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On multigraphs with a given partition

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Relationships between the numbers of general graphs and the numbers of multigraphs are set up with a view to enumerating multigraphs with a given partition. In the process of doing so certain structures which we call graph-lattices evolve. The principle of inclusion-exclusion plays an important part in the formulation of theorems and actual numerical results are computed with the aid of S-functions.

1. Introduction

Among the unsolved problems listed in Harary's review article [1] was the enumeration of graphs with a given partition. The relationships derived in this paper provide a method of solving this problem for multigraphs.

Our approach involves removing loops from general graphs and the correspondences between the general graphs and the resulting multigraphs are characterised by means of structures called graph-lattices, here-in-after called lattices. The principle of inclusion-exclusion plays a major part in the formulation of our theorems and numerical results are computed by means of the Superposition Theorem [4] and S-functions [3, 5]. Theorems with short proofs are included in entirety; in other cases, hints at the method of procedure for the proof are given. The reader is referred to [2] for basic definitions on graphs; however, to avoid

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confusion in terminology we include here a few definitions which are useful for our purposes.

DEFINITION 1. A node of a graph which has valency k is said to be *k-valent*. For $k = 1, 2, 3$ we have respectively *monovalent*, *bivalent* and *trivalent* nodes; $k = 0$ gives isolated nodes.

DEFINITION 2. A graph in which all nodes are trivalent is called a *cubic* graph.

DEFINITION 3. A *general* graph is a graph in which multiple edges and loops are allowed.

DEFINITION 4. A *multigraph* is a graph in which multiple edges but no loops are allowed.

The graphs which are discussed are not necessarily connected.

2. The simple lattice

In order to fix our ideas we commence with the simple case of cubic graphs on m (even and finite) nodes.

We introduce the notion of a graph-lattice by displaying such a structure. Figure 1 is the lattice for $m = 4$.

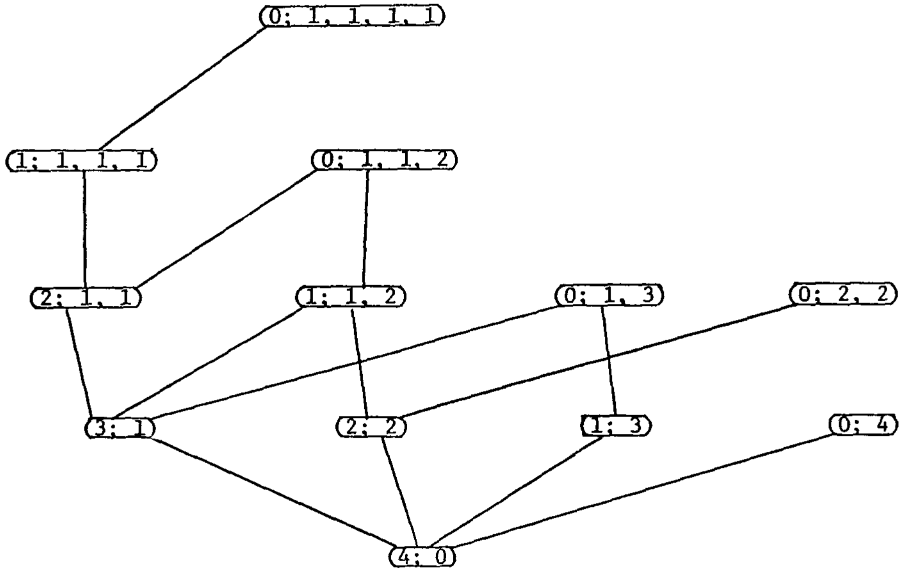

Figure 1.

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We now give a general description of such a structure and the process leading up to its evolution.

The lattice for m trivalent nodes is denoted by $GL(m : 3)$. Typical *elements* $\beta, \gamma \in GL(m : 3)$ are of the form:

$$(2.1)\qquad \beta = \{m-q;\ (\rho)\}\ ,\quad (\rho) = \left\{1^{j_1}2^{j_2}\ \dots\ q^{j_q}\right\}\ ,$$
$$\gamma = \{m-v;\ (\sigma)\}\ ,\quad (\sigma) = \left\{1^{k_1}2^{k_2}\ \dots\ v^{k_v}\right\}\ .$$

Naturally,

$$(2.2)\qquad q = \sum_{i=1}^{q} ij_i \quad\text{and}\quad v = \sum_{i=1}^{v} ik_i\ ,$$

and we let

$$(2.3)\qquad p = \sum_{i=1}^{q} j_i \quad\text{and}\quad s = \sum_{i=1}^{v} k_i\ .$$

The elements β, γ are then said to be on *levels* p and s respectively in the lattice.

The element $\alpha = (m : 0)$ is called the *root*.

The element β represents $m - q$ trivalent nodes and several *kinds* of monovalent nodes of which there are j_i of the i-th *kind* (colour) $(i = 1, 2, \dots, q)$; γ has a similar interpretation. $GL(m : 3)$ evolved in the following way: we consider the general graphs on β and denote the number of such graphs by $L(\beta)$; the number of multigraphs on β is denoted by $M(\beta)$. Let β_k $(k = 1, 2, \dots, m-q)$ be the elements on level $(p+1)$ which are joined to β by an edge. Such elements are said to be generated by β .

We strip the $L(\beta)$ graphs of their loops (if any) (at most one loop is connected to each node) and we obtain the following equation which characterises the relationships between elements on consecutive levels in the lattice.

$$(2.4)\qquad L(\beta) = M(\beta) + \sum_{k=1}^{m-q} M(\beta_k)\ ,$$

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where the partition (ρ_k) of β_k is given by

$$(2.5) \qquad (\rho_k) = \begin{cases} \left(1^{j_1} 2^{j_2} \ldots s^{j_s+1} \ldots q^{j_q}\right) & \text{if } 1 \le s \le q , \\ \left(1^{j_1} 2^{j_2} \ldots q^{j_q}, s\right) & \text{if } s > q . \end{cases}$$

We note that s assumes all integral values between 1 and $m - q$ and must be distinguished from the integers already in (ρ) since it is obtained from a further stripping.

The numbers $L(\beta)$ can all be obtained by means of the Superposition Theorem [4], thus our next step would be to invert equation (2.4) with a view to finding $M(\beta)$.

3. The inversion formula

Let $s > p$. We say that β, γ are *comparable*, written $\beta \prec \gamma$, if and only if every integer in (ρ) occurs in (σ) and $k_i \ge j_i$ $(1 \le i \le q)$, and *incomparable* otherwise; consequently, β and γ are comparable in the lattice if and only if there is a path between β and γ .

Let

$$(3.1) \qquad n_i = \begin{cases} k_i - j_i , & 1 \le i \le q , \\ k_i , & q < i \le v , \end{cases}$$

$$n = \sum_{i=1}^{v} n_i .$$

We define $\mu(\beta, \gamma)$ by the relation:

$$(3.2) \qquad \mu(\beta, \gamma) = (-1)^n \frac{n!}{n_1! n_2! \ldots n_v!} .$$

Then $\mu(\beta, \gamma)$ is the Möbius function [6] for the lattice as β, γ range over its elements.

Let $d(\beta, \gamma)$ denote the number (≥ 0) of edges in any path (all such paths are of the same length) joining β to γ and let

$\nu(\beta, \gamma) = (-1)^n \mu(\beta, \gamma)$. Then equation (2.4) can be written in the form:

$$L(\beta) = M(\beta) + \sum_{\substack{\beta \prec \gamma \\ d(\beta,\gamma)=1}} \nu(\beta, \gamma) M(\gamma) . \tag{3.3}$$

We now state our first theorem.

THEOREM 1.

$$M(\alpha) = \sum_{u=0}^{m} \left\{ \sum_{d(\alpha,\beta)=u} \mu(\alpha, \beta) L(\beta) \right\} ,$$

where $\beta = \alpha$ *when* $u = 0$.

Proof. From equation (3.3) we have

$$M(\beta) = L(\beta) - \sum_{\substack{\beta \prec \gamma \\ d(\beta,\gamma)=1}} \nu(\beta, \gamma) M(\gamma) .$$

Similarly,

$$M(\gamma) = L(\gamma) - \sum_{\substack{\gamma \prec \theta \\ d(\gamma,\theta)=1}} \nu(\gamma, \theta) M(\theta) .$$

Combining these two equations gives

$$M(\beta) = L(\beta) - \sum_{\substack{\beta \prec \gamma \\ d(\beta,\gamma)=1}} \nu(\beta, \gamma) L(\gamma) + \sum_{\substack{\beta \prec \gamma \\ d(\beta,\gamma)=2}} \nu(\beta, \gamma) M(\gamma) .$$

After $m - q$ steps, we obtain,

$$M(\beta) = \sum_{u=0}^{m-q} \left\{ \sum_{\substack{\beta \prec \gamma \\ d(\beta,\gamma)=u}} (-1)^u \nu(\beta, \gamma) L(\gamma) \right\} ,$$

with $\gamma = \beta$ when $u = 0$.

Now putting $\beta = \alpha$ and observing that $\alpha \prec \beta$ for all β in the lattice completes the proof. The result of Theorem 1 exhibits an inclusion-exclusion pattern.

4. An isomorphism between lattices

We consider the general lattice $GL(m : k)$, $k > 3$, on m k-valent nodes. Each node may contain as many as r loops, where

$$(4.1) \qquad r = \begin{cases} (k-1)/2 & \text{if } k \text{ is odd,} \\ k/2 & \text{if } k \text{ is even,} \end{cases}$$

and there are $(r+1)$ categories (including that for k-valent nodes) into which the numbers of nodes can be put after the loops, if any, are removed from the general graphs. Accordingly, the root of this lattice may be represented by

$$(4.2) \qquad \alpha' = (m; \underbrace{0; 0; \ldots; 0}_{r}) ,$$

where the categories are separated by semi-colons. In a typical element of the lattice the i-th category from the left contains numbers of $|k-2(i-1)|$-valent nodes $(i = 1, 2, \ldots, r+1)$.

We underline once more that our aim is to derive for $GL(m : k)$ a result similar to Theorem 1. Indeed, the general approach is the same, the underlying principle being inclusion-exclusion which is instrumental in establishing the following lemma.

LEMMA 1. *Let G be a lattice. If $\beta \in G$ is joined to root α by two paths which differ in lengths by an odd integer, then its contribution to $M(\alpha)$ is zero.*

Proof. Let l_1 and l_2 be the lengths of the two paths, where $l_1 - l_2$ is an odd integer. Then the contribution to $M(\alpha)$ is, by Theorem 1, $L(\beta)\left\{(-1)^{l_1}+(-1)^{l_2}\right\} = 0$. Hence the result of the lemma.

In circumstances in which the conditions of Lemma 1 obtain, we eliminate β from G and after all such β's are removed, we denote the remaining lattice by G^* . Such instances do occur in $GL(m : k)$, as we shall see shortly, and the lattice corresponding to $GL(m : k)$ in this way is denoted by $G^*L(m : k)$.

THEOREM 2. *There exists an isomorphism between $G^*L(m : k)$ and $GL(m : 3)$.*

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Proof. $G^*L(m : k)$ is generated by induction. A typical element on level p of $G^*L(m : k)$ is assumed to be of the form

$$\beta' = \{m-q;\ (\rho);\ \underbrace{0;\ 0;\ \ldots;\ 0}_{r-1}\}$$

where (ρ) and p are defined as in (2.1) and (2.3). β' generates on level $(p+1)$ several elements of which we single out $\gamma' = \{m-q-s;\ (\rho),\ s;\ 0;\ 0;\ \ldots;\ 0\}$. Fix s. Now γ' generates on level $(p+2)$ all elements generated by β' of the form $\left\{m-q-t;\ (\rho_1);\ (\rho_2);\ \ldots;\ \left(\rho_r\right)\right\}$ where $\left(\rho_i\right)$ are partitions, $s \le t \le m-q$ and at least one of $\left(\rho_j\right)$ $(j = 2,\ \ldots,\ r)$ is non-null. Hence from Lemma 1 all such elements are to be removed from $G^*L(m : k)$. In particular, when $s = m - q$, there are no elements generated by γ' in the final stage. Thus in $G^*L(m : k)$, β' generates all elements of the form $\gamma' = \left\{m-q-s;\ \left(\rho_k\right);\ \underbrace{0;\ 0;\ \ldots;\ 0}_{r-1}\right\}$ where $\left(\rho_k\right)$ is defined as in (2.5) and is generated by each of

$$\gamma'' = \left\{m-q+s;\ \left(\rho_l\right);\ \underbrace{0;\ 0;\ \ldots;\ 0}_{r-1}\right\},$$

where

$$\left(\rho_l\right) = \left\{1^{j_1}\ 2^{j_2}\ \ldots\ s^{j_s-1}\ \ldots\ q^{j_q}\right\} \quad (s = 1,\ 2,\ \ldots,\ q)$$

provided s occurs in (ρ). Now starting from the root α' we can generate $G^*L(m : k)$ in entirety.

Let the mapping $\chi : G^*L(m : k) \xrightarrow{\text{onto}} GL(m : 3)$ be defined by the relation $\chi(\gamma') = \gamma$, where $\gamma' \in G^*L(m : k)$ and $\gamma \in GL(m : 3)$ have identical partitions (σ) in the second category. Since (σ) occurs once only in each lattice, χ is one-to-one, and hence is an isomorphism. This proves the theorem.

The result of Theorem 2 indicates that we may use $GL(m : 3)$ to represent $G^*L(m : k)$ provided the two categories in $GL(m : 3)$ represent k-valent and $(k-2)$-valent nodes from left to right. Note that

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$G^*L(m : 3) = GL(m : 3)$. This representation establishes the fact that the result of Theorem 1 can be used to count multigraphs on k-valent nodes.

5. The lattice for any given partition

In this section we discuss the lattice for any given partition by first considering the product of two lattices $G^*L(m : k)$ and $G^*L(n : h)$, $h, k \geq 3$. The final result is obtained by straightforward induction on any number of such lattices.

Let $\beta \in G^*L(m : k)$, $\gamma \in G^*L(n : h)$ where β is defined as in (2.1) and

(5.1) $$\gamma = \{n-v; (\sigma)\} ; \quad (\sigma) \text{ as in (2.1).}$$

Here we assume a representation of each lattice by a replica of $GL(m : 3)$ and $k > h$. Then we define the product of β and γ by the relation:

(5.2) $$\beta \wedge \gamma = \{m-q; n-v; (\rho); (\sigma)\} = \gamma \wedge \beta .$$

In (5.2), valencies are arranged from left to right in descending order of magnitude.

Using (5.2) we define the product of β and $G^*L(n : h)$ as the structure obtained from $G^*L(n : h)$ when each $\gamma \in G^*L(n : h)$ is 'multiplied' by β and the connectivity is preserved.

Now we join the 'roots' of these new structures as the β's are joined in $G^*L(m : k)$ to obtain the product of $G^*L(m : k)$ and $G^*L(n : h)$ denoted by $G^*L(m : k) \times G^*L(n : h)$. Note that the product of the lattices as defined is not a commutative operation.

Let the G^*-lattice for m k-valent nodes and n h-valent nodes be denoted by $G^*L(m : k | n : h)$. Let $G = \text{rep}(G')$ stand for *lattice* G *is a representation of lattice* G' . Then we have,

LEMMA 2. $G^*L(m : k) \times G^*L(n : h) = \text{rep}\left(G^*L(m : k | n : h)\right)$.

The proof of this lemma follows along identical lines to that of Theorem 2 and is therefore omitted.

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Let $G_i^* = G^*L(m_i : k_i)$ $(i = 1, 2, \ldots, r)$.

The result of Lemma 2 is extended in a natural way by induction to give,

THEOREM 3. $\prod_{i=1}^{r} G_i^* = \text{rep}\left\{G^*L\left(m_1 : k_1 \mid m_2 : k_2 \mid \ldots \mid m_r : k_r\right)\right\}$ *where* $\prod_{i=1}^{r} G_i^* = G_1^* \times G_2^* \times \ldots \times G_r^*$.

Again Theorem 1 can be applied here with α, β defined appropriately for the lattice.

For $k = h$ Lemma 2 is still valid but in this case we are dealing with two kinds of k-valent nodes. We now consider applications of the preceding theorems.

6. Applications

The formula in Theorem 1 is expressed in terms of the numbers of general graphs. These numbers can all be found by the Superposition Theorem [4] which states that the number of general graphs on ρ_i nodes of valency i $(i = 1, 2, \ldots)$ is given by

(6.1) $$N\left\{\left(h_{\rho_1}[h_1].h_{\rho_2}[h_2] \ldots\right)*h_l[h_2]\right\} ,$$

where the number l of edges in the graph satisfies the equation

$$2l = \sum_{i=1} i\rho_i .$$

Equation (6.1) is written in the notation compatible with S-functions [3, 5] which have the important orthogonality property namely,

$$N\left(\{\lambda\}*\{\mu\}\right) = \begin{cases} 1 & \text{if } (\lambda) = (\mu) , \\ 0 & \text{if } (\lambda) \neq (\mu) , \end{cases}$$

where (λ) and (μ) are partitions of the same integer.

It is well known [3] that $h_l[h_2]$ contains all the S-functions which display even partitions only of the integer $2l$, all S-functions

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having coefficient unity, hence formula (6.1) may be replaced by

(6.2) $$N_e\left(h_{\rho_1}[h_1]\cdot h_{\rho_2}[h_2]\cdot \;\ldots\right) ,$$

where $N_e(A)$ means the number of S-functions in A with even partitions.

The following result, which has appeared in the literature, is relevant to our applications:

(6.3) $$a_m[h_r] = (-1)^m \sum \frac{(-1)^{\Sigma k}(\Sigma k)!}{k_1!k_2!\ldots} \prod_{i=1} \left(h_i[h_r]\right)^{k_i} ,$$

where (Σk) denotes the sum of the k_i's and $m = \sum_{i=1}^{m} ik_i$.

We now apply Theorem 1 to the result in Theorem 3.

A typical element of $G^*L(m_i : k_i)$ is $\beta_i = \left[m_i - q_i ; \; (\rho_i)\right]$, where the partition (ρ_i) of the integer q_i is obtained by replacing each subscript n of j_n in partition (ρ) defined in (2.1) by n_i , $(n = 1, 2, \ldots, q)$. Hence, $q_i = \sum_{n_i=1}^{q_i} n_i j_{n_i}$, and $p_i = \sum_{n_i=1}^{q_i} j_{n_i}$.

For $\prod_{i=1}^{r} G^*L(m_i : k_i)$, $\alpha = \bigwedge_{i=1}^{r} \alpha_i$ and $\beta = \bigwedge_{i=1}^{r} \beta_i$, where α_i is root of $G^*L(m_i : k_i)$.

Let $p = \sum_{i=1}^{r} p_i$ and $m = \sum_{i=1}^{r} m_i$. With these definitions, Theorem 1 gives,

LEMMA 3. *$M(\alpha) = \sum_{p=0}^{m} \left\{ \sum_{d(\alpha,\beta)=p} \left[\prod_{i=1}^{r} \mu(\alpha_i, \beta_i) \right] L(\beta) \right\}$ with $L(\beta) = L(\alpha)$ when $p = 0$.*

MAIN THEOREM. *The number of multigraphs on m_i k_i-valent nodes $(i = 1, 2, \ldots, r)$ is*

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$$N_e\left\{\prod_{i=1}^{r}\left[\sum_{q_i=0}^{m_i}(-1)^{q_i}h_{m_i-q_i}\left[h_{k_i}\right]a_{q_i}\left[h_{k_i-2}\right]\right]\right\} .$$

Proof. The number of such graphs is, using Lemma 3,

$$N_e\left[\sum_{p=0}^{m}\left\{\sum_{d(\alpha,\beta)=p}\left\{\prod_{i=1}^{r}\mu\left(\alpha_i,\ \beta_i\right)\ L(\beta)\right\}\right\}\right]$$
$$= N_e\left[\prod_{i=1}^{r}\left\{\sum_{q_i=0}^{m_i}\left\{\left(\frac{(-1)^{p_i}p_i!}{\prod_{\lambda=1}^{q_i}j_\lambda!}\prod_{\lambda=1}^{q_i}h_\lambda^{j_\lambda}\left[h_{k_i-2}\right]\right)h_{m_i-q_i}\left[h_{k_i}\right]\right\}\right\}\right]$$
$$= N_e\left[\prod_{i=1}^{r}\left\{\sum_{q_i=0}^{m_i}\left((-1)^{q_i}a_{q_i}\left[h_{k_i-2}\right]h_{m_i-q_i}\left[h_{k_i}\right]\right)\right\}\right] ,$$

by (6.3).

We stipulate that for $r \geq 2$, the factors in this formula must be multiplied before even parts are extracted.

We now apply the main theorem to obtain more explicit results. Let $M(m : k)$ denote the number of multigraphs on m k-valent nodes. Then, we have,

THEOREM 4. $M(m : 3) = N_e\left\{\sum_{q=0}^{m}(-1)^q h_{m-q}[h_3]a_q\right\}$.

Proof. The result follows from the main theorem by putting $r = 1$, $m_1 = m$, $q_1 = q$, $k_1 = 3$ and from the identity $a_q[h_1] = a_q$.

THEOREM 5. $M(m : 2) = N_e\left\{h_m[h_2] - h_{m-1}[h_2]\right\}$.

Proof. Similar to that of Theorem 4, with

$$a_q\left[h_0\right] = \begin{cases} 1 & \text{if } q = 0 \text{ or } 1 , \\ 0 & \text{otherwise.}\end{cases}$$

Theorem 5 indicates that the two-element lattice containing the root

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$(m\ ;\ 0)$ and the element $(m-1;\ 1)$ is sufficient to enumerate graphs on bivalent nodes. This is a particularly straightforward formula to apply. We now present an example of the result of the main theorem.

EXAMPLE. We enumerate the multigraphs on 2 4-valent nodes and 2 trivalent nodes.

The number of such graphs is

$$N_e\Big\{\big(h_2[h_4]-h_4h_2+a_2[h_2]\big)\big(h_2[h_3]-h_3a_1+a_2\big)\Big\}$$

$$= N_e\big(h_2[h_4]h_2[h_3]-h_2[h_4]h_3a_1+h_2[h_4]a_2-h_4h_2h_2[h_3]+h_4h_3h_2a_1 - h_4h_2a_2+a_2[h_2]h_2[h_3]-a_2[h_2]h_3a_1+a_2[h_2]a_2\big)$$

$$= 33 - 16 + 0 - 27 + 16 - 1 + 4 - 4 + 1$$

$$= 6\ .$$

The six graphs are displayed in figure 2.

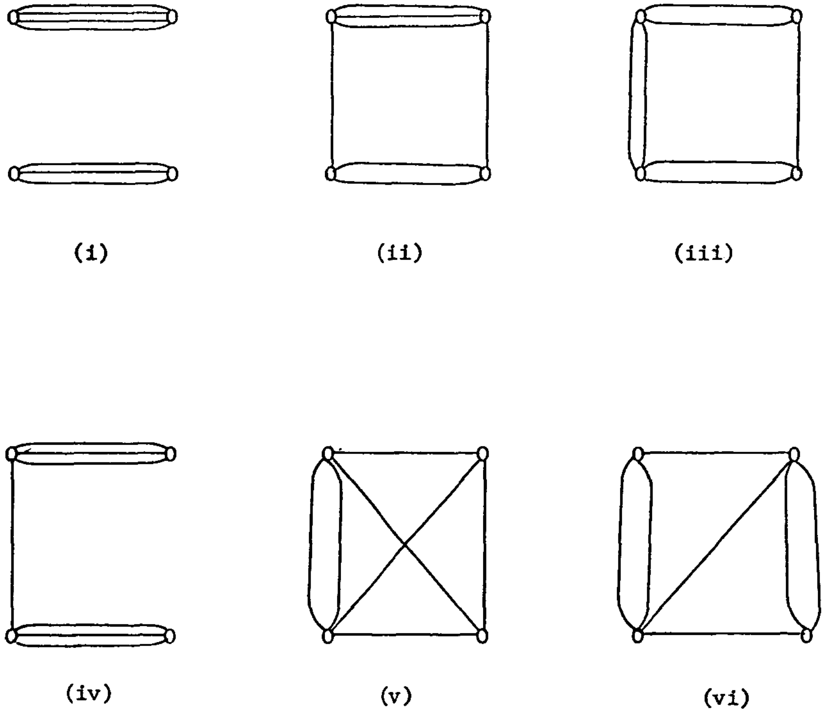

Figure 2.

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