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A BOUND ON THE SCHUR MULTIPLIER OF A PRIME-POWER GROUP

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For Bernhard Neumann on his 90th birthday

The paper improves on an upper bound for the order of the Schur multiplier of a finite p-group given by Wiegold in 1969. The new bound is applied to the problem of classifying p-groups according to the size of their Schur multipliers.

In a paper [**6**] dedicated to B.H. Neumann's sixtieth birthday, the second author used results of [**5**] to show that a d-generator group G of prime-power order p^n has Schur multiplier $M(G)$ of order at most $p^{(d-1)(2n-d)/2}$. In this article we use results of the first author [**3**] to obtain a reduction of this bound. The reduced bound is then applied to the problem of classifying p-groups according to the orders of their Schur multipliers, at least in the case where the multipliers are large.

We begin by blending parts (i) and (ii) of Proposition 5 in [**3**] to produce the following proposition.

PROPOSITION 1. [**3**] *Let G be a finite p-group with centre $Z(G)$ and lower central series $1 = \gamma_{c+1}G \trianglelefteq \gamma_c G \trianglelefteq \cdots \trianglelefteq \gamma_1 G = G$. Set $\overline{G} = G/Z(G)$ and consider the homomorphism*

$$\Psi\colon \overline{G}^{ab} \otimes \overline{G}^{ab} \otimes \overline{G}^{ab} \longrightarrow \frac{\gamma_2 G}{\gamma_3 G} \otimes \overline{G}^{ab},\ \overline{x} \otimes \overline{y} \otimes \overline{z} \mapsto [x,y]_\gamma \otimes \overline{z} + [y,z]_\gamma \otimes \overline{x} + [z,x]_\gamma \otimes \overline{y}.$$

Here $\overline{x}$ denotes the image in $\overline{G}$ of the element $x \in G$, and $[x,y]_\gamma$ denotes the image in $\gamma_2 G/\gamma_3 G$ of the commutator $[x,y] \in G$. Then

$$(1)\quad |M(G)|\,|\gamma_2 G|\,|\mathrm{image}(\Psi)| \leqslant |M(G^{ab})| \left|\frac{\gamma_2 G}{\gamma_3 G} \otimes \overline{G}^{ab}\right| \left|\frac{\gamma_3 G}{\gamma_4 G} \otimes \overline{G}^{ab}\right| \cdots |\gamma_c G \otimes \overline{G}^{ab}|.$$

Proposition 1 leads to the following numerical bound on the order of the Schur multiplier.

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THEOREM 2. *Let G be a d-generator group of order p^n. Suppose that the Abelianisation G^{ab} has order p^m and exponent p^e, and that the central quotient $G/Z(G)$ is a δ-generator group. Then*

$$|M(G)| \leqslant p^{d(m-e)/2\ +\ (\delta-1)(n-m)\ -\ \max\{0,\delta-2\}}. \tag{2}$$

Since $e \geqslant m/d$ and $d \geqslant \delta$, inequality (2) implies

$$|M(G)| \leqslant p^{(d-1)(2n-m)/2}. \tag{3}$$

Bound (3) is attained if $G = C_{p^e} \times C_{p^e} \times \cdots \times C_{p^e}$.

PROOF: Recall that $M(G^{ab})$ is isomorphic to the exterior square $G^{ab} \wedge G^{ab}$ of Abelian groups [2]. Suppose that $G^{ab} \cong C_{p^{n_1}} \times C_{p^{n_2}} \times \cdots \times C_{p^{n_d}}$ where $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_d = e$ and $n_1 + n_2 + \cdots + n_d =$ m. Then $M(G)$ has order p^a, where

$$\begin{aligned} a &= (d-1)n_1 + (d-2)n_2 + \cdots + n_{d-1} \\ &= d(n_1 + n_2 + \cdots + n_{d-1}) - \big(n_1 + 2n_2 + \cdots + (d-1)n_{d-1}\big) \\ &= d(m-e) - \big(n_1 + 2n_2 + \cdots + (d-1)n_{d-1}\big) \\ &\leqslant d(m-e) - \frac{m-e}{d-1}\big(1 + 2 + \cdots + (d-1)\big) \\ &= d(m-e)/2. \end{aligned} \tag{4}$$

Since the tensor product is distributive with respect to direct sums, we have

$$\left|\frac{\gamma_2 G}{\gamma_3 G} \otimes \overline{G}^{ab}\right|\left|\frac{\gamma_3 G}{\gamma_4 G} \otimes \overline{G}^{ab}\right| \cdots |\gamma_c G \otimes \overline{G}^{ab}| = \left|\left(\frac{\gamma_2 G}{\gamma_3 G} \oplus \cdots \oplus \gamma_c G\right) \otimes \overline{G}^{ab}\right| \leqslant p^{\delta(n-m)}. \tag{5}$$

Suppose next that $\delta \geqslant 3$. Since $\gamma_2 G/\gamma_3 G$ is non-trivial, we can choose a generating set $\{\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_\delta\}$ for $G/Z(G)$ such that $[x_1, x_2]_\gamma$ is a non-trivial element of $\gamma_2 G/\gamma_3 G$ and indeed is not a pth power of any element there since pth powers lie in the Frattini subgroup. We shall establish now the critical point of the proof, namely that the $\delta - 2$ elements

$$\Psi(\overline{x}_1 \otimes \overline{x}_2 \otimes \overline{x}_3), \Psi(\overline{x}_1 \otimes \overline{x}_2 \otimes \overline{x}_4), \cdots, \Psi(\overline{x}_1 \otimes \overline{x}_2 \otimes \overline{x}_\delta) \tag{6}$$

constitute $\delta - 2$ linearly independent elements in the Abelian group $\gamma_2 G/\gamma_3 G \otimes \overline{G}^{ab}$. Setting $A := \gamma_2 G/\gamma_3 G$ temporarily, we see that

$$A \otimes \overline{G}^{ab} \cong \big(A \otimes \langle \overline{x}_1 \rangle\big) \times \cdots \times \big(A \otimes \langle \overline{x}_\delta \rangle\big),$$

and that $\Psi(\bar{x}_1 \otimes \bar{x}_2 \otimes \bar{x}_i)$ is the only one of the $\delta - 2$ elements listed in (6) above to have a non-trivial projection in $A \otimes \langle \bar{x}_i \rangle$, so that these $\delta - 2$ elements are indeed linearly independent and we have

$$|\text{image}(\Psi)| \geqslant p^{\delta-2}. \tag{7}$$

Inequality (2) is obtained by substituting inequalities (4), (5) and (7) into (1). ∎

The methods in [3] show that the quantity $|\text{image}(\Psi)|$ could be replaced by a (larger) product $|\text{image}(\Psi)||\text{image}(\Psi_3)| \cdots |\text{image}(\Psi_c)|$ in inequality (1), thus leading to an improvement in the bounds of Theorem 2.

On substituting the inequalities $d \leqslant m \leqslant n$ into (3) we obtain a well-known result of J.A. Green, namely that $|M(G)| \leqslant p^{n(n-1)/2}$ for any group G of order p^n. In other words, for any group G of order p^n there is an integer $t \geqslant 0$ such that $|M(G)| = p^{n(n-1)/2-t}$. Those finite p-groups with $t = 0, 1$ have been classified by Berkovich [1]. The classification has been extended to $t = 2$ by Zhou [7], and to $t = 3$ by the first author [4]. In light of this work we make the following formal definition.

DEFINITION: Let the *corank* of a finite p-group G be the integer $t = \text{corank}(G)$ for which $|M(G)| = p^{n(n-1)/2-t}$ with $n = \log_p |G|$.

The known classifications of finite p-groups by corank are summarised in the following table. All groups G with $\text{corank}(G) \leqslant 3$ are listed. In the table C_{p^i} denotes the cyclic group of order p^i, D denotes the dihedral group of order 8, Q denotes the quaternion group of order 8, E_1 denotes the extraspecial group of order p^3 with odd exponent p, and E_2 denotes the extraspecial group of order p^3 with odd exponent p^2.

corank (G)	$p = 2$	$p =$ odd prime
$t = 0$	$(C_2)^k,\ k \geqslant 1$	$(C_p)^k,\ k \geqslant 1$
$t = 1$	C_4	$C_{p^2},\ E_1$
$t = 2$	$C_2 \times C_4,\ D$	$C_p \times C_{p^2},\ E_1$
$t = 3$	$C_8,\ C_2 \times C_2 \times C_4,\ Q,\ D \times C_2$	$C_{p^3},\ C_p \times C_p \times C_{p^2},\ E_2,\ E_1 \times C_p \times C_p$

In [4] it is shown how the information in this table can be derived from a bound on the Schur multiplier due to Gaschütz, Neubüser and Yen [5]. Since inequality (2) is slightly sharper than the bound of Gaschütz *et alia*, it too has ramifications for the classification of p-groups by corank. Some of these are listed in the following proposition. An interesting corollary to the proposition is that, for any given prime p and integer $t \geqslant 1$, there are only finitely many p-groups G with $\text{corank}(G) = t$.

PROPOSITION 3. *Let G be a non-cyclic d-generator group of order p^n, with commutator subgroup $[G, G]$ of order p^c, and Frattini subgroup $[G, G]G^p$ of order p^a.*

Suppose that the Abelianisation G^{ab} *has exponent* p^e, *and that the central quotient* $G/Z(G)$ *is a* δ*-generator group. Furthermore, suppose that* corank$(G) = t$ *where* $t \geqslant 1$. *Then:*

(i) $0 \leqslant c \leqslant t$.

(ii) $c \leqslant a \leqslant \sqrt{2t - c}$.

(iii) $2 \leqslant d \leqslant \big(2(t+a) - a^2 - 3c\big)/(a-c)$ *whenever* $a \neq c$.

(iv) $2 \leqslant d \leqslant t + 2 - \big(a^2 + a\big)/2$ *whenever* $a = c$.

(v) $\big(a^2 - 2a + (d+3)c + ad - 2(t+1)\big)/(2c-1) \leqslant \delta \leqslant d$ *whenever* $c \neq 0$.

(vi) $1 \leqslant e \leqslant \big(2t - 2(d+1-\delta)c - d(a-c-1) - \big(a^2 - a\big) - 2\max\{0, \delta - 2\}\big)/d$.

(vii) $\big(1 + \sqrt{1+4t}\big)/2 \leqslant n \leqslant \big(2t + a(c+e) + 2(\delta - 1)c - 2\max\{0, \delta-2\}\big)/(c+e+a-1)$.

Proof: Note that $a \geqslant c \geqslant 0$, $d \geqslant \delta \geqslant 0$, $e \geqslant m/d \geqslant 1$ and $d \geqslant 2$. On substituting $n = a + d, m = a + d - c$ into (2) we obtain

$$a^2 - a \leqslant 2\big(t - (d+1-\delta)c\big) + d(c + 1 - a - e) - 2\max\{0, \delta - 2\}. \tag{8}$$

We derive the inequality

$$a^2 - a \leqslant 2(t-c) - (a-c)(d-1) - 2\max\{0, \delta - 2\} \tag{9}$$

from (8) by substituting $d \geqslant \delta$, $e \geqslant m/d$. Since $a^2 - a \geqslant 0$, inequality (9) implies (i). Since $d - 1 \geqslant 1$, inequality (9) implies $a^2 \leqslant 2t - c$, from which we deduce (ii). We also deduce (iii) from (9). On substituting $a = c$, $e = 1$, $\delta \geqslant 2$ into (8), we obtain

$$d + (d - \delta)(a-1) \leqslant t + 2 - \frac{a^2 + a}{2}. \tag{10}$$

The inequality $\delta \geqslant 2$ corresponds to the fact [2] that no non-trivial cyclic group is itself a central quotient. Inequality (10) implies (iv). Inequality (9) implies (v), the condition $c \neq 0$ being used to obtain $\delta \geqslant 2$. Inequality (8) implies (vi) and the right-hand inequality of (vii). The left-hand inequality of (vii) follows immediately from the definition of corank. ∎

Corollary 4.

(i) *For each prime* p *and integer* $t \geqslant 0$ *there exists at least one* p*-group with corank equal to* t.

(ii) *For each prime* p *and integer* $t \geqslant 1$ *there are only finitely many* p*-groups with corank equal to* t.

Proof: The formula for the Schur multiplier of a direct product [2], namely $M(G \times H) \cong M(G) \oplus M(H) \oplus \big(G^{ab} \otimes H^{ab}\big)$, can be used to show that the Abelian group $(C_p)^{t-1} \times C_{p^2}$ has corank equal to t for each $t \geqslant 1$. Any elementary Abelian group has corank equal to 0. This proves part (i).

Suppose that G is a p-group with $\operatorname{corank}(G) = t \geqslant 1$. Proposition 3 implies that the order of G is bounded by a number, say $f(t)$, that depends only on t. There are only finitely many groups of order at most $f(t)$. This proves part (ii). ▯

The following modification to the definition of corank provides a single numerical parameter for measuring how far a p-group 'deviates' from being elementary Abelian.

DEFINITION: The *relative corank* of a finite p-group G is the number

$$\operatorname{rcrank}(G) = \frac{\operatorname{corank}(G)}{\log_p |G|}.$$

Thus the relative corank is a rational number lying in the range

$$0 \leqslant \operatorname{rcrank}(G) \leqslant \frac{\log_p |G| - 1}{2}.$$

Proposition 3(ii) shows that groups with a small relative corank also have a relatively small Frattini subgroup. But relative corank captures more than the size of the Frattini subgroup. For example, the dihedral and quaternion groups of order eight have $\operatorname{rcrank}(D) = 2/3$ and $\operatorname{rcrank}(Q) = 1$. For certain families of groups it is fairly straightforward to compute the relative corank. For instance, letting $ES(p,k)$ denote an arbitrary extraspecial p-group of order p^{2k+1}, we have:

$$\begin{aligned}
&\operatorname{rcrank}\big((C_p)^n\big) = 0,\\
&\operatorname{rcrank}(C_{p^n}) = (n-1)/2,\\
&\operatorname{rcrank}\big((C_p)^{n-2} \times C_{p^2}\big) = (n-1)/n,\\
&\operatorname{rcrank}\big((C_{p^2})^{n/2}\big) = n/4,\\
&\operatorname{rcrank}\big(ES(p,k)\big) = 1, \text{ for } k \geqslant 2,\\
&\operatorname{rcrank}\big(ES(p,k) \times ES(p,k)\big) = 2 + 1/(4k+2), \text{ for } k \geqslant 2.
\end{aligned}$$

To obtain the last two calculations we have used the description of the Schur multipliers of extraspecial p-groups given in [**2**], together with the following simple lemma whose proof is left to the reader.

LEMMA 5. *Let G and H be groups of orders p^n and p^m. Then*

$$\operatorname{rcrank}(G \times H) = \frac{n}{n+m}\operatorname{rcrank}(G) + \frac{m}{n+m}\operatorname{rcrank}(H) + \frac{nm - \log_p |G^{ab} \times H^{ab}|}{n+m}.$$

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References

[1] Ya.G. Berkovich, 'On the order of the commutator subgroup and Schur multiplier of a finite p-group', *J. Algebra* **144** (1991), 269–272.

[2] F.R. Beyl and J. Tappe, *Group extensions, representations, and the Schur multiplicator*, Lecture Notes in Math. **958** (Springer-Verlag, Berlin, Heidelberg, New York, 1982).

[3] G. Ellis, 'A bound for the derived and Frattini subgroups of a prime-power group', *Proc. Amer. Math. Soc.* **126** (1998), 2513–2523.

[4] G. Ellis, 'On the Schur multiplier of p-groups', *Comm. Algebra* (to appear).

[5] W. Gaschütz, J. Neubüser and Ti Yen, 'Über den Multiplikator von p-Gruppen', *Math. Z.* **100** (1967), 93–96.

[6] J. Wiegold, 'Commutator subgroups of finite p-groups', *J. Austral. Math. Soc.* **10** (1969), 480–484.

[7] X. Zhou, 'On the order of Schur multipliers of finite p-groups', *Comm. Algebra* **22** (1994), 1-8.

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