L_{\mathcal{M}}(X,Y)$ for *any* pair of simple modules X and Y which coincides with $\Lambda(X,Y)$ provided X, Y commutes and one of them is affreal.

To the best of the authors' knowledge, the positivity result in part (C) is new. We can understand result (A) that a quasi-Laurent family is a generalization of a cluster in the categorical view point, and that the positivity conjecture can be extended to elements corresponding to simple modules in all skew-symmetrizable types.

In [FZ07, Qin17], Fomin-Zelevinsky and Qin defined a pointed (respectively, copointed) element $\mathbf{x}$ in a cluster algebra and its degree $\mathbf{deg}_{\mathcal{S}}(\mathbf{x}) \in \mathbb{Z}^{\oplus \mathsf{K}}$ (respectively, codegree $\mathbf{codeg}_{\mathcal{S}}(\mathbf{x}) \in \mathbb{Z}^{\oplus \mathsf{K}}$) depending on the choice of a seed $\mathcal{S}$ (see also [Qin20] for codegree and [Tra11] for degree elements in a quantum cluster algebra). With a fixed choice of a seed, it is proved in [Tra11] that every cluster monomial is pointed, and in [DWZ10, GHKK18] that cluster monomials are determined by their degrees.

For a given quasi-Laurent family $\mathcal{M}$ and a simple module $X \in \mathscr{C}_w$, we define vectors $\mathbf{g}^R_{\mathcal{M}}(X), \mathbf{g}^L_{\mathcal{M}}(X) \in \mathbb{Z}^{\oplus J}$ in Definition 3.7 by using the $\mathbb{Z}^{\oplus J}_{\geqslant 0}$-vectors in Lemma 3.3 and guaranteeing its well-definedness in Lemma 3.1. We then prove that, for every simple module $X \in \mathscr{C}_w$, the element $[X]$ in $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ is (co)pointed with respect to the GLS seed $\mathcal{S}^{\boldsymbol{i}}$ and that $\mathbf{g}^R_{\mathcal{M}^{\boldsymbol{i}}}(X)$ and $\mathbf{g}^L_{\mathcal{M}^{\boldsymbol{i}}}(X)$ coincide with $\mathbf{deg}_{\mathcal{S}^{\boldsymbol{i}}}([X])$ and $\mathbf{codeg}_{\mathcal{S}^{\boldsymbol{i}}}([X])$, respectively.

Utilizing the vectors $\mathbf{g}^R_{\mathcal{M}}(X)$ and $\mathbf{g}^L_{\mathcal{M}}(X)$, we define skew-symmetric $\mathbb{Z}$-valued forms $\mathrm{G}^R_{\mathcal{M}}(-,-)$ and $\mathrm{G}^L_{\mathcal{M}}(-,-)$ on the pairs (X,Y) of simple modules. Then we compare $\mathrm{G}^R_{\mathcal{M}}(X,Y)$ and $\mathrm{G}^L_{\mathcal{M}}(X,Y)$ with the $\mathbb{Z}$-invariant $\Lambda(X,Y)$ when the pair of simple module (X,Y) admits the $\mathbb{Z}$-invariant $\Lambda(X,Y)$. It is proved in Proposition 5.3 that $\mathrm{G}^R_{\mathcal{M}}(X,Y)$ and $\mathrm{G}^L_{\mathcal{M}}(X,Y)$ give lower bounds of $\Lambda(X,Y)$, and in Proposition 5.4 that $\mathrm{G}^R_{\mathcal{M}}(X,Y) = \mathrm{G}^L_{\mathcal{M}}(X,Y) = \Lambda(X,Y)$ when (X,Y) is a commuting pair. Here we would like to emphasize that (1) $\mathrm{G}^R_{\mathcal{M}}(X,Y)$ and $\mathrm{G}^L_{\mathcal{M}}(X,Y)$ are defined even for pairs (X,Y) we do not know whether they admit $\Lambda(X,Y)$ or not, and (2) the $\mathbb{Z}$-values $\mathrm{G}^R_{\mathcal{M}}(X,Y)$ and $\mathrm{G}^L_{\mathcal{M}}(X,Y)$ *do depend* on the choice of $\mathcal{M}$ as (co)degree does on the one of seeds (Remark 5.5).

This paper is organized as follows. In § 2, we give preliminaries. In § 3, we define the notions of quasi-Laurent and Laurent families, and investigate their properties. Then we define $\mathbf{g}^R_{\mathcal{M}}(X)$ and $\mathbf{g}^L_{\mathcal{M}}(X)$, and prove that $\mathbf{g}^R_{\mathcal{M}}(X)$ and $\mathbf{g}^L_{\mathcal{M}}(X)$ determine the isomorphism class of X. In § 4, we prove that $\mathcal{M}^{\boldsymbol{i}}$ is Laurent by studying PBW decomposition vectors of simple modules. In § 5, we define the skew-symmetric pairings on pairs of simple modules and investigate the relationships among the pairings and Λ-invariants.

Convention. Throughout this paper, we use the following convention.

(i) For a statement $\mathtt{P}$, we set $\delta(\mathtt{P})$ to be 1 or 0 depending on whether $\mathtt{P}$ is true or not. As a special case, we use the notation $\delta_{i,j} := \delta(i = j)$ (Kronecker's delta).

(ii) For integers $a, b \in \mathbb{Z}$, we set

$$[a,b] := \{x \in \mathbb{Z} \mid a \leqslant x \leqslant b\}.$$

We refer to the subset as an *interval* and understand it as an empty set if $a > b$.

(iii) Let $\mathbf{x} = (x_j)_{j \in J}$ be a family parameterized by an index set J. Then for any $j \in J$, we set

$$(\mathbf{x})_j := x_j.$$

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2. Preliminaries

In this preliminary section, we briefly review the basic material of this paper. We refer the reader to [BZ05, FZ07, KL09, Rou08, Kim12, GLS13a, KKKO18, GY17, KiOy21, KP18, KKOP18, KK19, GHKK18] for more details.

2.1 Quantum cluster algebras

Fix a finite index set $\mathsf{K} = \mathsf{K}_{\mathrm{ex}} \sqcup \mathsf{K}_{\mathrm{fr}}$ with a decomposition into the set K_{ex} of exchangeable indices and the set K_{fr} of frozen indices. Let $L = (l_{ij})_{i,j\in\mathsf{K}}$ be a skew-symmetric integer-valued matrix and let q be an indeterminate. We set $\mathbb{A} := \mathbb{Z}[q^{\pm 1/2}]$ where $q^{1/2}$ denotes the formal square root of q.

Definition 2.1. We define the *quantum torus* $\mathcal{T}(L)$ to be the $\mathbb{A}$-algebra generated by a finite family of elements $\{X_k^{\pm 1}\}_{k\in\mathsf{K}}$ subject to the following defining relations:

$$X_j X_j^{-1} = X_j^{-1} X_j = 1 \quad \text{and} \quad X_i X_j = q^{l_{ij}} X_j X_i \quad \text{for } i,j \in \mathsf{K}.$$

For $\mathbf{a} = (\mathbf{a}_i)_{i\in\mathsf{K}} \in \mathbb{Z}^{\mathsf{K}}$, we define the element $X^{\mathbf{a}}$ of $\mathcal{T}(L)$ as

$$X^{\mathbf{a}} = q^{(1/2)\sum_{i>j}\mathbf{a}_i\mathbf{a}_j l_{ij}} \overrightarrow{\prod_{i\in\mathsf{K}}} X_i^{\mathbf{a}_i}.$$

Here $\overrightarrow{\prod}_{i\in\mathsf{K}} X_i^{\mathbf{a}_i} := X_{i_1}^{\mathbf{a}_{i_1}} \cdots X_{i_r}^{\mathbf{a}_{i_r}}$, where $\mathsf{K} = \{i_1,\ldots,i_r\}$ with a total order $i_1 < \cdots < i_r$. Note that $X^{\mathbf{a}}$ does not depend on the choice of a total order $<$ on K. Then $\{X^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}^{\mathsf{K}}\}$ forms an $\mathbb{A}$-basis of $\mathcal{T}(L)$. Since $\mathcal{T}(L)$ is an Ore domain, it is embedded into the skew field of fractions $\mathbb{F}(\mathcal{T}(L))$.

Let $\widetilde{B} = (b_{ij})_{i\in\mathsf{K},j\in\mathsf{K}_{\mathrm{ex}}}$ be an integer-valued $\mathsf{K}\times\mathsf{K}_{\mathrm{ex}}$-matrix whose principal part $B = (b_{ij})_{i,j\in\mathsf{K}_{\mathrm{ex}}}$ is skew-symmetrizable, i.e. there exists a diagonal matrix D with a positive integer entries such that DB is skew-symmetric. Such a matrix $\widetilde{B}$ is called an *exchange matrix*. We say that a pair (L, B) is *compatible* if

$$\sum_{k\in\mathsf{K}} b_{ki} l_{kj} = d_i \delta_{i,j} \quad \text{for any } i \in \mathsf{K}_{\mathrm{ex}} \text{ and } j \in \mathsf{K}$$

for some positive integers $\{d_i\}_{i\in\mathsf{K}_{\mathrm{ex}}}$. We call the triple $\mathcal{S} = (\{X_k\}_{k\in\mathsf{K}}, L, \widetilde{B})$ a *quantum seed* in the quantum torus $\mathcal{T}(L)$ and $\{X_k\}_{k\in\mathsf{K}}$ a *quantum cluster*.

For $k \in \mathsf{K}_{\mathrm{ex}}$, the *mutation* $\mu_k(L,\widetilde{B}) := (\mu_k(L), \mu_k(B))$ of a compatible pair $(L,\widetilde{B})$ *in a direction* k is defined in a combinatorial way (see [BZ05]). Note that (i) the pair $(\mu_k(L), \mu_k(B))$ is also compatible with the same positive integers $\{d_i\}_{i\in\mathsf{K}}$ and (ii) the operation μ_k is an involution, i.e. $\mu_k(\mu_k(L,\widetilde{B})) = (L,\widetilde{B})$. We define an isomorphism of $\mathbb{Q}(q^{1/2})$-algebras $\mu_k^*\colon \mathbb{F}(\mathcal{T}(\mu_k L)) \xrightarrow{\sim} \mathbb{F}(\mathcal{T}(L))$ by

$$\mu_k^*(X_j) := \begin{cases} X^{\mathbf{a}'} + X^{\mathbf{a}''} & \text{if } j = k, \\ X_j & \text{if } j \neq k, \end{cases}$$

where

$$\mathbf{a}_i' = \begin{cases} -1 & \text{if } i = k, \\ \max(0, b_{ik}) & \text{if } i \neq k, \end{cases} \quad \text{and} \quad \mathbf{a}_i'' = \begin{cases} -1 & \text{if } i = k, \\ \max(0, -b_{ik}) & \text{if } i \neq k. \end{cases}$$

Then the *mutation* $\mu_k(\mathcal{S})$ *of the quantum seed* $\mathcal{S}$ *in a direction* k is defined to be the triple $(\{X_i\}_{i\neq k} \sqcup \{\mu_k^*(X_k)\}, \mu_k(L), \mu_k(\widetilde{B}))$.

For a quantum seed $\mathcal{S} = (\{X_k\}_{k\in\mathsf{K}}, L, \widetilde{B})$, an element in $\mathbb{F}(\mathcal{T}(L))$ is called a *quantum cluster variable* (respectively, *quantum cluster monomial*) if it is of the form

$$\mu^*_{k_1}\cdots\mu^*_{k_\ell}(X_j) \quad \text{(respectively, } \mu^*_{k_1}\cdots\mu^*_{k_\ell}(X^{\mathbf{a}}))$$

for some finite sequence $(k_1,\dots,k_\ell) \in \mathsf{K}_{\rm ex}^{\ell}$ ($\ell \in \mathbb{Z}_{\geqslant 0}$) and $j \in \mathsf{K}$ (respectively, $\mathbf{a} \in \mathbb{Z}^{\mathsf{K}}_{\geqslant 0}$). For a quantum seed $\mathcal{S} = (\{X_k\}_{k\in\mathsf{K}}, L, \widetilde{B})$, the *quantum cluster algebra* $\mathscr{A}_q(\mathcal{S})$ is the $\mathbb{A}$-subalgebra of $\mathbb{F}(\mathcal{T}(L))$ generated by all the quantum cluster variables. Note that $\mathscr{A}_q(\mathcal{S}) \simeq \mathscr{A}_q(\boldsymbol{\mu}(\mathcal{S}))$ for any sequence $\boldsymbol{\mu}$ of mutations.

The *quantum Laurent phenomenon*, proved by Berenstein and Zelevinsky in [BZ05], says that the quantum cluster algebra $\mathscr{A}_q(\mathcal{S})$ is indeed contained in $\mathcal{T}(L)$.

For a quantum seed $\mathcal{S}$ with a compatible pair $(L, \widetilde{B})$, an element $\mathbf{x} \in \mathcal{T}(L)$ is called *pointed* (respectively, *copointed*) if it is of the following form:

$$\mathbf{x} = q^a X^{\mathbf{g}^R} + \sum_{\mathbf{c}\in\mathbb{Z}^{\mathsf{K}_{\rm ex}}_{\geqslant 0}\setminus\{0\}} p_{\mathbf{c}} X^{\mathbf{g}^R + \widetilde{B}\mathbf{c}} \quad \left(\text{respectively, } \mathbf{x} = q^a X^{\mathbf{g}^L} + \sum_{\mathbf{c}\in\mathbb{Z}^{\mathsf{K}_{\rm ex}}_{\leqslant 0}\setminus\{0\}} p_{\mathbf{c}} X^{\mathbf{g}^L + \widetilde{B}\mathbf{c}}\right) \tag{2.1}$$

for some $a \in \frac{1}{2}\mathbb{Z}$, $\mathbf{g}^R \in \mathbb{Z}^{\mathsf{K}}$ (respectively, $\mathbf{g}^L \in \mathbb{Z}^{\mathsf{K}}$) and $p_{\mathbf{c}} \in \mathbb{A}$. In this case, we call $\mathbf{g}^R$ the *degree* (respectively, *codegree*) of the pointed (respectively, copointed) element $\mathbf{x}$ and denote it by $\mathbf{deg}_{\mathcal{S}}(\mathbf{x})$ (respectively, $\mathbf{codeg}_{\mathcal{S}}(\mathbf{x})$). The degree (respectively, codegree) is often the called *g-vector* (respectively, *dual g-vector*) of $\mathbf{x}$ (see [Qin17, Definition 3.1.4] and [Qin20, Definition 3.1.3]). It is worth remarking that the notion of g-vector (respectively, dual g-vector) *does depend on* the compatible pair $(L, \widetilde{B})$ and, hence, on the seed $\mathcal{S}$. It is proved in [Tra11, Theorem 5.3] that every quantum cluster monomial in $\mathscr{A}_q(\mathcal{S})$ is pointed.

We say that an $\mathbb{A}$-algebra R has a *quantum cluster algebra structure* if there exists a quantum seed $\mathcal{S}$ and an $\mathbb{A}$-algebra isomorphism $\Omega : \mathscr{A}_q(\mathcal{S}) \xrightarrow{\sim} R$. In the case, a *quantum seed of* R refers to the image of a quantum seed in $\mathscr{A}_q(\mathcal{S})$, which is obtained by a sequence of mutations.

2.2 Quantum unipotent coordinate rings

Let I be an index set. A *Cartan datum* $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ consists of:

(a) a symmetrizable Cartan matrix $\mathsf{A} = (a_{i,j})_{i,j\in I}$, i.e. DA is symmetric for a diagonal matrix $\mathsf{D} = \mathrm{diag}(\mathsf{d}_i \mid i \in I)$ with $\mathsf{d}_i \in \mathbb{Z}_{>0}$;
(b) a free abelian group P, called the *weight lattice*;
(c) $\Pi = \{\alpha_i \mid i \in I\} \subset \mathsf{P}$, called the set of *simple roots*;
(d) $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathsf{P}^\vee := \mathrm{Hom}(\mathsf{P}, \mathbb{Z})$, called the set of *simple coroots*;
(e) a $\mathbb{Q}$-valued symmetric bilinear form $(\cdot,\cdot)$ on P;

satisfying the standard properties (see [KKKO18, § 1.1] for instance). Here we take $\mathsf{D} = \mathrm{diag}(\mathsf{d}_i \mid i \in I)$ such that $\mathsf{d}_i := (\alpha_i, \alpha_i)/2 \in \mathbb{Z}_{>0}$ ($i \in I$) in this paper.

For $i \in I$, we choose $\varpi_i \in \mathsf{P}$ such that $\langle h_i, \varpi_j\rangle = \delta_{ij}$ for any $j \in I$ and call it the *ith fundamental weight*. The free abelian group $\mathsf{Q} := \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ is called the *root lattice* and we set $\mathsf{Q}^+ = \sum_{i\in I}\mathbb{Z}_{\geqslant 0}\alpha_i \subset \mathsf{Q}$ and $\mathsf{Q}^- = \sum_{i\in I}\mathbb{Z}_{\leqslant 0}\alpha_i \subset \mathsf{Q}$. We denote by Δ the set of *roots* and by $\Delta^\pm$ the set of *positive* roots (respectively, *negative* roots). For $\beta \in \sum_{i\in I} m_i\alpha_i \in \mathsf{Q}^+$, we set $|\beta| := \sum_{i\in I} m_i$, $\mathrm{supp}(\beta) := \{i \in I \mid m_i \neq 0\}$ and $I^\beta := \{\nu = (\nu_1,\dots,\nu_{|\beta|}) \in I^{|\beta|} \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_{|\beta|}} = \beta\}$. Note that the symmetric group $\mathfrak{S}_{|\beta|} = \langle r_1,\dots,r_{|\beta|}\rangle$ acts on I^β by the place permutations.

Let $\mathfrak{g}$ be the Kac–Moody algebra associated with the Cartan datum $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$, and W the *Weyl group* of $\mathfrak{g}$. It is generated by the simple reflections $s_i \in \mathrm{Aut}(\mathsf{P})$ ($i \in I$) defined by

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$s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle \alpha_i$ for $\lambda \in \mathsf{P}$. For a sequence $\boldsymbol{i} = (i_1, \dots, i_r) \in I^r$, we call it a *reduced sequence* of $w \in \mathsf{W}$ if $s_{i_1} \dots s_{i_r}$ is a reduced expression of w. For $w, v \in \mathsf{W}$, we write $w \geqslant v$ if there is a reduced sequence of v which appears in a reduced sequence of w as a subsequence.

For $\lambda, \mu \in \mathsf{P}$, we write $\lambda \preccurlyeq \mu$ if there exists a sequence of real positive roots β_k $(1 \leqslant k \leqslant l)$ such that $\lambda = s_{\beta_l} \cdots s_{\beta_1}\mu$ and $(\beta_k, s_{\beta_{k-1}} \cdots s_{\beta_1}\mu) > 0$ for $1 \leqslant k \leqslant l$. When $\Lambda \in \mathsf{P}^+$ and $\lambda, \mu \in \mathsf{W}\Lambda$ the relation $\lambda \preccurlyeq \mu$ holds if and only if there exist $w, v \in \mathsf{W}$ such that $\lambda = w\Lambda$, $\mu = v\Lambda$ and $v \leqslant w$.

Let $\mathcal{U}_q(\mathfrak{g})$ be the quantum group of $\mathfrak{g}$ over $\mathbb{Q}(q^{1/2})$, generated by e_i, f_i $(i \in I)$ and q^h $(h \in \mathsf{P}^\vee)$. We denote by $\mathcal{U}_q^+(\mathfrak{g})$ the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by e_i and $\mathcal{U}_{\mathbb{A}}^+(\mathfrak{g})$ the $\mathbb{A}$-subalgebra of $\mathcal{U}_q(\mathfrak{g})^+$ generated by $e_i^n/[n]_i!$ $(i \in I,\ n \in \mathbb{Z}_{>0})$, where

$$q_i := q^{\mathsf{d}_i}, \quad [k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}} \quad \text{and} \quad [k]_i! = \prod_{s=1}^{k} [s]_i.$$

Set

$$\mathcal{A}_q(\mathfrak{n}) = \bigoplus_{\beta \in \mathsf{Q}^-} \mathcal{A}_q(\mathfrak{n})_\beta \quad \text{where } \mathcal{A}_q(\mathfrak{n})_\beta := \mathrm{Hom}_{\mathbb{Q}(q^{1/2})}(\mathcal{U}_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q^{1/2})),$$

where $\mathcal{U}_q^+(\mathfrak{g})_{-\beta}$ denotes the $(-\beta)$-root space of $\mathcal{U}_q^+(\mathfrak{g})$. Then $\mathcal{A}_q(\mathfrak{n})$ also has an algebra structure and is called the *quantum unipotent coordinate ring* of $\mathfrak{g}$. We denote by $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ the $\mathbb{A}$-submodule of $\mathcal{A}_q(\mathfrak{n})$ generated by $\psi \in \mathcal{A}_q(\mathfrak{n})$ such that $\psi(\mathcal{U}_{\mathbb{A}}^+(\mathfrak{g})) \subset \mathbb{A}$. Then, $\mathcal{A}_q(\mathfrak{n})$ is an $\mathbb{A}$-subalgebra with a $\mathcal{U}_{\mathbb{A}}^+(\mathfrak{g})$-bimodule structure.

For each $\lambda \in \mathsf{P}^+ := \sum_{i \in I} \mathbb{Z}_{\geqslant 0}\varpi_i$ and Weyl group elements $w, w' \in \mathsf{W}$, we can define a specific homogeneous element $D(w\lambda, w'\lambda)$ of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$, called a *unipotent quantum minor* (see, for example, [KKKO18, § 9]).

For $w \in \mathsf{W}$, we denote by $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ the $\mathbb{A}$-submodule of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ consisting of elements ψ such that $e_{i_1} \cdots e_{i_{|\beta|}}\psi = 0$ for any $\beta \in \mathsf{Q}^+ \setminus w\mathsf{Q}^-$ and $(\nu_{i_1}, \dots, \nu_{i_{|\beta|}}) \in I^\beta$. Then it is an $\mathbb{A}$-subalgebra and we call it the *quantum unipotent coordinate ring associated with* w.

For a reduced sequence $\boldsymbol{i} = (i_1, \dots, i_r)$ of $w \in \mathsf{W}$ and $1 \leqslant k \leqslant r$, define $w_{\leqslant k}^{\boldsymbol{i}} = s_{i_1} \cdots s_{i_k}$ and $w_{<k}^{\boldsymbol{i}} = s_{i_1} \cdots s_{i_{k-1}}$. Then $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ is generated by the set of unipotent quantum minors $\{D(w_{\leqslant k}^{\boldsymbol{i}}\varpi_{i_k}, w_{<k}^{\boldsymbol{i}}\varpi_{i_k}) \mid 1 \leqslant k \leqslant r\}$ as an $\mathbb{A}$-algebra.

It is proved in [GLS13a, GY17, KKKO18, Qin20] that $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ has a quantum cluster algebra structure, one of whose quantum seeds $\mathcal{S}^{\boldsymbol{i}}$ can be obtained from a reduced sequence $\boldsymbol{i} = (i_1, \dots, i_r)$ of w. To introduce $\mathcal{S}^{\boldsymbol{i}}$, we need preparations.

Let $\boldsymbol{j} = (j_1, \dots, j_l)$ be a sequence of indices in I. For $1 \leqslant k \leqslant l$ and $j \in I$, we set

$$k_+^{\boldsymbol{j}} := \min(\{u \mid k < u \leqslant l,\ j_u = j_k\} \cup \{l+1\}),$$
$$k_-^{\boldsymbol{j}} := \max(\{u \mid 1 \leqslant u < k,\ j_u = j_k\} \cup \{0\}).$$

We also set

$$k_{\min}^{\boldsymbol{j}} := \min\{u \mid 1 \leqslant u \leqslant k,\ j_u = j_k\} \quad \text{and} \quad k_{\max}^{\boldsymbol{j}} := \max\{u \mid k \leqslant u \leqslant l,\ j_u = j_k\}.$$

We sometimes drop $^{\boldsymbol{j}}$ in the above notation if there is no danger of confusion.

Take $\mathsf{K} = [1, r]$ as an index set and decompose K into

$$\mathsf{K}_{\mathrm{fr}} = \{k \mid 1 \leqslant k \leqslant r,\ k_+^{\boldsymbol{i}} = r+1\} \quad \text{and} \quad \mathsf{K}_{\mathrm{ex}} := \mathsf{K} \setminus \mathsf{K}_{\mathrm{fr}}.$$

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We define the $\mathbb{Z}$-valued $\mathsf{K}\times\mathsf{K}_{\mathrm{ex}}$ matrix $\widetilde{B}^{\boldsymbol{i}} = (b^{\boldsymbol{i}}_{st})_{s\in\mathsf{K},t\in\mathsf{K}_{\mathrm{ex}}}$ and the $\mathbb{Z}$-valued skew-symmetric $\mathsf{K}\times\mathsf{K}$ matrix $L^{\boldsymbol{i}} = (l^{\boldsymbol{i}}_{st})_{s,t\in\mathsf{K}}$ as follows:

$$b^{\boldsymbol{i}}_{st} = \begin{cases} \pm 1 & \text{if } s = t^{\boldsymbol{i}}_{\pm}, \\ -a_{i_s,i_t} & \text{if } s < t < s^{\boldsymbol{i}}_{+} < t^{\boldsymbol{i}}_{+}, \\ a_{i_s,i_t} & \text{if } t < s < t^{\boldsymbol{i}}_{+} < s^{\boldsymbol{i}}_{+}, \\ 0 & \text{otherwise,} \end{cases}$$

$$l^{\boldsymbol{i}}_{st} = (\varpi_{i_s} - w^{\boldsymbol{i}}_{\leqslant s}\varpi_{i_s}, \varpi_{i_t} + w^{\boldsymbol{i}}_{\leqslant t}\varpi_{i_t}) \quad \text{for } s < t.$$

Then the quantum seed $\mathcal{S}^{\boldsymbol{i}}$ of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ is given as follows:

$$\mathcal{S}^{\boldsymbol{i}} := \Big(\{q^{c^{\boldsymbol{i}}_k} D(w^{\boldsymbol{i}}_{\leqslant k}\varpi_{i_k}, \varpi_{i_k})\}_{k\in\mathsf{K}}, L^{\boldsymbol{i}}, \widetilde{B}^{\boldsymbol{i}} \Big), \tag{2.2}$$

where $c^{\boldsymbol{i}}_s = \frac{1}{4}(\varpi_{i_s} - w^{\boldsymbol{i}}_{\leqslant s}\varpi_{i_s}, \varpi_{i_s} - w^{\boldsymbol{i}}_{\leqslant s}\varpi_{i_s}) \in \mathbb{Z}/2$. Note that $(L^{\boldsymbol{i}}\widetilde{B}^{\boldsymbol{i}})_{ab} = -2\mathsf{d}_{i_a} \times \delta_{a,b}$ for $(a,b) \in \mathsf{K}\times\mathsf{K}_{\mathrm{ex}}$, $\mathrm{wt}(D(w^{\boldsymbol{i}}_{\leqslant k}\varpi_{i_k}, \varpi_{i_k})) = -\varpi_{i_s} + w^{\boldsymbol{i}}_{\leqslant s}\varpi_{i_s}$, and

$$\big\{q^{c^{\boldsymbol{i}}_k} D(w^{\boldsymbol{i}}_{\leqslant k}\varpi_{i_k}, \varpi_{i_k}) = q^{c^{\boldsymbol{i}}_k} D(w\varpi_{i_k}, \varpi_{i_k}) \mid k \in \mathsf{K}_{\mathrm{fr}}\big\}$$

forms the set of frozen variables of the quantum cluster algebra $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$. We call $\mathcal{S}^{\boldsymbol{i}}$ the *GLS seed* (associated with $\boldsymbol{i}$).

We set $\mathcal{D}(w) := \{q^m D(w\varpi, \varpi) \mid m \in \mathbb{Z}/2,\ \varpi \in \mathsf{P}^+\}$. Then it is well-known that $\mathcal{D}(w)$ consists of q-central elements of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ and, hence, forms an Ore set. We denote by $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}^w)$ the quotient ring of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$ by the Ore set $\mathcal{D}(w)$. Then $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}^w)$ has also the quantum cluster algebra structure with the *invertible* frozen variables $\{q^{c^{\boldsymbol{i}}_k} D(w^{\boldsymbol{i}}_{\leqslant k}\varpi_{i_k}, \varpi_{i_k})\}_{k\in\mathsf{K}_{\mathrm{fr}}}$ in the sense of [BZ05].

2.3 Quiver Hecke algebras and categorifications

Let $\mathbf{k}$ be a base field. For $i,j\in I$, we choose polynomials $\mathcal{Q}_{i,j}(u,v) \in \mathbf{k}[u,v]$ such that (a) $\mathcal{Q}_{i,j}(u,v) = \mathcal{Q}_{j,i}(v,u)$ and (b) each $\mathcal{Q}_{i,j}(u,v)$ is of the following form:

$$\mathcal{Q}_{i,j}(u,v) = \delta(i\neq j) \sum_{p(\alpha_i,\alpha_i)+q(\alpha_j,\alpha_j) = -2(\alpha_i,\alpha_j)} t_{i,j;p,q} u^p v^q \quad \text{where } t_{i,j;-a_{i,j},0} \in \mathbf{k}^{\times}.$$

For a Cartan datum $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ and $\beta \in \mathsf{Q}^+$, the *quiver Hecke algebra $R(\beta)$ associated with* $(\mathcal{Q}_{i,j})_{i,j\in I}$ is the $\mathbb{Z}$-graded algebra over $\mathbf{k}$ generated by the elements

$$\{e(\nu)\}_{\nu\in I^\beta}, \quad \{x_k\}_{1\leqslant k\leqslant|\beta|}, \quad \{\tau_m\}_{1\leqslant m<|\beta|}$$

subject to certain defining relations (see [KKOP21, Definition 1.1] for instance). Note that the $\mathbb{Z}$-grading of $R(\beta)$ is determined by the degrees of following elements:

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \text{and} \quad \deg(\tau_m e(\nu)) = -(\alpha_{\nu_m}, \alpha_{\nu_{m+1}}).$$

We say that $R(\beta)$ is *symmetric* if $\mathcal{Q}_{i,j}(u,v) \in \mathbf{k}[u-v]$ for $i,j\in \mathrm{supp}(\beta)$.

We denote by $R(\beta)$-gmod the category of finite-dimensional graded $R(\beta)$-modules with homomorphisms of degree 0. For $M \in R(\beta)$-gmod, we set $\mathrm{wt}(M) := -\beta \in \mathsf{Q}^-$. Note that there exists the *degree shift functor*, denoted by q, such that $(qM)_n = M_{n-1}$ for $M = \bigoplus_{k\in\mathbb{Z}} M_k \in R(\beta)$-gmod.

Throughout this paper, we usually deal with graded $R(\beta)$-modules ($\beta \in \mathsf{Q}^+$) and sometimes skip grading shifts. Thus, we usually say that M is an R-module instead of saying that M is a graded $R(\beta)$-module and $f\colon M \to N$ is a homomorphism if $f\colon q^a M \to N$ is a morphism in

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$R(\beta)$-gmod. We set

$$\mathrm{HOM}_{R(\beta)}(M,N) := \bigoplus_{a\in\mathbb{Z}} \mathrm{HOM}_{R(\beta)}(M,N)_a$$

with $\mathrm{HOM}_{R(\beta)}(M,N)_a := \mathrm{Hom}_{R(\beta)\text{-gmod}}(q^a M, N)_a$. We write $\deg(f) := a$ for an $f \in \mathrm{HOM}_{R(\beta)}(M,N)_a$.

For an $R(\beta)$-module M and an $R(\gamma)$-module N, we can obtain $R(\beta+\gamma)$-module

$$M \circ N := R(\beta+\gamma)e(\beta,\gamma) \underset{R(\beta)\otimes R(\gamma)}{\otimes} (M \otimes N),$$

where $e(\beta,\gamma) := \sum_{\nu\in I^\beta, \nu'\in I^\gamma} e(\nu * \nu') \in R(\beta+\gamma)$. Here $\nu * \nu'$ denotes the concatenation of ν and ν', and $\circ$ is called the *convolution product*. We say that two simple R-modules M and N *strongly commute* if $M \circ N$ is simple. If a simple module M strongly commutes with itself, then M is called *real*. A simple R-module M is said to be *prime* if there are no non-trivial simple R-modules N_1 and N_2 such that $M \simeq N_1 \circ N_2$.

For an $R(\beta)$-module M, the dual space $M^* := \mathrm{Hom}_{\mathbf{k}}(M, \mathbf{k})$ admits an $R(\beta)$-module structure via

$$(r \cdot f)(u) = f(\psi(r)u) \quad (r \in R(\beta),\ u \in M,\ f \in M^*).$$

Here ψ denotes the $\mathbf{k}$-algebra anti-involution $R(\beta)$ which fixes the generators of $R(\beta)$. A simple $R(\beta)$-module M is called *self-dual* if $M^* \simeq M$.

Set $R\text{-gmod} := \bigoplus_{\beta\in\mathsf{Q}^+} R(\beta)\text{-gmod}$. Then the category R-gmod is a monoidal category with the tensor product $\circ$ and the unit object $\mathbf{1} := \mathbf{k} \in R(0)$-gmod. Hence, the Grothendieck group $K(R\text{-gmod})$ has the $\mathbb{Z}[q^{\pm1}]$-algebra structure derived from $\circ$ and the degree shift functors $q^{\pm1}$.

For a monoidal abelian subcategory $\mathcal{C}$ of R-gmod stable by grading shifts, we set

$$\mathcal{K}_{\mathbb{A}}(\mathcal{C}) := \mathbb{A} \underset{\mathbb{Z}[q^{\pm1}]}{\otimes} K(\mathcal{C}),$$

where $K(\mathcal{C})$ denotes the Grothendieck ring of $\mathcal{C}$. For a subcategory $\mathcal{C}$ of R-gmod, we denote by $\mathrm{Irr}(\mathcal{C})$ the set of the isomorphism classes of self-dual modules in $\mathcal{C}$. Note that $\mathrm{Irr}(R\text{-gmod})$ forms an $\mathbb{A}$-basis of $\mathcal{K}_{\mathbb{A}}(R\text{-gmod})$.

It is proved in [KL09, KL11, Rou08] that there exists an $\mathbb{A}$-algebra isomorphism

$$\Omega\colon \mathcal{K}_{\mathbb{A}}(R\text{-gmod}) \xrightarrow{\sim} \mathcal{A}_{\mathbb{A}}(\mathfrak{n}). \tag{2.3}$$

PROPOSITION 2.2 [KKOP18, Proposition 4.1]. *For $\varpi \in \mathsf{P}^+$ and $\mu, \zeta \in \mathsf{W}\varpi$ with $\mu \preccurlyeq \zeta$, there exists a self-dual real simple $R(\zeta-\mu)$-module $M(\mu,\zeta)$ such that*

$$\Omega([M(\mu,\zeta)]) = D(\mu,\zeta).$$

Here, $[M(\mu,\zeta)]$ denotes the isomorphism class of $M(\mu,\zeta)$ which is called the determinantal module *associated with $D(\mu,\zeta)$.*

For an $R(\beta)$-module M, we define

$$W(M) := \{\gamma \in \mathsf{Q}^+ \cap (\beta - \mathsf{Q}^+) \mid e(\gamma, \beta-\gamma)M \neq 0\},$$
$$W^*(M) := \{\gamma \in \mathsf{Q}^+ \cap (\beta - \mathsf{Q}^+) \mid e(\beta-\gamma, \gamma)M \neq 0\}.$$

An ordered pair (M,N) of R-modules is called *unmixed* [TW16, Definition 2.5] if

$$W^*(M) \cap W(N) \subset \{0\}.$$

For $w \in \mathsf{W}$, we denote by $\mathscr{C}_w$ the full subcategory of R-gmod whose objects M satisfy $W(M) \subset \mathsf{Q}^+ \cap w\mathsf{Q}^-$. Then the category $\mathscr{C}_w$ is the smallest monoidal abelian category of

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R-gmod which (i) is stable under taking subquotients, extensions, grading shifts and (ii) contains $\{S_k^{\boldsymbol{i}} := M(w_{\leqslant k}^{\boldsymbol{i}}\varpi_{i_k}, w_{<k}^{\boldsymbol{i}}\varpi_{i_k}) \mid 1 \leqslant k \leqslant r\}$ for any reduced sequence $\boldsymbol{i}$ of w. We call $S_k^{\boldsymbol{i}}$ the kth *cuspidal module associated with* $\boldsymbol{i}$. Defining $\beta_k^{\boldsymbol{i}} := w_{<k}^{\boldsymbol{i}}\alpha_{i_k}$ for $1 \leqslant k \leqslant r$, one can see that $\{\beta_k^{\boldsymbol{i}} \mid 1 \leqslant k \leqslant r\} = \Delta^+ \cap w\Delta^-$, and $-\mathrm{wt}(S_k^{\boldsymbol{i}}) = \beta_k^{\boldsymbol{i}}$. Then we have [Kim12, § 4]

$$\Omega(\mathcal{K}_{\mathbb{A}}(\mathscr{C}_w)) \simeq \mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w)).$$

2.4 *R*-matrices and affreal simple modules

For $\beta \in \mathsf{Q}^+$ and $i \in I$, let

$$\mathfrak{p}_{i,\beta} = \sum_{\eta \in I^\beta} \Bigg(\prod_{a \in [1,|\beta|];\ \eta_a = i} x_a \Bigg) e(\eta) \in \mathcal{Z}(R(\beta)),$$

where $\mathcal{Z}(R(\beta))$ denotes the center of $R(\beta)$.

Definition 2.3 [KP18, Definition 2.2]. For an $R(\beta)$-module M, we say that M *admits an affinization* if there exists an $R(\beta)$-module $\widehat{M}$ satisfying the condition: there exists an endomorphism $z_{\widehat{M}}$ of degree $t \in \mathbb{Z}_{>0}$ such that $\widehat{M}/z_{\widehat{M}}\widehat{M} \simeq M$ and:

(i) $\widehat{M}$ is a finitely generated free module over the polynomial ring $\mathbf{k}[z_{\widehat{M}}]$;
(ii) $\mathfrak{p}_{i,\beta}\widehat{M} \neq 0$ for all $i \in I$.

We say that a simple $R(\beta)$-module M is *affreal* if M is real and admits an affinization.

It is known that any $M \in R(\beta)$-gmod admits an affinization if $R(\beta)$ is symmetric. However, when $R(\beta)$ is not symmetric, it is widely open whether an $R(\beta)$-module M admits an affinization or not.

Theorem 2.4 [KKOP21, Theorem 3.26]. *For $\varpi \in \mathsf{P}^+$ and $\mu, \zeta \in \mathsf{W}\varpi$ such that $\mu \preccurlyeq \zeta$, the determinantal module $M(\mu, \zeta)$ is affreal.*

Proposition 2.5 [KKKO18, KKOP21]. *Let M and N be simple modules such that one of them is affreal. Then there exists a unique R-module homomorphism $\mathbf{r}_{M,N} \in \mathrm{Hom}_R(M, N)$ satisfying*

$$\mathrm{Hom}_R(M \circ N, N \circ M) = \mathbf{k}\,\mathbf{r}_{M,N}.$$

We call the homomorphism $\mathbf{r}_{M,N}$ the *R-matrix*.

Definition 2.6. For simple R-modules M and N such that one of them is affreal, we define

$$\Lambda(M, N) := \deg(\mathbf{r}_{M,N}),$$
$$\widetilde{\Lambda}(M, N) := \tfrac{1}{2}\big(\Lambda(M, N) + \big(\mathrm{wt}(M), \mathrm{wt}(N)\big)\big),$$
$$\mathfrak{d}(M, N) := \tfrac{1}{2}\big(\Lambda(M, N) + \Lambda(N, M)\big).$$

It is proved in [KKKO18, KKOP21] that the invariants $\widetilde{\Lambda}(M, N)$ and $\mathfrak{d}(M, N)$ in Definition 2.6 belong to $\mathbb{Z}_{\geqslant 0}$.

For simple modules M and N, $M \nabla N$ and $M \Delta N$ denote the head and the socle of $M \circ N$, respectively.

Proposition 2.7 [KKKO15, Lemma 3.1.4] and [KKKO18, Corollary 4.1.2]. *Let M and N be simple R-modules such that one of them is affreal.*

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(i) *The image of* $\mathbf{r}_{M,N}$ *is equal to* $M \nabla N$ *and* $N \Delta M$.
(ii) *The head* $M \nabla N$ *and socle* $M \Delta N$ *are simple modules and each of them appears exactly once in the composition series of* $M \circ N$ *(up to a grading shift).*
(iii) *Assume that* N *is affreal.*
   (a) *If a simple subquotient* S *of* $M \circ N$ *is not isomorphic to* $M \nabla N$*, then* $\Lambda(S, N) < \Lambda(M \nabla N, N) = \Lambda(M, N)$.
   (b) *If a simple subquotient* S *of* $M \circ N$ *is not isomorphic to* $M \Delta N$*, then* $\Lambda(N, S) < \Lambda(N, M \Delta N) = \Lambda(N, M)$.
(iv) *If* M *and* N *are self-dual, then* $q^{\widetilde{\Lambda}(M,N)} M \nabla N$ *is a self-dual simple module.*
(v) *The following conditions are equivalent:*
   (a) $M \circ N \simeq N \circ M$ *up to a grading shift;*
   (b) $M \circ N$ *is simple;*
   (c) $\mathfrak{d}(M, N) = 0$*;*
   (d) $M \nabla N \simeq M \Delta N$ *up to a grading shift.*

Proposition 2.8 [KKOP21, Corollary 3.18]. *Let* M *be an affreal simple module. Let* X *be an* R*-module in* R*-gmod. Let* $n \in \mathbb{Z}_{>0}$ *and assume that any simple subquotient* S *of* X *satisfies* $\mathfrak{d}(M, S) \leqslant n$*. Then any simple subquotient* N *of* $M \circ X$ *satisfies* $\mathfrak{d}(M, N) < n$*. In particular, any simple subquotient of* $M^{\circ n} \circ X$ *strongly commutes with* M.

An ordered sequence of simple modules $\underline{L} = (L_1, \ldots, L_r)$ is called *almost affreal* if all L_i $(1 \leqslant i \leqslant r)$ are affreal except for at most one.

Definition 2.9. An almost affreal sequence $\underline{L}$ of simple modules is called a *normal sequence* if the composition of R-matrices

$$\mathbf{r}_{\underline{L}} := \prod_{1 \leqslant i < k \leqslant r} \mathbf{r}_{L_i, L_k} = (\mathbf{r}_{L_{r-1}, L_r}) \circ \cdots (\mathbf{r}_{L_2, L_r} \circ \cdots \circ \mathbf{r}_{L_2, L_3}) \circ (\mathbf{r}_{L_1, L_r} \circ \cdots \circ \mathbf{r}_{L_1, L_2})$$
$$: \ q^{\sum_{1 \leqslant i < k \leqslant r} \Lambda(L_i, L_k)} L_1 \circ \cdots \circ L_r \longrightarrow L_r \circ \cdots \circ L_1 \quad \text{does not vanish.}$$

Lemma 2.10 [KK19, § 2.3] and [KKOP23, § 2.2]. *Let* $\underline{L}$ *be an almost affreal sequence of simple modules. If* $\underline{L}$ *is normal, then the image of* $\mathbf{r}_{\underline{L}}$ *is simple and coincides with the head of* $L_1 \circ \cdots \circ L_r$ *and also with the socle of* $L_r \circ \cdots \circ L_1$*, up to grading shifts.*

Lemma 2.11 [KKKO18, Proposition 3.2.13]. *Let* (A, B, C) *be an almost affreal sequence. Then we have the following:*

(i) $\Lambda(A, B \nabla C) = \Lambda(A, B) + \Lambda(A, C)$ *if* A *and* B *commute;*
(ii) $\Lambda(A \nabla B, C) = \Lambda(A, C) + \Lambda(B, C)$ *if* B *and* C *commute.*

For a given almost affreal sequence $\underline{L}$ of R-modules, the sufficient conditions for $\underline{L}$ being normal are studied in [KK19, KKOP23]. In this paper, we will use the conditions frequently.

2.5 Commuting families

Let J be an index set. We say that a family of affreal simple modules $\mathcal{M} = \{M_j\}_{j \in J}$ in R-gmod is a *commuting family* if

$$M_i \circ M_j \simeq M_j \circ M_i \quad \text{up to a grading shift for any } i, j \in J.$$

For a commuting family $\mathcal{M} = \{M_j \mid j \in J\}$ in R-gmod, let us take $\lambda \colon \mathbb{Z}^{\oplus J} \times \mathbb{Z}^{\oplus J} \to \mathbb{Z}$ such that

$$\lambda(\mathbf{e}_i, \mathbf{e}_j) - \lambda(\mathbf{e}_j, \mathbf{e}_i) = \Lambda(M_i, M_j) \quad \text{for any } i, j \in J. \tag{2.4}$$

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Here $\{\mathbf{e}_j \mid j \in J\}$ is the standard basis of $\mathbb{Z}^{\oplus J}$. Then there exists a family $\{\mathcal{M}_\lambda(\mathbf{a}) \mid \mathbf{a} \in \mathbb{Z}^{\oplus J}_{\geqslant 0}\}$ of simple modules in $\mathscr{C}_w$ such that

$$\begin{aligned} &\mathcal{M}_\lambda(0) = \mathbf{1}, \quad \mathcal{M}_\lambda(\mathbf{e}_j) = M_j && \text{for any } j \in J, \\ &\mathcal{M}_\lambda(\mathbf{a}) \circ \mathcal{M}_\lambda(\mathbf{b}) \simeq q^{-\lambda(a,b)}\mathcal{M}_\lambda(\mathbf{a}+\mathbf{b}) && \text{for any } \mathbf{a},\mathbf{b} \in \mathbb{Z}^{J}_{\geqslant 0}. \end{aligned} \tag{2.5}$$

We sometimes omit $_\lambda$ for notational simplicity.

Remark 2.12. Note that $\lambda(\mathbf{e}_i, \mathbf{e}_j) = \widetilde{\Lambda}(M_i, M_j)$ satisfies condition (2.4). Moreover, if all the M_i are self-dual, then $\mathcal{M}_\lambda(\mathbf{a})$ is self-dual for any $\mathbf{a} \in \mathbb{Z}^{J}_{\geqslant 0}$. We usually take this choice of λ.

Definition 2.13. A commuting family $\mathcal{M} = \{M_j \mid j \in J\}$ is called *independent* if $\mathbf{a},\mathbf{b} \in \mathbb{Z}^{\oplus J}_{\geqslant 0}$ satisfies $\mathcal{M}(\mathbf{a}) \simeq q^s \mathcal{M}(\mathbf{b})$ for some $s \in \mathbb{Z}$, then we have $\mathbf{a} = \mathbf{b}$.

The following lemma is obvious.

Lemma 2.14. *Let $\mathcal{M} = \{M_j \mid j \in J\}$ be a commuting family. Then it is independent if and only if the set $\{[\mathcal{M}(\mathbf{a})] \mid \mathbf{a} \in \mathbb{Z}^{\oplus J}_{\geqslant 0}\}$ in $K(R\text{-gmod})$ is linearly independent over $\mathbb{Z}[q^{\pm 1}]$.*

2.6 Localization of $\mathscr{C}_w$

Throughout this subsection, we fix $w \in \mathsf{W}$ and set

$$I_w := \{i \in I \mid w\varpi_i \neq \varpi_i\}.$$

For notational simplicity, let us write

$$C_i := M(w\varpi_i, \varpi_i) \in R\text{-gmod} \quad \text{for } i \in I.$$

Then $\{\Omega([C_i]) \mid i \in I_w\}$ forms the set of frozen variables of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$. For each $\mu = \sum_{i \in I} \mu_i \varpi_i \in \mathsf{P}^+$, we set $C_\mu = M(w\mu, \mu)$, which is a self-dual convolution product $q^c \circ_{i \in I} C_i^{\circ \mu_i}$ for some $c \in \mathbb{Z}$.

It is proved in [KKOP21, KKOP23, KKOP24a] that there exists a monoidal abelian category $\widetilde{\mathscr{C}}_w = \mathscr{C}_w[C_i^{\circ -1} \mid i \in I]$ with a tensor product $\circ$, a degree shift functor q and a monoidal *exact fully faithful* functor $\Phi_w \colon \mathscr{C}_w \to \widetilde{\mathscr{C}}_w$ satisfying the following properties.

(A) The objects $\Phi_w(C_i)$ are invertible objects in $\widetilde{\mathscr{C}}_w$; that is, there exists an object $\Phi_w(C_i)^{-1}$ in $\widetilde{\mathscr{C}}_w$ such that $\Phi(C_i) \circ \Phi_w(C_i)^{-1} \simeq 1$ and $\Phi(C_i)^{-1} \circ \Phi(C_i) \simeq 1$.

(B) The category $\widetilde{\mathscr{C}}_w$ is universal to $\mathscr{C}_w$ in the following sense: for any monoidal functor $\Psi \colon \mathscr{C}_w \to \mathcal{T}$ to another monoidal category $\mathcal{T}$ in which $\Psi(C_i)$ is invertible for every $i \in I$, there exists a monoidal functor $\Psi' \colon \widetilde{\mathscr{C}}_w \to \mathcal{T}$ such that $\Psi \simeq \Psi' \circ \Phi_w$. Moreover, Ψ' is unique up to a unique isomorphism. (2.6)

(C) There exists a commuting family of simple objects $\{\widetilde{C}_\mu \mid \mu \in \mathsf{P}\}$ such that $\widetilde{C}_\mu \simeq \Phi_w(C_\mu)$ for every $\mu \in \mathsf{P}^+$ and $\widetilde{C}_\mu \circ \widetilde{C}_{\mu'} \simeq q^{\mathsf{H}(\mu,\mu')}\widetilde{C}_{\mu+\mu'}$ for every $\mu, \mu' \in \mathsf{P}$. Here H denotes the bilinear form on P given by $\mathsf{H}(\mu,\mu') = (\mu, w\mu' - \mu')$.

(D) Every simple object in $\widetilde{\mathscr{C}}_w$ is isomorphic to $\Phi_w(S) \circ \widetilde{C}_\mu$ for some simple object $S \in \mathscr{C}_w$ and $\mu \in \mathsf{P}$.

(For the precise properties, see [KKOP21, KKOP23, KKOP24a].)

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THEOREM 2.15 ([KKOP21] (see also [KKOP23, Remark 3.6])). *There exists an* $\mathbb{A}$*-algebra isomorphism*

$$\widetilde{\Omega}\colon \mathcal{K}_{\mathbb{A}}(\widetilde{\mathscr{C}}_w) := \mathbb{A} \underset{\mathbb{Z}[q^{\pm 1}]}{\otimes} K(\widetilde{\mathscr{C}}_w) \xrightarrow{\sim} \mathcal{A}_{\mathbb{A}}(\mathfrak{n}^w) \quad \text{such that} \quad \widetilde{\Omega}|_{\mathcal{K}_{\mathbb{A}}(\mathscr{C}_w)} = \Omega.$$

Here $K(\widetilde{\mathscr{C}}_w)$ *denotes the Grothendieck ring of* $\widetilde{\mathscr{C}}_w$.

A pair $(\varepsilon\colon X \otimes Y \to \mathbf{1},\ \eta\colon \mathbf{1} \to Y \otimes X)$ of morphisms in a monoidal category with a unit object $\mathbf{1}$ is called an *adjunction* if the following two conditions hold.

(a) The composition $X \simeq X \otimes \mathbf{1} \xrightarrow{X\otimes\eta} X \otimes Y \otimes X \xrightarrow{\varepsilon\otimes X} \mathbf{1} \otimes X \simeq X$ is the identity.
(b) The composition $Y \simeq \mathbf{1} \otimes Y \xrightarrow{\eta\otimes Y} Y \otimes X \otimes Y \xrightarrow{Y\otimes\varepsilon} Y \otimes \mathbf{1} \simeq Y$ is the identity.

In the case when (ε, η) is an adjunction, we say that X is a *left dual to* Y, Y is a *right dual to* X and (X, Y) is a *dual pair.*

THEOREM 2.16 [KKOP21, KKOP23]. *The monoidal category* $\widetilde{\mathscr{C}}_w$ *is rigid; i.e. every object of* $\widetilde{\mathscr{C}}_w$ *has a right dual and a left dual in* $\widetilde{\mathscr{C}}_w$.

2.7 Determinantal modules and monoidal clusters

In this subsection, we denote by $\mathcal{C}$ the category of $\mathscr{C}_w$ or $\widetilde{\mathscr{C}}_w$. Recall that

$$\mathscr{A} = \mathcal{K}_{\mathbb{A}}(\mathcal{C}) \text{ has a quantum cluster algebra structure via an isomorphism}$$
$$\Omega = \Omega \text{ or } \widetilde{\Omega}.$$

Let $\boldsymbol{i} = (i_1, \ldots, i_r)$ be a reduced sequence of $w \in \mathsf{W}$. For k such that $1 \leqslant k \leqslant r$, set

$$M_k^{\boldsymbol{i}} = M(w_{\leqslant k}^{\boldsymbol{i}} \varpi_{i_k}, \varpi_{i_k})$$

(see Proposition 2.2 for the notation).

PROPOSITION 2.17 [KKOP18, Theorem 4.12]. *Let* $\boldsymbol{i} = (i_1, \ldots, i_r)$ *be a reduced sequence of* $w \in \mathsf{W}$. *For* $s < t \in \mathsf{K}$, *we have*

$$-\Lambda(M_s^{\boldsymbol{i}}, M_t^{\boldsymbol{i}}) = (\varpi_{i_s} - w_{\leqslant s}^{\boldsymbol{i}} \varpi_{i_s}, \varpi_{i_t} + w_{\leqslant t}^{\boldsymbol{i}} \varpi_{i_t}) = l_{st}^{\boldsymbol{i}} = (L^{\boldsymbol{i}})_{st}.$$

We say that a commuting family $\mathcal{M} = \{M_i\}_{i\in\mathsf{K}}$ in $\mathcal{C}$ is a *monoidal cluster* if there exists a quantum seed $(\{X_i\}_{i\in\mathsf{K}}, L = (l_{i,j})_{i,j\in\mathsf{K}}, \widetilde{B} = (b_{i,j})_{i\in\mathsf{K}, j\in\mathsf{K}_{\mathrm{ex}}})$ of $\mathscr{A}$ such that

$$X_i = \Omega(q^{1/4(\mathrm{wt}(M_i), \mathrm{wt}(M_i))}[M_i]) \quad \text{and} \quad l_{i,j} = -\Lambda(M_i, M_j).$$

Note that every monoidal seed is independent since the quantum cluster monomials in a cluster are linearly independent over $\mathbb{A}$ by the definition.

With Proposition 2.2 and (2.2), Proposition 2.17 says that

$$\mathcal{M}^{\boldsymbol{i}} := \{M_k^{\boldsymbol{i}}\}_{1\leqslant k\leqslant r} \text{ is a monoidal cluster in } \mathscr{C}_w, \tag{2.7}$$

for any reduced sequence $\boldsymbol{i} = (i_1, \ldots, i_r)$ of w. We call $\mathcal{M}^{\boldsymbol{i}}$ the *GLS cluster* (associated with $\boldsymbol{i}$).

3. Quasi-Laurent family and Laurent family

In this section, we introduce the notions of quasi-Laurent families and Laurent families, which allow us to associate two vectors in $\mathbb{Z}^J$ with each simple module.

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3.1 Definition

Let J be a finite index set. Let $\mathscr{C}$ be a full monoidal subcategory of R-gmod stable by taking subquotients, extensions and grading shifts.

Lemma 3.1. *Let $\mathcal{M} = \{M_j \mid j \in J\}$ be an* independent *commuting family of affreal simple objects in $\mathscr{C}$ and X a simple module in $\mathscr{C}$.*

(i) *If $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, $\mathbf{b}' \in \mathbb{Z}^J_{\geqslant 0}$ satisfy $X \nabla \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b})$ and $X \nabla \mathcal{M}(\mathbf{a}') \simeq \mathcal{M}(\mathbf{b}')$ up to grading shifts, then one has $\mathbf{b} - \mathbf{a} = \mathbf{b}' - \mathbf{a}'$.*

(ii) *If $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, $\mathbf{b}' \in \mathbb{Z}^J_{\geqslant 0}$ satisfy $\mathcal{M}(\mathbf{a}) \nabla X \simeq \mathcal{M}(\mathbf{b})$ and $\mathcal{M}(\mathbf{a}') \nabla X \simeq \mathcal{M}(\mathbf{b}')$ up to grading shifts, then one has $\mathbf{b} - \mathbf{a} = \mathbf{b}' - \mathbf{a}'$.*

Proof. Since the proof are similar, we prove only part (i). We have

$$\begin{aligned}\mathcal{M}(\mathbf{b}+\mathbf{a}') &\simeq \mathcal{M}(\mathbf{b}) \nabla \mathcal{M}(\mathbf{a}') \simeq (X \nabla \mathcal{M}(\mathbf{a})) \nabla \mathcal{M}(\mathbf{a}') \simeq \mathrm{hd}(X \circ \mathcal{M}(\mathbf{a}') \circ \mathcal{M}(\mathbf{a})) \\ &\simeq (X \nabla \mathcal{M}(\mathbf{a}')) \nabla \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b}'+\mathbf{a}),\end{aligned}$$

and, hence, we have $\mathbf{a}'+\mathbf{b} = \mathbf{a}+\mathbf{b}'$ since $\mathcal{M}$ is independent. □

Definition 3.2. We say that a commuting family $\mathcal{M} = \{M_j \mid j \in J\}$ of affreal simple objects of $\mathscr{C}$ is a *quasi-Laurent family* in $\mathscr{C}$ if $\mathcal{M}$ satisfies the following conditions:

(a) $\mathcal{M}$ is independent; and

(b) if a simple module X commutes with all M_j $(j \in J)$, then there exist $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^J_{\geqslant 0}$ such that

$$X \circ \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b}) \quad \text{up to a grading shift.}$$

If $\mathcal{M}$ satisfies part (a) and part (c) below, then we say that $\mathcal{M}$ is a *Laurent family*:

(c) if a simple module X commutes with all M_j $(j \in J)$, then there exists $\mathbf{b} \in \mathbb{Z}^J_{\geqslant 0}$ such that $X \simeq \mathcal{M}(\mathbf{b})$.

Note that a Laurent family is a quasi-Laurent family.

Lemma 3.3. *Let $\mathcal{M} = \{M_j \mid j \in J\}$ be a quasi-Laurent family in $\mathscr{C}$. Then we have the following:*

(i) *for any simple module $X \in \mathscr{C}$, there exist $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^J_{\geqslant 0}$ such that $X \nabla \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b})$ up to a grading shift;*

(ii) *for any simple module $X \in \mathscr{C}$, there exist $\mathbf{a}', \mathbf{b}' \in \mathbb{Z}^J_{\geqslant 0}$ such that $X \,\Delta\, \mathcal{M}(\mathbf{a}') \simeq \mathcal{M}(\mathbf{b}')$ up to a grading shift.*

Proof. Since the proofs are similar, we shall only prove the first statement. Let us take $\mathbf{a}^{(1)} \in \mathbb{Z}^J_{\geqslant 0}$ such that $\mathbf{a}^{(1)}_j \gg 0$ for all $j \in J$. Then Proposition 2.8 says that the simple module $Y := X \nabla \mathcal{M}(\mathbf{a}^{(1)})$ commutes with all M_j. Since $\mathcal{M}$ is a quasi-Laurent family, there exists $\mathbf{a}^{(2)}, \mathbf{b} \in \mathbb{Z}^J_{\geq 0}$ such that $Y \circ \mathcal{M}(\mathbf{a}^{(2)}) \simeq \mathcal{M}(\mathbf{b})$. Hence, by taking $\mathbf{a} = \mathbf{a}^{(1)} + \mathbf{a}^{(2)}$, we have

$$X \nabla \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b}),$$

as desired. □

By applying the similar argument as in the lemma above to composition factors of $X \circ \mathcal{M}(\mathbf{a})$, we have the following corollary.

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COROLLARY 3.4. *Let $\mathcal{M}$ be a quasi-Laurent family in $\mathscr{C}$. Then, for any $X \in \mathscr{C}$, there exist $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}$, a finite index set S, $c(s) \in \mathbb{Z}$ and $\mathbf{b}(s) \in \mathbb{Z}^J_{\geqslant 0}$ $(s \in \mathsf{S})$ such that*

$$[X \circ \mathcal{M}(\mathbf{a})] = \sum_{s \in \mathsf{S}} q^{c(s)} [\mathcal{M}(\mathbf{b}(s))]. \tag{3.1}$$

Remark 3.5. The above corollary says that for every module X in $\mathscr{C}$ and a quasi-Laurent family $\mathcal{M} = \{M_j \mid j \in J\}$, the isomorphism class $[X]$ of X in $K(\mathscr{C})$ can be expressed as an element in the Laurent polynomial ring $\mathbb{Z}[q^{\pm 1}][[M_j]^{\pm 1} \mid j \in J]$ with positive coefficients.

PROPOSITION 3.6. *Assume that $\mathcal{K}_{\mathbb{A}}(\mathscr{C})$ has a quantum cluster algebra structure, and let $\mathcal{M} = \{M_k \mid k \in \mathsf{K}\}$ be a monoidal cluster in $\mathscr{C}$. Then the commuting family $\mathcal{M}$ is a quasi-Laurent family. In particular, the isomorphism class $[X]$ of X in $K(\mathscr{C})$ can be expressed as an element in the Laurent polynomial $\mathbb{Z}[q^{\pm 1}][\,[M_j] \mid j \in J]$ with positive coefficients.*

If, moreover, every $[M_k]$ is prime in $K(\mathscr{C})|_{q=1}$, then $\mathcal{M}$ is a Laurent family.

Note that if $K(\mathscr{C})|_{q=1}$ is factorial, then every $[M_k]$ is prime in $K(\mathscr{C})|_{q=1}$ (see [GLS13b]).

Proof. Let X be a simple module in $\mathscr{C}$ commuting with all M_k $(k \in \mathsf{K})$. The quantum Laurent phenomenon states that there exists $\mathbf{a} \in \mathbb{Z}^{\mathsf{K}}_{\geqslant 0}$ such that

$$[X] = \frac{\sum_{s \in \mathsf{S}} c(s)[\mathcal{M}(\mathbf{b}^{(s)})]}{[\mathcal{M}(\mathbf{a})]} \iff [X \circ \mathcal{M}(\mathbf{a})] = \sum_{s \in \mathsf{S}} c(s)[\mathcal{M}(\mathbf{b}^{(s)})] \tag{3.2}$$

for some $\ell \in \mathbb{Z}_{\geqslant 1}$, $\mathbf{b}^{(s)} \in \mathbb{Z}^{\mathsf{K}}_{\geqslant 0}$ and $c(s) \in \mathbb{Z}[q^{\pm 1/2}]$. Since $X \circ \mathcal{M}(\mathbf{a})$ is simple, the right-hand side of (3.2) must coincide with $q^c \mathcal{M}(\mathbf{c})$ for some $\mathbf{c} \in \mathbb{Z}^J_{\geqslant 0}$ and $c \in \mathbb{Z}$. Hence, $\mathcal{M}$ is a quasi-Laurent family.

Let us show that $\mathcal{M}$ is a Laurent family. If $X \circ \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b})$, then we have $\mathbf{a}_k \leqslant \mathbf{b}_k$ for all k, since each $[M_k]$ is a prime element of $K(\mathscr{C})|_{q=1}$. Hence, setting $\mathbf{c} = \mathbf{b} - \mathbf{a} \in \mathbb{Z}^{\mathsf{K}}_{\geq 0}$, we have $X \circ \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{c}) \circ \mathcal{M}(\mathbf{a})$, which implies that $X \simeq \mathcal{M}(\mathbf{c})$. Hence, $\mathcal{M}$ is a Laurent family. □

DEFINITION 3.7. For a simple module $X \in \mathscr{C}$ and a quasi-Laurent family $\mathcal{M} = \{M_j \mid j \in J\}$, take $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}' \in \mathbb{Z}^J_{\geqslant 0}$ such that $X \nabla \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b})$ and $X \Delta \mathcal{M}(\mathbf{a}') \simeq \mathcal{M}(\mathbf{b}')$ up to grading shifts. Then we define

$$\mathbf{g}^R_{\mathcal{M}}(X) := \mathbf{b} - \mathbf{a} \quad \text{and} \quad \mathbf{g}^L_{\mathcal{M}}(X) := \mathbf{b}' - \mathbf{a}' \in \mathbb{Z}^J.$$

Remark 3.8.

(1) For a quasi-Laurent family $\mathcal{M}$ in $\mathscr{C}$, $\mathbf{g}^R_{\mathcal{M}}$ and $\mathbf{g}^L_{\mathcal{M}}$ are well-defined by Lemma 3.1.
(2) For a reduced sequence $\boldsymbol{i}$ of w and its quasi-Laurent family $\mathcal{M}^{\boldsymbol{i}}$, we write $\mathbf{g}^R_{\boldsymbol{i}}$ and $\mathbf{g}^L_{\boldsymbol{i}}$ instead of $\mathbf{g}^R_{\mathcal{M}^{\boldsymbol{i}}}$ and $\mathbf{g}^L_{\mathcal{M}^{\boldsymbol{i}}}$, respectively.
(3) The map $\mathbf{g}^R_{\boldsymbol{i}}$ and $\mathbf{g}^L_{\boldsymbol{i}}$ for the quasi-Laurent family $\mathcal{M}^{\boldsymbol{i}}$ can be extended to the set $\mathrm{Irr}(\widetilde{\mathscr{C}}_w)$ of the isomorphism classes of self-dual simples in $\widetilde{\mathscr{C}}_w$.

The following lemma can be proved by the same arguments in [KK19].

LEMMA 3.9. *Let $\mathcal{M}$ be a quasi-Laurent family in $\mathscr{C}$ and X a simple module in $\mathscr{C}$.*

(i) *If $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^J_{\geqslant 0}$ satisfy $\mathbf{b} - \mathbf{a} = \mathbf{g}^R_{\mathcal{M}}(X)$ (respectively, $\mathbf{b} - \mathbf{a} = \mathbf{g}^L_{\mathcal{M}}(X)$), then we have*

$$X \nabla \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b}) \text{ (respectively, } X \Delta \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b})\text{) up to a grading shift.}$$

(ii) *For any $\mathbf{a} \in \mathbb{Z}^J_{\geqslant 0}$, we have*

$$\mathbf{g}^R_{\mathcal{M}}(X \nabla \mathcal{M}(\mathbf{a})) = \mathbf{g}^R_{\mathcal{M}}(X) + \mathbf{a} \quad \text{and} \quad \mathbf{g}^L_{\mathcal{M}}(X \Delta \mathcal{M}(\mathbf{a})) = \mathbf{g}^L_{\mathcal{M}}(X) + \mathbf{a}.$$

(iii) *The maps $\mathbf{g}^R_{\mathcal{M}}$ and $\mathbf{g}^L_{\mathcal{M}}$ from $\mathrm{Irr}(\mathscr{C})$ to $\mathbb{Z}^J$ are injective.*

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4. PBW decomposition vector and GLS seed

In this section, we recall the PBW basis, and we investigate the relationship between the g-vectors and the PBW decomposition vectors.

4.1 PBW decomposition vector

Let us take $w \in \mathsf{W}$ and its reduced sequence $\boldsymbol{i} = (i_1, \ldots, i_r)$. Recall the following.

(a) We take an index set $\mathsf{K} = [1, r]$ with a decomposition

$$\mathsf{K}_{\mathrm{ex}} \sqcup \mathsf{K}_{\mathrm{fr}} \quad \text{where } \mathsf{K}_{\mathrm{ex}} = \{k \in \mathsf{K} \mid k_+ \leqslant r\}.$$

(b) For each $1 \leqslant k \leqslant r$, we set $\beta_k^{\boldsymbol{i}} \in \Delta^+ \cap w\Delta^-$ and define simple modules $S_k^{\boldsymbol{i}} = M(w_{\leqslant k}^{\boldsymbol{i}} \varpi_{i_k}, w_{<k}^{\boldsymbol{i}} \varpi_{i_k})$ and $M_k^{\boldsymbol{i}} = M(w_{\leqslant k}^{\boldsymbol{i}} \varpi_{i_k}, \varpi_{i_k})$. Note that

$$M_k^{\boldsymbol{i}} \simeq S_k^{\boldsymbol{i}} \nabla M_{k_-}^{\boldsymbol{i}} \quad \text{and} \quad \mathcal{M}^{\boldsymbol{i}} := \{M_k^{\boldsymbol{i}} \mid k \in \mathsf{K}\} \text{ forms a commuting family.} \tag{4.1}$$

For any $\mathbf{a} = (\mathbf{a}_k)_{1 \leqslant k \leqslant r} \in \mathbb{Z}_{\geqslant 0}^{\mathsf{K}}$, the convolution product

$$P_{\boldsymbol{i}}(\mathbf{a}) := q^{(1/2)\sum_{k=1}^{r} \mathbf{a}_k(\mathbf{a}_k - 1)\mathsf{d}_{i_k}} {S_r^{\boldsymbol{i}}}^{\circ \mathbf{a}_r} \circ \cdots \circ {S_1^{\boldsymbol{i}}}^{\circ \mathbf{a}_1}$$

has a self-dual simple head. Conversely, every self-dual simple module in $\mathscr{C}_w$ is isomorphic to $\mathrm{hd}(P_{\boldsymbol{i}}(\mathbf{a}))$ for some $\mathbf{a} \in \mathbb{Z}_{\geq 0}^{\mathsf{K}}$ in a unique way (see [McN15, Theorem 3.1] and [TW16, Theorem 2.19]). We call $\{P_{\boldsymbol{i}}(\mathbf{a}) \mid \mathbf{a} \in \mathbb{Z}_{\geq 0}^{\mathsf{K}}\}$ the *PBW basis of $\mathscr{C}_w$ associated with $\boldsymbol{i}$.*

For a simple module X such that $X \simeq \mathrm{hd}(P_{\boldsymbol{i}}(\mathbf{a}))$, we set

$$\mathrm{PBW}_{\boldsymbol{i}}(X) := \mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_r).$$

The following lemma says that the operation $\mathrm{PBW}_{\boldsymbol{i}}(- \nabla \mathcal{M}^{\boldsymbol{i}}(\mathbf{a}))$ on the set of simple modules behaves very nicely, where $\mathcal{M}^{\boldsymbol{i}}(\mathbf{a})$ is defined in (2.5).

Lemma 4.1 (cf. [KK19, Lemma 3.11 and Proposition 3.14]). *For $M = \mathcal{M}^{\boldsymbol{i}}(\mathbf{a})$ with $\mathbf{a} \in \mathbb{Z}_{\geq 0}^{\mathsf{K}}$ and a simple module X, we have*

$$\mathrm{PBW}_{\boldsymbol{i}}(X \nabla M) = \mathrm{PBW}_{\boldsymbol{i}}(X) + \mathrm{PBW}_{\boldsymbol{i}}(M).$$

In particular, $\mathbf{c} := \mathrm{PBW}_{\boldsymbol{i}}\big(\mathcal{M}^{\boldsymbol{i}}(\mathbf{a})\big)$ is given by $\mathbf{c}_k = \sum_j \mathbf{a}_j$ where j ranges over $j \in [1, r]$ such that $j \geqslant k$ and $\boldsymbol{i}_j = \boldsymbol{i}_k$.

Proof. It is enough to show it when $M = M_k^{\boldsymbol{i}}$. Note that

$$X \simeq \mathrm{hd}\Big(\mathop{\overrightarrow{\circ}}_{1 \leqslant k \leqslant r} {S_k^{\boldsymbol{i}}}^{\circ \mathbf{n}_k} \Big) = {S_r^{\boldsymbol{i}}}^{\circ \mathbf{n}_r} \nabla Y,$$

where $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_r) = \mathrm{PBW}_{\boldsymbol{i}}(X)$, $Y \simeq \mathrm{hd}\Big(\mathop{\overrightarrow{\circ}}_{1 \leqslant k \leqslant r-1} {S_k^{\boldsymbol{i}}}^{\circ \mathbf{n}_k} \Big)$ and $\mathop{\overrightarrow{\circ}}_{p \leqslant k \leqslant q} X_k$ denotes the ordered convolution product $X_q \circ X_{q-1} \circ \cdots X_{p+1} \circ X_p$ for X_k in R-gmod.

If $r > k$, then $({S_r^{\boldsymbol{i}}}^{\circ \mathbf{n}_r}, M_k^{\boldsymbol{i}})$ is unmixed and, hence, ${S_r^{\boldsymbol{i}}}^{\circ \mathbf{n}_r} \circ Y \circ M_k^{\boldsymbol{i}}$ has a simple head. We have

$$\begin{aligned} X \nabla M_k^{\boldsymbol{i}} &\simeq \mathrm{hd}({S_r^{\boldsymbol{i}}}^{\circ \mathbf{n}_r} \circ Y \circ M_k^{\boldsymbol{i}}) \\ &\simeq {S_r^{\boldsymbol{i}}}^{\circ \mathbf{n}_r} \nabla (Y \nabla M_k^{\boldsymbol{i}}) \simeq {S_r^{\boldsymbol{i}}}^{\circ \mathbf{n}_r} \nabla \mathrm{hd}\Big(\mathop{\overrightarrow{\circ}}_{1 \leqslant k \leqslant r-1} {S_k^{\boldsymbol{i}}}^{\circ \mathbf{c}_k} \Big), \end{aligned}$$

where $\mathbf{c} = \mathrm{PBW}_{\boldsymbol{i}}(Y) + \mathrm{PBW}_{\boldsymbol{i}}(M_k^{\boldsymbol{i}})$ by induction on r. Thus, our assertion follows in this case.

If $r = k$, then $M_r^{\boldsymbol{i}}$ commutes with all the objects of $\mathscr{C}_w$ and, hence, we have

$$X \nabla M_r^{\boldsymbol{i}} \simeq \mathrm{hd}({S_r^{\boldsymbol{i}}}^{\circ \mathbf{n}_r} \circ M_r^{\boldsymbol{i}} \circ {S_{r-1}^{\boldsymbol{i}}}^{\circ \mathbf{n}_{r-1}} \circ \cdots {S_1^{\boldsymbol{i}}}^{\circ \mathbf{n}_1})$$

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$$\simeq \mathrm{hd}(S_r^{\boldsymbol{i}\,\circ \mathbf{n}_r} \circ M_r^{\boldsymbol{i}}) \nabla Y \simeq \mathrm{hd}(S_r^{\boldsymbol{i}\,\circ \mathbf{n}_r} \circ (S_r^{\boldsymbol{i}} \nabla M_{r_-}^{\boldsymbol{i}})) \nabla Y$$
$$\simeq \mathrm{hd}(S_r^{\boldsymbol{i}\,\circ \mathbf{n}_r+1} \nabla M_{r_-}^{\boldsymbol{i}}) \nabla Y \simeq S_r^{\boldsymbol{i}\,\circ \mathbf{n}_r+1} \nabla (M_{r_-}^{\boldsymbol{i}} \circ Y),$$

where the last isomorphism follows from the commutativity of $M_{r_-}^{\boldsymbol{i}}$ and Y. Then our assertion follows from the induction hypothesis. □

The lemma above gives a direct proof of the following corollary although it follows immediately from Proposition 3.6.

Corollary 4.2. *For any reduced sequence $\boldsymbol{i}$ for w, the commuting family $\mathcal{M}^{\boldsymbol{i}}$ is independent.*

The following proposition is proved in [KK19, Proposition 2.11] for symmetric quiver Hecke algebra and the same proof also works for the general case.

Proposition 4.3. *For any $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_r)$, $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_r) \in \mathbb{Z}_{\geqslant 0}^{\mathsf{K}}$, the ordered sequence*

$$(S_r^{\boldsymbol{i}\,\circ \mathbf{a}_r}, (S_{r-1}^{\boldsymbol{i}})^{\circ \mathbf{a}_{r-1}}, \dots, S_1^{\boldsymbol{i}\,\circ \mathbf{a}_1}, M_1^{\boldsymbol{i}\,\circ \mathbf{b}_1}, M_2^{\boldsymbol{i}\,\circ \mathbf{b}_2}, \dots, M_r^{\boldsymbol{i}\,\circ \mathbf{b}_r})$$

is a normal sequence.

The statement and proof of following proposition are the same as [KK19, Proposition 3.14] even though [KK19] dealt only with symmetric quiver Hecke algebras. Here we repeat it in order to show relations between explicit $\mathbb{Z}_{\geqslant 0}$-vectors associated with a simple module X in $\mathscr{C}_w$ for the readers' convenience.

Proposition 4.4. *For a simple module X in $\mathscr{C}_w$ or $\widetilde{\mathscr{C}}_w$, there exist $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{\mathsf{K}}$ such that*

$$X \nabla \mathcal{M}^{\boldsymbol{i}}(\mathbf{a}) \simeq \mathcal{M}^{\boldsymbol{i}}(\mathbf{b}) \quad \text{up to a grading shift.}$$

Proof. In this proof, we sometimes drop $^{\boldsymbol{i}}$ for notational simplicity. It is enough to consider when $X \in \mathscr{C}_w$ by part (D) in (2.6). Note that there exists a unique $\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_r) \in \mathbb{Z}_{\geqslant 0}^{\mathsf{K}}$ such that

$$X \simeq \mathrm{hd}(P_{\boldsymbol{i}}(\mathbf{c})) \simeq \mathrm{hd}(S_r^{\boldsymbol{i}\,\circ \mathbf{c}_r} \circ \cdots \circ S_1^{\boldsymbol{i}\,\circ \mathbf{c}_1}).$$

Set $\mathbf{c}_+ := \sum_{k=1}^r \mathbf{c}_{k_+} \mathbf{e}_k = \sum_{j \in \mathsf{K}} \mathbf{c}_k \mathbf{e}_{k_-}$, where $\{\mathbf{e}_j \mid j \in \mathsf{K}\}$ is the standard basis of $\mathbb{Z}^{\mathsf{K}}$ such that $\mathbf{c} = \sum_{j \in \mathsf{K}} \mathbf{c}_j \mathbf{e}_j$. Then we have $\mathcal{M}^{\boldsymbol{i}}(\mathbf{c}_+) \simeq M_{1_-}^{\circ \mathbf{c}_1} \circ \cdots \circ M_{r_-}^{\circ \mathbf{c}_r}$. By (4.1) and Proposition 4.3, we have

$$\begin{aligned}
X \nabla \mathcal{M}^{\boldsymbol{i}}(\mathbf{c}_+) &\simeq \mathrm{hd}(S_r^{\circ \mathbf{c}_r} \circ S_{r-1}^{\circ \mathbf{c}_{r-1}} \circ \cdots \circ S_1^{\circ \mathbf{c}_1} \circ \mathcal{M}^{\boldsymbol{i}}(\mathbf{c}_+)) \\
&\simeq \mathrm{hd}(S_r^{\circ \mathbf{c}_r} \circ S_{r-1}^{\circ \mathbf{c}_{r-1}} \circ \cdots \circ S_2^{\circ \mathbf{c}_2} \circ S_1^{\circ \mathbf{c}_1} \circ M_{1_-}^{\circ \mathbf{c}_1} \circ M_{2_-}^{\circ \mathbf{c}_2} \circ \cdots \circ M_{r_-}^{\circ \mathbf{c}_r}) \\
&\simeq \mathrm{hd}((S_r^{\circ \mathbf{c}_r} \circ S_{r-1}^{\circ \mathbf{c}_{r-1}} \circ \cdots \circ S_2^{\circ \mathbf{c}_2}) \circ (S_1^{\circ \mathbf{c}_1} \nabla M_{1_-}^{\circ \mathbf{c}_1}) \circ (M_{2_-}^{\circ \mathbf{c}_2} \circ \cdots \circ M_{r_-}^{\circ \mathbf{c}_r})) \\
&\simeq \mathrm{hd}((S_r^{\circ \mathbf{c}_r} \circ S_{r-1}^{\circ \mathbf{c}_{r-1}} \circ \cdots \circ S_1^{\circ \mathbf{c}_2}) \circ M_1^{\circ \mathbf{c}_1} \circ (M_{2_-}^{\circ \mathbf{c}_2} \circ \cdots \circ M_{r_-}^{\circ \mathbf{c}_r})) \\
&\simeq \mathrm{hd}((S_r^{\circ \mathbf{c}_r} \circ S_{r-1}^{\circ \mathbf{c}_{r-1}} \circ \cdots \circ S_2^{\circ \mathbf{c}_2}) \circ (M_{2_-}^{\circ \mathbf{c}_2} \circ \cdots \circ M_{r_-}^{\circ \mathbf{c}_r}) \circ M_1^{\circ \mathbf{c}_1}) \\
&\simeq \cdots \simeq \mathrm{hd}(M_r^{\circ \mathbf{c}_r} \circ \cdots \circ M_1^{\circ \mathbf{c}_1}) \simeq \mathcal{M}^{\boldsymbol{i}}(\mathbf{c}),
\end{aligned}$$

which implies our assertion. □

As seen by the proof of the above proposition and Proposition 3.6, we have the following.

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Proposition 4.5. *The commuting family $\mathcal{M}^{\boldsymbol{i}}$ is a Laurent family. Moreover, for a simple module M, two vectors $\mathbf{a} = \mathrm{PBW}_{\boldsymbol{i}}(M)$ and $\mathbf{g} := \mathbf{g}_{\boldsymbol{i}}^R(M)$ are related by*

$$\mathbf{g}_k = \mathbf{a}_k - \mathbf{a}_{k_+}, \quad \mathbf{a}_k = \sum_{j;j \geqslant k,\ \boldsymbol{i}_j = \boldsymbol{i}_k} \mathbf{g}_j,$$

where $\mathbf{a}_{r+1} = 0$.

The following corollary can be proved by the same arguments in [KK19].

Corollary 4.6. *Let $\boldsymbol{i}$ be a reduced sequence of w.*

(i) *For a dual pair of simples (L, R) in $\widetilde{\mathscr{C}}_w$, we have*

$$\mathbf{g}_{\boldsymbol{i}}^R(L) + \mathbf{g}_{\boldsymbol{i}}^L(R) = 0.$$

(ii) *The maps $\mathbf{g}_{\boldsymbol{i}}^R, \mathbf{g}_{\boldsymbol{i}}^L \colon \mathrm{Irr}(\widetilde{\mathscr{C}}_w) \to \mathbb{Z}^{\mathsf{K}}$ are bijective.*

5. Skew-symmetric pairings

In this section, we study skew-symmetric pairings induced by the $\mathbb{Z}$-vectors associated with simple modules.

5.1 Skew-symmetric pairing associated with a quasi-Laurent family

Let $\mathscr{C}$ be a full monoidal subcategory of R-gmod stable by taking subquotients, extensions and grading shifts, and let $\mathcal{M} = \{M_j \mid j \in J\}$ be a quasi-Laurent family in $\mathscr{C}$ labeled by a finite index set J.

For $X, Y \in \mathrm{Irr}(\mathscr{C})$, let us define

$$\begin{aligned} \mathrm{G}_{\mathcal{M}}^R(X, Y) &:= \sum_{a,b \in J} (\mathbf{g}_{\mathcal{M}}^R(X))_a (\mathbf{g}_{\mathcal{M}}^R(Y))_b \, \Lambda(M_a, M_b) \quad \text{and} \\ \mathrm{G}_{\mathcal{M}}^L(X, Y) &:= \sum_{a,b \in J} (\mathbf{g}_{\mathcal{M}}^L(X))_a (\mathbf{g}_{\mathcal{M}}^L(Y))_b \, \Lambda(M_a, M_b). \end{aligned} \tag{5.1}$$

The following lemma immediately follows from Lemma 3.9.

Lemma 5.1. *For $M = \mathcal{M}(\mathbf{a})$ with $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^J$ and $X, Y \in \mathrm{Irr}(\mathscr{C})$, we have*

$$\mathrm{G}_{\mathcal{M}}^R(X \nabla M, Y) = \mathrm{G}_{\mathcal{M}}^R(X, Y) + \mathrm{G}_{\mathcal{M}}^R(M, Y), \quad \mathrm{G}_{\mathcal{M}}^R(X, Y) = -\mathrm{G}_{\mathcal{M}}^R(Y, X),$$
$$\mathrm{G}_{\mathcal{M}}^L(X \Delta M, Y) = \mathrm{G}_{\mathcal{M}}^L(X, Y) + \mathrm{G}_{\mathcal{M}}^L(M, Y), \quad \mathrm{G}_{\mathcal{M}}^L(X, Y) = -\mathrm{G}_{\mathcal{M}}^L(Y, X).$$

Proposition 5.2. *Let X be a simple module in $\mathscr{C}$. Then for any $\mathbf{c} \in \mathbb{Z}_{\geq 0}^J$, we have*

$$\text{(i) } \Lambda(X, \mathcal{M}(\mathbf{c})) = \mathrm{G}_{\mathcal{M}}^R(X, \mathcal{M}(\mathbf{c})) \quad \textit{and} \quad \text{(ii) } \Lambda(\mathcal{M}(\mathbf{c}), X) = \mathrm{G}_{\mathcal{M}}^L(\mathcal{M}(\mathbf{c}), X).$$

Proof. If X is also of the form $\mathcal{M}(\mathbf{d})$, it is obvious. Set $Y = \mathcal{M}(\mathbf{c})$.

(i) Note that there exist $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geqslant 0}^J$ such that $X \nabla \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b})$. Then we have

$$\begin{aligned} \Lambda(X, Y) + \Lambda(\mathcal{M}(\mathbf{a}), Y) = \Lambda(X \nabla \mathcal{M}(\mathbf{a}), Y) &= \mathrm{G}_{\mathcal{M}}^R(X \nabla \mathcal{M}(\mathbf{a}), Y) \\ &= \mathrm{G}_{\mathcal{M}}^R(X, Y) + \mathrm{G}_{\mathcal{M}}^R(\mathcal{M}(\mathbf{a}), Y). \end{aligned}$$

Since $\Lambda(\mathcal{M}(\mathbf{a}), Y) = \mathrm{G}_{\mathcal{M}}^R(\mathcal{M}(\mathbf{a}), Y)$, our assertion follows.

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(ii) Similarly, there exist $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{J}$ such that $\mathcal{M}(\mathbf{a}) \nabla X \simeq \mathcal{M}(\mathbf{b})$. Then we have

$$\Lambda(Y, \mathcal{M}(\mathbf{a})) + \Lambda(Y, X) = \Lambda(Y, \mathcal{M}(\mathbf{a}) \nabla X) = \mathrm{G}^{L}_{\mathcal{M}}(Y, \mathcal{M}(\mathbf{a}) \nabla X) = \mathrm{G}^{L}_{\mathcal{M}}(Y, \mathcal{M}(\mathbf{a})) + \mathrm{G}^{L}_{\mathcal{M}}(Y, X).$$

Then our assertion follows from the fact that $\Lambda(Y, \mathcal{M}(\mathbf{a})) = \mathrm{G}^{L}_{\mathcal{M}}(Y, \mathcal{M}(\mathbf{a}))$. □

PROPOSITION 5.3. *For any simple modules X, Y in $\mathscr{C}$ such that one of them is affreal, we have*

$$\mathrm{G}^{R}_{\mathcal{M}}(X, Y), \mathrm{G}^{L}_{\mathcal{M}}(X, Y) \leqslant \Lambda(X, Y).$$

Proof. Since the proofs are similar, we will consider the case of $\mathrm{G}^{R}_{\mathcal{M}}$. Take $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geq 0}^{J}$ such that $Y \nabla \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b})$. Then we have

$$\begin{aligned} \mathrm{G}^{R}_{\mathcal{M}}(X, Y) + \mathrm{G}^{R}_{\mathcal{M}}(X, \mathcal{M}(\mathbf{a})) &= \mathrm{G}^{R}_{\mathcal{M}}(X, Y \nabla \mathcal{M}(\mathbf{a})) \\ &= \Lambda(X, Y \nabla \mathcal{M}(\mathbf{a})) \leqslant \Lambda(X, Y) + \Lambda(X, \mathcal{M}(\mathbf{a})) \\ &= \Lambda(X, Y) + \mathrm{G}^{R}_{\mathcal{M}}(X, \mathcal{M}(\mathbf{a})), \end{aligned}$$

which yields our assertion. Here, the inequality follows from [KKKO18, Proposition 3.2.10]. □

PROPOSITION 5.4. *If simple modules X and Y in $\mathscr{C}$ commute and one of them is affreal, then we have*

$$\Lambda(X, Y) = \mathrm{G}^{R}_{\mathcal{M}}(X, Y) = \mathrm{G}^{L}_{\mathcal{M}}(X, Y).$$

Proof. Since the proofs are similar, we will only give the proof for $\mathrm{G}^{R}_{\mathcal{M}}$. By the preceding proposition, we have

$$\begin{aligned} 0 &= \big(\Lambda(X, Y) + \Lambda(Y, X)\big) - \big(\mathrm{G}^{R}_{\mathcal{M}}(X, Y) + \mathrm{G}^{R}_{\mathcal{M}}(Y, X)\big) \\ &= \big(\Lambda(X, Y) - \mathrm{G}^{R}_{\mathcal{M}}(X, Y)\big) + \big(\Lambda(Y, X) - \mathrm{G}^{R}_{\mathcal{M}}(Y, X)\big) \geq 0, \end{aligned}$$

which implies $\Lambda(X, Y) - \mathrm{G}^{R}_{\mathcal{M}}(X, Y) = 0$. □

Remark 5.5. The two invariants $\mathrm{G}^{R}_{\mathcal{M}}(X, Y)$ and $\mathrm{G}^{L}_{\mathcal{M}}(X, Y)$ are different in general and depend on the choice of $\mathcal{M}$.

Let w_0 be the longest element of finite type A_2. For a reduced sequence $\boldsymbol{i} = (1, 2, 1)$ of w_0, we have

$$\{S^{\boldsymbol{i}}_1 = \langle 1 \rangle, S^{\boldsymbol{i}}_2 = \langle 12 \rangle, S^{\boldsymbol{i}}_3 = \langle 2 \rangle\} \quad \text{and} \quad \mathcal{M}^{\boldsymbol{i}} = \{M^{\boldsymbol{i}}_1 = \langle 1 \rangle, M^{\boldsymbol{i}}_2 = \langle 12 \rangle, M^{\boldsymbol{i}}_3 = \langle 21 \rangle\}, \tag{5.2}$$

while

$$\{S^{\boldsymbol{j}}_1 = \langle 2 \rangle, S^{\boldsymbol{j}}_2 = \langle 21 \rangle, S^{\boldsymbol{j}}_3 = \langle 1 \rangle\} \quad \text{and} \quad \mathcal{M}^{\boldsymbol{j}} = \{M^{\boldsymbol{j}}_1 = \langle 2 \rangle, M^{\boldsymbol{j}}_2 = \langle 21 \rangle, M^{\boldsymbol{j}}_3 = \langle 12 \rangle\} \tag{5.3}$$

for the other reduced sequence $\boldsymbol{j} = (2, 1, 2)$ of w_0. Here $\langle k \rangle$ $(k = 1, 2)$ is a one-dimensional $R(\alpha_k)$-module, and $\langle 12 \rangle$ and $\langle 21 \rangle$ are one-dimensional $R(\alpha_1 + \alpha_2)$-modules (see [KKK18] for more details on these modules).

Since A_2 is symmetric, $\mathcal{M}' := \mu_1(\mathcal{M}^{\boldsymbol{i}})$ is also a Laurent family given as follows:

$$\mathcal{M}' = \{M'_1 = \mu_1(M^{\boldsymbol{i}}_1) \simeq \langle 2 \rangle, M'_2 = M^{\boldsymbol{i}}_2 \simeq \langle 12 \rangle, M'_3 = M^{\boldsymbol{i}}_3 \simeq \langle 21 \rangle\}.$$

Note that $\mathcal{M}' = \mathcal{M}^{\boldsymbol{j}}$ (up to an index permutation). We have

$$\begin{aligned} &\mathbf{g}^{R}_{\mathcal{M}^{i}}(\langle 1 \rangle) = \mathbf{g}^{L}_{\mathcal{M}^{i}}(\langle 1 \rangle) = (1, 0, 0), \quad \mathbf{g}^{R}_{\mathcal{M}^{i}}(\langle 2 \rangle) = (-1, 0, 1), \quad \mathbf{g}^{L}_{\mathcal{M}^{i}}(\langle 2 \rangle) = (-1, 1, 0), \\ &\mathbf{g}^{R}_{\mathcal{M}'}(\langle 1 \rangle) = (-1, 1, 0), \quad \mathbf{g}^{L}_{\mathcal{M}'}(\langle 1 \rangle) = (-1, 0, 1), \quad \mathbf{g}^{R}_{\mathcal{M}'}(\langle 2 \rangle) = \mathbf{g}^{L}_{\mathcal{M}'}(\langle 2 \rangle) = (1, 0, 0), \end{aligned}$$

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and

$$\Lambda(\langle 1\rangle,\langle 2\rangle)=\Lambda(\langle 2\rangle,\langle 1\rangle)=1,\quad \Lambda(\langle 1\rangle,\langle 12\rangle)=1,$$
$$\Lambda(\langle 1\rangle,\langle 21\rangle)=-1,\quad \Lambda(\langle 12\rangle,\langle 2\rangle)=1,\quad \Lambda(\langle 21\rangle,\langle 2\rangle)=-1.$$

Note that $\Lambda(X,X)=0$ for an affreal simple module X. Thus, we have

$$\mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}(\langle 1\rangle,\langle 2\rangle)=1\times\Lambda(\langle 1\rangle,\langle 21\rangle)=-1,\quad \mathrm{G}^R_{\mathcal{M}'}(\langle 1\rangle,\langle 2\rangle)=1\times\Lambda(\langle 12\rangle,\langle 2\rangle)=1,$$
$$\mathrm{G}^L_{\mathcal{M}^{\boldsymbol{i}}}(\langle 1\rangle,\langle 2\rangle)=1\times\Lambda(\langle 1\rangle,\langle 12\rangle)=1,\quad \mathrm{G}^L_{\mathcal{M}'}(\langle 1\rangle,\langle 2\rangle)=1\times\Lambda(\langle 21\rangle,\langle 2\rangle)=-1.$$

Thus, for a non-commuting pair of simple modules (X,Y) in $\mathscr{C}$, the $\mathbb{Z}$-values $\mathrm{G}^R_{\mathcal{M}}(X,Y)$ and $\mathrm{G}^L_{\mathcal{M}}(X,Y)$ *do depend on* the choice of a quasi-Laurent commuting family $\mathcal{M}$.

5.2 Skew-symmetric pairing associated with the GLS cluster

Let $w\in\mathsf{W}$ and $\boldsymbol{i}=(i_1,\dots,i_r)$ a reduced sequence of w. Let $\mathcal{M}^{\boldsymbol{i}}$ be the associated GLS cluster. For such a Laurent family, we can define $\mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}$ in terms of PBW decompositions.

We define a skew-symmetric $\mathbb{Z}$-valued map $\lambda^{\boldsymbol{i}}\colon[1,r]\times[1,r]\to\mathbb{Z}$ by

$$\lambda^{\boldsymbol{i}}_{a,b}:=(-1)^{\delta(a>b)}\delta(a\neq b)(\beta^{\boldsymbol{i}}_a,\beta^{\boldsymbol{i}}_b) \tag{5.4}$$

for $1\leqslant a,b\leqslant r$.

Remark 5.6. The skew-symmetric map $\lambda^{\boldsymbol{i}}$ in (5.4) is known when $\mathfrak{g}$ is of finite type and $\boldsymbol{i}$ is *adapted to a Q-datum* (see [HL15, Proposition 3.2], [FO21, Proposition 5.21] and [KaOh23, Theorem 5.4]).

Let us recall the notion of $\boldsymbol{i}$-box and an affreal simple module $M^{\boldsymbol{i}}[a,b]$ in $\mathscr{C}_w$ for an $\boldsymbol{i}$-box $[a,b]$, which are introduced in [KKOP24b].

(a) For $1\leqslant a\leqslant b\leqslant r$ such that $i_a=i_b$, we call an interval $[a,b]$ an *$\boldsymbol{i}$-box*.
(b) For an $\boldsymbol{i}$-box $[a,b]$, we set $[a,b]_{\boldsymbol{i}}:=\{u\mid a\leqslant u\leqslant b,\ i_a=i_u\}$.
(c) For an $\boldsymbol{i}$-box $[a,b]$, we set

$$M^{\boldsymbol{i}}[a,b]:=M(w^{\boldsymbol{i}}_{\leqslant b}\varpi_{i_a},w^{\boldsymbol{i}}_{<a}\varpi_{i_a})\simeq\mathrm{hd}\Big(\overset{\to}{\underset{u\in[a,b]_{\boldsymbol{i}}}{\circ}}S^{\boldsymbol{i}}_u\Big)$$
$$\simeq S^{\boldsymbol{i}}_b\nabla M^{\boldsymbol{i}}[a,b_-]\simeq M^{\boldsymbol{i}}[a_+,b]\nabla S^{\boldsymbol{i}}_a,$$

up to grading shifts. In particular, $M^{\boldsymbol{i}}_k=M^{\boldsymbol{i}}[k_{\min},k]$ and $S^{\boldsymbol{i}}_k=M^{\boldsymbol{i}}[k,k]$.

Note that $M^{\boldsymbol{i}}[a,b]$ is an affreal simple module in $\mathscr{C}_w$.

PROPOSITION 5.7. *For $\boldsymbol{i}$-boxes $[x,y]$ and $[x',y']$ in an interval $[1,r]$, assume that*

$$\text{(a)}\quad x>x'_-\quad\text{or}\quad\text{(b)}\quad y_+>y'. \tag{5.5}$$

Then we have

$$\Lambda(M^{\boldsymbol{i}}[x,y],M^{\boldsymbol{i}}[x',y'])=\sum_{u\in[x,y]_{\boldsymbol{i}},\ v\in[x',y']_{\boldsymbol{i}}}\lambda_{u,v}. \tag{5.6}$$

Proof. Since the proof is similar, we shall give only the proof of case (a). Let us divide into sub-cases as below.

(i) $[x=y>x'_-]$ If $x>x'$, we have

$$\Lambda(S^{\boldsymbol{i}}_x,M^{\boldsymbol{i}}[x',y'])=\Lambda(S^{\boldsymbol{i}}_x,M^{\boldsymbol{i}}[x'_+,y']\nabla S^{\boldsymbol{i}}_{x'})$$
$$\underset{(1)}{=}\Lambda(S^{\boldsymbol{i}}_x,M^{\boldsymbol{i}}[x'_+,y'])+\Lambda(S^{\boldsymbol{i}}_x,S^{\boldsymbol{i}}_{x'})=\Lambda(S^{\boldsymbol{i}}_x,M^{\boldsymbol{i}}[x'_+,y'])+\lambda^{\boldsymbol{i}}_{x,x'}.$$

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Here $\underset{(1)}{=}$ holds by [KKOP23, Proposition 2.12] and the fact that $(S^{\boldsymbol{i}}_x, S^{\boldsymbol{i}}_{x'})$ is an unmixed pair. Then by the induction hypothesis on $|[x', y']_{\boldsymbol{i}}|$, we have

$$\Lambda(S^{\boldsymbol{i}}_x, M^{\boldsymbol{i}}[x', y']) = \lambda^{\boldsymbol{i}}_{x,x'} + \sum_{v \in [x'_+, y']_{\boldsymbol{i}}} \lambda^{\boldsymbol{i}}_{x,v} = \sum_{v \in [x', y']_{\boldsymbol{i}}} \lambda^{\boldsymbol{i}}_{x,v},$$

as we desired.

Now, the remainder of case (i) can be described as follows:

$$x'_- < x = y \leqslant x' \leqslant y'.$$

Since $S^{\boldsymbol{i}}_x$ commutes with $M^{\boldsymbol{i}}[x', y']$ and $M^{\boldsymbol{i}}[x', y'_-]$ by [KKOP21, Proposition 3.27],

$$\begin{aligned}\Lambda(S^{\boldsymbol{i}}_x, M^{\boldsymbol{i}}[x', y']) &= -\Lambda(M^{\boldsymbol{i}}[x', y'], S^{\boldsymbol{i}}_x) = -\Lambda(S^{\boldsymbol{i}}_{y'} \nabla M^{\boldsymbol{i}}[x', y'_-], S^{\boldsymbol{i}}_x) \\ &= -\Lambda(S^{\boldsymbol{i}}_{y'}, S^{\boldsymbol{i}}_x) - \Lambda(M^{\boldsymbol{i}}[x', y'_-], S^{\boldsymbol{i}}_x) \\ &= (\beta_{y'}, \beta_x) + \Lambda(S^{\boldsymbol{i}}_x, M^{\boldsymbol{i}}[x', y'_-]) = \lambda^{\boldsymbol{i}}_{x,y'} + \Lambda(S^{\boldsymbol{i}}_x, M^{\boldsymbol{i}}[x', y'_-]),\end{aligned}$$

then our assertion follows from the induction hypothesis on $|\,[x', y']_{\boldsymbol{i}}|$.

(ii) $[x < y]$ Assume first that $y > y'$. Then we have

$$\begin{aligned}\Lambda(M^{\boldsymbol{i}}[x, y], M^{\boldsymbol{i}}[x', y']) &= \Lambda(S^{\boldsymbol{i}}_y \nabla M^{\boldsymbol{i}}[x, y_-], M^{\boldsymbol{i}}[x', y']) \\ &= \Lambda(S^{\boldsymbol{i}}_y, M^{\boldsymbol{i}}[x', y']) + \Lambda(M^{\boldsymbol{i}}[x, y_-], M^{\boldsymbol{i}}[x', y']).\end{aligned}$$

Note that $(S^{\boldsymbol{i}}_y, M^{\boldsymbol{i}}[x', y'])$ is an unmixed pair. Then, by induction on $|[x, y]_{\boldsymbol{i}}|$, we have

$$\begin{aligned}\Lambda(M^{\boldsymbol{i}}[x, y], M^{\boldsymbol{i}}[x', y']) &= \Lambda(S^{\boldsymbol{i}}_y \nabla M^{\boldsymbol{i}}[x, y_-], M^{\boldsymbol{i}}[x', y']) \\ &= \sum_{v \in [x', y']_{\boldsymbol{i}}} \lambda^{\boldsymbol{i}}_{y,v} + \sum_{u \in [x, y_-]_{\boldsymbol{i}};\ v \in [x', y']_{\boldsymbol{i}}} \lambda^{\boldsymbol{i}}_{u,v},\end{aligned}$$

which yields our assertion for this case.

Now let us assume that $y \leqslant y'$. Then we have

$$x'_- < x < y \leqslant y'.$$

Then for any $u \in [x, y]_{\boldsymbol{i}}$, $S^{\boldsymbol{i}}_u$ commutes with $M^{\boldsymbol{i}}[x', y']$ by [KKOP21, Proposition 3.27]. By [KKKO18, Proposition 3.2.13], we have

$$\Lambda(M^{\boldsymbol{i}}[x, y], M^{\boldsymbol{i}}[x', y']) = \sum_{u \in [x, y]_{\boldsymbol{i}}} \Lambda(S^{\boldsymbol{i}}_u, M^{\boldsymbol{i}}[x', y']).$$

Then our assertion follows from case (i). □

We say that $\boldsymbol{i}$-boxes $[a_1, b_1]$ and $[a_2, b_2]$ *commute* if we have either

$$(a_1)_- < a_2 \leqslant b_2 < (b_1)_+ \quad \text{or} \quad (a_2)_- < a_1 \leqslant b_1 < (b_2)_+.$$

The following corollary is proved in [KKOP24b, Theorem 4.21] in the quantum affine case.

COROLLARY 5.8. *For commuting $\boldsymbol{i}$-boxes $[a_1, b_1]$ and $[a_2, b_2]$, the modules $M^{\boldsymbol{i}}[a_1, b_1]$ and $M^{\boldsymbol{i}}[a_2, b_2]$ commute.*

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Proof. By Proposition 5.7, we have

$$\Lambda(M^{\boldsymbol{i}}[a_1,b_1],M^{\boldsymbol{i}}[a_2,b_2])=\sum_{\substack{u\in[a_1,b_1]_{\boldsymbol{i}}\\ v\in[a_2,b_2]_{\boldsymbol{i}}}}\lambda^{\boldsymbol{i}}_{u,v}=-\Lambda(M^{\boldsymbol{i}}[a_2,b_2],M^{\boldsymbol{i}}[a_1,b_1]),$$

which implies $\mathfrak{d}(M^{\boldsymbol{i}}[a_1,b_1],M^{\boldsymbol{i}}[a_2,b_2])=0$. Thus, our assertion follows from Proposition 2.7(v). □

Proposition 5.9. *For a commuting pair* $(M^{\boldsymbol{i}}[x,y],M^{\boldsymbol{i}}[x',y'])$, (5.6) *holds.*

Proof. If the $\boldsymbol{i}$-boxes $[x,y]$ and $[x',y']$ satisfy (5.5), our assertion holds. Thus, it is enough to consider when $x\leqslant x'_-$ and $y_+\leqslant y'$. Since they commute,

$$\Lambda(M^{\boldsymbol{i}}[x,y],M^{\boldsymbol{i}}[x',y'])=-\Lambda(M^{\boldsymbol{i}}[x',y'],M^{\boldsymbol{i}}[x,y]).$$

If $x'>x_-$ or $y'_+>y$, Proposition 5.7 says that

$$\Lambda(M^{\boldsymbol{i}}[x,y],M^{\boldsymbol{i}}[x',y'])=-\sum_{u\in[x,y]_{\boldsymbol{i}};\ v\in[x',y']_{\boldsymbol{i}}}\lambda^{\boldsymbol{i}}_{v,u}=\sum_{u\in[x,y]_{\boldsymbol{i}};\ v\in[x',y']_{\boldsymbol{i}}}\lambda^{\boldsymbol{i}}_{u,v},$$

which implies the assertion. Thus, we may assume that $x'\leqslant x_-$. However, in this case, we have

$$x'\leqslant x_-\leqslant x\leqslant x'_-,$$

which yields a contradiction. □

Let us define the skew-symmetric pairing $\mathrm{L}_{\boldsymbol{i}}$ on $\mathrm{Irr}(\mathscr{C}_w)$ as follows:

$$\mathrm{L}_{\boldsymbol{i}}(X,Y):=\sum_{1\leqslant a,b\leqslant r}(\mathrm{PBW}_{\boldsymbol{i}}(X))_a(\mathrm{PBW}_{\boldsymbol{i}}(Y))_b\,\lambda^{\boldsymbol{i}}_{a,b}. \tag{5.7}$$

The following lemma follows from Lemma 4.1 and (5.7).

Lemma 5.10. *For* $M=\mathcal{M}^{\boldsymbol{i}}(\mathbf{a})$ *with* $\mathbf{a}\in\mathbb{Z}^{\mathsf{K}}_{\geq 0}$, *we have*

$$\mathrm{L}_{\boldsymbol{i}}(X\nabla M,Y)=\mathrm{L}_{\boldsymbol{i}}(X,Y)+\mathrm{L}_{\boldsymbol{i}}(M,Y)\quad\textit{and}\quad \mathrm{L}_{\boldsymbol{i}}(X,Y)=-\mathrm{L}_{\boldsymbol{i}}(Y,X).$$

Proposition 5.11. *For any simple* X,Y *in* $\mathscr{C}_w$, *we have*

$$\mathrm{L}_{\boldsymbol{i}}(X,Y)=\mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}(X,Y).$$

Proof. Let $\mathcal{S}$ be the set of simple modules Y in $\mathscr{C}_w$ such that $\mathrm{L}_{\boldsymbol{i}}(X,Y)=\mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}(X,Y)$ for any simple $X\in\mathscr{C}_w$, and let $\mathcal{S}'$ be the set of simple modules Y in $\mathscr{C}_w$ such that $\mathrm{L}_{\boldsymbol{i}}(\mathcal{M}^{\boldsymbol{i}}(\mathbf{a}),Y)=\mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}(\mathcal{M}^{\boldsymbol{i}}(\mathbf{a}),Y)$ for any $\mathbf{a}\in\mathbb{Z}^{\mathsf{K}}_{\geq 0}$. By Proposition 5.9, we have

$$\mathrm{L}_{\boldsymbol{i}}(M^{\boldsymbol{i}}_s,M^{\boldsymbol{i}}_t)=\Lambda(M^{\boldsymbol{i}}_s,M^{\boldsymbol{i}}_t)\quad\text{for any } s,t\in\mathsf{K}.$$

Thus, we have $\mathcal{M}^{\boldsymbol{i}}(\mathbf{a})\in\mathcal{S}'$ by Lemma 5.10. Now, let us show $\mathcal{S}'\subset\mathcal{S}$. Let $Y\in\mathcal{S}'$. For any simple X, there exist $\mathbf{a},\mathbf{b}\in\mathbb{Z}^{\mathsf{K}}_{\geq 0}$ such that $X\nabla\mathcal{M}^{\boldsymbol{i}}(\mathbf{a})\simeq\mathcal{M}^{\boldsymbol{i}}(\mathbf{b})$. Hence, we have

$$\begin{aligned}\mathrm{L}_{\boldsymbol{i}}(X,Y)+\mathrm{L}_{\boldsymbol{i}}(\mathcal{M}^{\boldsymbol{i}}(\mathbf{a}),Y)&\underset{(1)}{=}\mathrm{L}_{\boldsymbol{i}}(\mathcal{M}^{\boldsymbol{i}}(\mathbf{b}),Y)=\mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}(\mathcal{M}^{\boldsymbol{i}}(\mathbf{b}),Y)\\ &\underset{(2)}{=}\mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}(X,Y)+\mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}(\mathcal{M}^{\boldsymbol{i}}(\mathbf{a}),Y)=\mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}(X,Y)+\mathrm{L}_{\boldsymbol{i}}(\mathcal{M}^{\boldsymbol{i}}(\mathbf{a}),Y).\end{aligned}$$

Here $\underset{(1)}{=}$ follows from Lemma 5.10 and $\underset{(2)}{=}$ follows from Lemma 5.1. Hence, we have $\mathrm{L}_{\boldsymbol{i}}(X,Y)=\mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}(X,Y)$. Thus, we have proved $\mathcal{S}'\subset\mathcal{S}$.

Since $\mathcal{M}^{\boldsymbol{i}}(\mathbf{a}) \in \mathcal{S}'$, we have $\mathcal{M}^{\boldsymbol{i}}(\mathbf{a}) \in \mathcal{S}$, which implies that

$$\mathrm{L}_{\boldsymbol{i}}(Y, \mathcal{M}^{\boldsymbol{i}}(\mathbf{a})) = \mathrm{G}^R_{\mathcal{M}^{\boldsymbol{i}}}(Y, \mathcal{M}^{\boldsymbol{i}}(\mathbf{a}))$$

for any simple Y. Hence any simple Y belongs to $\mathcal{S}'$ and hence to $\mathcal{S}$. □

5.3 Degree and codegree

In this subsection, we see the relationship between $\mathbf{g}^R_{\mathcal{M}}(X)$ (respectively, $\mathbf{g}^L_{\mathcal{M}}(X)$) and the degree (respectively, codegree) in the (quantum) cluster algebra theory. For a commuting family $\mathcal{M} = \{M_j \mid j \in J\}$ labeled by a finite index set J, let us recall the preorder $\preccurlyeq_{\mathcal{M}}$ on $\mathbb{Z}^J_{\geqslant 0}$ given in [KK19, § 3.3] (see also [Qin17, Definition 3.1.1]):

$$\mathbf{b}' \preccurlyeq_{\mathcal{M}} \mathbf{b} \quad \text{if and only if} \quad \begin{aligned} &(1)\ \mathrm{wt}(\mathcal{M}(\mathbf{b})) = \mathrm{wt}(\mathcal{M}(\mathbf{b}')), \\ &(2)\ \Lambda(\mathcal{M}(\mathbf{b}'), M_j) \leqslant \Lambda(\mathcal{M}(\mathbf{b}), M_j) \text{ for all } j \in J. \end{aligned}$$

The preorder $\preccurlyeq_{\mathcal{M}}$ can be extended to the one on $\mathbb{Z}^J$ as follows: for $\mathbf{b}, \mathbf{b}' \in \mathbb{Z}^J$,

$$\mathbf{b}' \preccurlyeq_{\mathcal{M}} \mathbf{b} \text{ if } \mathbf{b}' + \mathbf{a} \preccurlyeq_{\mathcal{M}} \mathbf{b} + \mathbf{a} \text{ for some } \mathbf{a} \in \mathbb{Z}^J_{\geqslant 0} \text{ such that } \mathbf{b} + \mathbf{a}, \mathbf{b}' + \mathbf{a} \in \mathbb{Z}^J_{\geqslant 0}.$$

We write $\mathbf{b}' \prec_{\mathcal{M}} \mathbf{b}$, if $\mathbf{b}' \preccurlyeq_{\mathcal{M}} \mathbf{b}$ holds but $\mathbf{b} \preccurlyeq_{\mathcal{M}} \mathbf{b}'$ does not hold. Hence, $\mathbf{b}' \prec_{\mathcal{M}} \mathbf{b}$ if and only if $\mathbf{b}' \preccurlyeq_{\mathcal{M}} \mathbf{b}$ and there exists $j \in J$ such that $\Lambda(\mathcal{M}(\mathbf{b}), M_j) < \Lambda(\mathcal{M}(\mathbf{b}'), M_j)$.

Lemma 5.12 (cf. [KK19, Lemma 3.6]). *Let X be a simple module and $\mathcal{M} = \{M_j \mid j \in J\}$ be a quasi-Laurent family in $\mathscr{C}$. Let $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}$, S, $c(s) \in \mathbb{Z}$ and $\mathbf{b}(s) \in \mathbb{Z}^J_{\geqslant 0}$ ($s \in \mathsf{S}$) be as in (3.1). Then we have the following.*

(i) *There exists a unique $s_0 \in \mathsf{S}$ such that $X \nabla \mathcal{M}(\mathbf{a}) \simeq q^{c(s_0)}\mathcal{M}(\mathbf{b}(s_0))$. Moreover, we have $\mathbf{b}(s) \prec_{\mathcal{M}} \mathbf{b}(s_0)$ for any $s \in \mathsf{S} \setminus \{s_0\}$.*
(ii) *There exists a unique $s_1 \in \mathsf{S}$ such that $X \Delta \mathcal{M}(\mathbf{a}) \simeq q^{c(s_1)}\mathcal{M}(\mathbf{b}(s_1))$. Moreover, we have $\mathbf{b}(s_1) \prec_{\mathcal{M}} \mathbf{b}(s)$ for any $s \in \mathsf{S} \setminus \{s_1\}$.*
(iii) *If $s_0 = s_1$, then $\mathsf{S} = \{s_0\}$ and $X \circ \mathcal{M}(\mathbf{a}) \simeq \mathcal{M}(\mathbf{b}(s_0))$.*
(iv) *If $s_0 \neq s_1$ and there exists no $\mathbf{c} \in \mathbb{Z}^{\mathsf{K}}$ such that $\mathbf{b}(s_1) - \mathbf{a} \prec_{\mathcal{M}} \mathbf{c} \prec_{\mathcal{M}} \mathbf{b}(s_0) - \mathbf{a}$, then*

$$[X \circ \mathcal{M}(\mathbf{a})] = [X \nabla \mathcal{M}(\mathbf{a})] + [X \Delta \mathcal{M}(\mathbf{a})] \quad \text{in } K(\mathscr{C}_w).$$

Proof. It follows from Proposition 2.7 and (3.1). □

The following proposition is proved for symmetric quiver Hecke algebras and can be extended to general quiver Hecke algebras using almost the same argument:

Proposition 5.13 [KK19, Proposition 3.3]. *For a monoidal cluster $\mathcal{M} = \{M_k \mid k \in \mathsf{K}\}$ associated with a quantum seed $(\{X_k\}_{k \in \mathsf{K}}, L, \widetilde{B})$,*

$$\mathbf{b}' \preccurlyeq_{\mathcal{M}} \mathbf{b} \quad \text{if and only if} \quad \mathbf{b} - \mathbf{b}' = \widetilde{B}\underline{v} \text{ for some } \underline{v} \in \mathbb{Z}^{\mathsf{K}_{\mathrm{ex}}}_{\geq 0}.$$

In particular, the relation $\preccurlyeq_{\mathcal{M}}$ is an order on $\mathbb{Z}^{\mathsf{K}}$.

Corollary 5.14. *Let $\mathcal{M} = \{M_k \mid k \in \mathsf{K}\}$ be a monoidal cluster associated with a quantum seed $\mathcal{S} = (\{X_k\}_{k \in \mathsf{K}}, L, B)$. Then $[X]$ is $\mathcal{T}(L)$-pointed and $\mathcal{T}(L)$-copointed for any simple module $X \in \mathscr{C}_w$.*

Remark 5.15. For a monoidal cluster $\mathcal{M}$ associated with a quantum seed $\mathcal{S}$ and a simple module $M \in \mathscr{C}$, the above corollary says that $\mathbf{g}^R_{\mathcal{M}}(M)$ and $\mathbf{g}^L_{\mathcal{M}}(M)$ coincide with the degree and codegree of $[M] \in \mathcal{K}_{\mathbb{A}}(\mathscr{C}) \simeq \mathscr{A}_{\mathbb{A}}(\mathcal{S})$, respectively.

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Conflicts of interest
None.

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