Glasgow Math. J. **58** (2016) 219–228.
doi:10.1017/S0017089515000166.

ON COLIMITS OVER ARBITRARY POSETS

M. DOKUCHAEV

Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, 05508-090 São Paulo, SP, Brasil
e-mail: dokucha@ime.usp.br

and B. NOVIKOV†

*Kharkov National University, Svobody sq., 4
61077, Kharkov, Ukraine*
e-mail: boris.v.novikov@univer.kharkov.ua

(Received 5 February 2014; accepted 28 February 2015; first published online 22 July 2015)

Abstract. We characterize those partially ordered sets I for which the canonical maps $M_i \to \operatorname{colim} M_j$ into colimits of abstract sets are always injective, provided that the transition maps are injective. We also obtain some consequences for colimits of vector spaces.

2010 *Mathematics Subject Classification.* 18A30, 06A06.

1. Introduction. Crowns arise in various problems related to partially ordered sets (posets). Thus, for example, they appear in the study of retracts and fixed points (see [**7**]), in calculation of the cohomological dimension (see [**5**]), in applications to homotopy theory (see [**11**]) and in the investigation of incidence algebras and their quotients (see [**1**] and [**6**]). At the same time, quite often they play a "negative" role: the absence of crowns of some kind ensures the existence of certain good properties of posets or constructions related to them. For instance, an incidence algebra $\kappa[S]$ of a finite poset S is completely separating if and only if S contains no crowns [**6**]. It is not surprising that such a situation arises in a problem of colimits which is discussed in this note: roughly speaking, the crowns are antagonists of directed sets for which colimits are usually considered and well understood (note that colimits over directed posets are called directed colimits, or direct limits, or inductive limits).

More precisely, if one takes a directed colimit $\operatorname{colim} M_i$ (also denoted by $\varinjlim M_i$), where i runs over a directed poset I, such that the transition maps $\varphi_{ij} : M_i \to M_j$ are injective, then the canonical maps $M_i \to \operatorname{colim} M_j$ are also injective, which is a crucial property for applications. Thus, one may wonder which are the posets I for which this always happens. We completely characterize such I in the case when the M_i's are abstract sets and obtain consequences for the colimits of vector spaces.

This problem is related to a similar question about the ring-theoretic version of cross-sectional algebras of Fell bundles over inverse semigroups studied in [**8**], since such algebras are epimorphic images of colimits of vector spaces over non-necessarily directed posets. In the C^*-algebraic context, it is proved that the fibres are canonically embedded into the cross-sectional algebra; however, the abstract ring theoretic version

†Deceased 30 March 2014.

of this fact is not established so far. Moreover, colimits are used in formal software development. This was inspired by J. Goguen in **[9]** (see also the fifths "dogma" in **[10]**), and being pursued by several authors (see, in particular, [**4**, **13**, **14**]). More precisely, colimits are the algebraic tools to compose specifications into new ones, and it is important to make sure that the initial specifications are canonically embedded into more complex ones obtained via general colimits. We believe that the present article may be useful for the above-mentioned problems.

2. Setting of the problem. Let I be a set with a partial order $\trianglelefteq$, considered as a small category, in particular, we interpret $i \trianglelefteq j$ as a (unique) arrow from i to j. Denote by Set the category of sets. In analogy with the case of directed posets (see [**12**, Chapter VIII]), by an *I-spectrum*[1] M_I we mean a diagram $M : I \to$ Set (i.e. M is a functor) such that each $M(i \trianglelefteq j) : M(i) \to M(j)$ is injective. Write, $M_i = M(i)$ and $\varphi_{ij} = M(i \trianglelefteq j)$. The maps $\varphi_{ij} : M_i \to M_j$, with $i \trianglelefteq j$, are called the *transition maps*.

Alternatively, an I-spectrum M_I consists of (possibly empty) sets M_i $(i \in I)$ and maps $\varphi_{ij} : M_i \to M_j$ $(i \trianglelefteq j)$ such that

$$\forall i \in I \ \ \varphi_{ii} = \text{id (the identity map)}, \tag{1}$$

$$(i \trianglelefteq j \trianglelefteq k) \Longrightarrow \varphi_{jk}\varphi_{ij} = \varphi_{ik}, \tag{2}$$

$$\forall i, j \in I \ (i \trianglelefteq j) \ \ \varphi_{ij} \text{ isinjective}. \tag{3}$$

We shall write $a \triangleleft b$ if $a \trianglelefteq b$ and $a \neq b$, and $a \trianglelefteq b$ will be also denoted by $b \trianglerighteq a$.

REMARK 1. If $M_i = \varnothing$ and $i \trianglelefteq j$, then φ_{ij} is an empty map and its composition with other maps is also empty. In particular, φ_{ii} is the identity map. On the other hand, if $M_i \neq \varnothing$ and $i \triangleleft j$, then M_j cannot be empty.

Given an I-spectrum, the colimit of the diagram $M \to$ Set will be denoted by colim M_i. We say that the I-spectrum M_I is *faithful* if all canonical maps $M_i \to$ colim M_j are injective. A poset I is called *faithful* if any I-spectrum is faithful.

Write $M = \coprod_{i \in I} M_i$ and let $\varepsilon_i : M_i \to M$ be the natural embeddings. Recall that by the construction of colimits in Set,

$$\operatorname{colim} M_i = M/\nu,$$

where ν is the smallest equivalence relation containing the following binary relation:

$$\mu = \{(\varepsilon_i(a_i), \varepsilon_j \circ \varphi_{ij}(a_i)) \mid i, j \in I, \ i \trianglelefteq j, \ a_i \in M_i\} \subseteq M \times M,$$

and the canonical maps $M_i \to \operatorname{colim} M_j = M/\nu$ are the compositions $M_i \hookrightarrow M \xrightarrow{\chi} M/\nu$, where χ is the natural projection. Therefore, I is faithful if and only if for every I-spectrum $M_I = (\{M_i \mid i \in I\}, \{\varphi_{ij} : M_i \to M_j \mid i \trianglelefteq j\})$ the maps $\psi_i = \chi \circ \varepsilon_i$ are all injective.

We shall consider the following:

PROBLEM. Characterize the faithful posets.

[1]One may also use the term *inductive system* from Category Theory.

https://doi.org/10.1017/S0017089515000166 Published online by Cambridge University Press

A similar problem can be formulated in the category of vector spaces. Then, we assume that each M_i is a vector space over a fixed field and all φ_{ij} are linear maps. In this case, we take $M = \oplus_{i\in I} M_i$ and the subspace $N \subseteq M$ generated by all $\varepsilon_i(a_i) - \varepsilon_j \circ \varphi_{ij}(a_i)$, $i, j \in I$, $i \trianglelefteq j$, $a_i \in M_i$. Then, colim $M_i = M/N$. We shall say that a poset I is *linearly faithful* if for any I-spectrum of vector spaces each ψ_i is injective.

In all what follows, M_i $(i \in I)$ will be abstract sets and $\varphi_{ij} : M_i \to M_j$ maps of sets.

If M is a set and κ a field, denote by $\langle M \rangle$ the vector κ-space whose base is M. Then, $\langle \_ \rangle$ determines a functor from the category of sets to that of κ-spaces. The forgetful functor from κ-spaces to sets is a right adjoint to $\langle \_ \rangle$, and by [**3**, Corollary 3.9] we have that $\langle \_ \rangle$ commutes with colimits. Let now $M_I = (\{M_i \mid i \in I\}, \{\varphi_{ij} \mid i \trianglelefteq j\})$ be an I-spectrum, and let $\tilde{\varphi}_{ij} : \langle M_i \rangle \to \langle M_j \rangle$, $(i, j \in I, i \trianglelefteq j)$ be the linear extension of φ_{ij}, i.e. $\tilde{\varphi}_{ij} = \langle \varphi_{ij} \rangle$.Then, obviously $V_I = (\{\langle M_i \rangle \mid i \in I\}, \{\tilde{\varphi}_{ij} \mid i \trianglelefteq j\})$ is an I-spectrum of vector spaces, and by the above

$$\text{colim } \langle M_i \rangle = \langle \text{colim } M_i \rangle.$$

Consequently, if M_I is not faithful, then neither is V_I, and thus we have the following:

PROPOSITION 1. *If the poset I is linearly faithful, then I is faithful.*

Recall that a subset F of a poset I is called a *filter* (resp. an *ideal*) if $a \trianglelefteq x$ (resp. if $a \trianglerighteq x$) implies $x \in F$ for any $a \in F$, $x \in I$. If $A \subseteq F$ and for every $x \in F$, there exists $a \in A$, such that $a \trianglelefteq x$, then we say that the filter F *is generated* by the subset A. If A consists of a single element, $A = \{a\}$, then F is called the *principal filter* generated by a.

The set of all common upper bounds of a subset $J \subseteq I$ will be denoted by $U(J)$. If, for example, $J = \{a, b\}$, we also write $U(a, b)$ instead of $U(J)$.

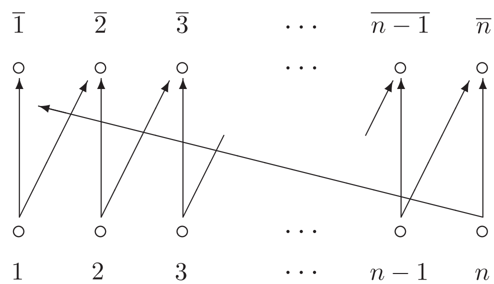
3. Crowns. A poset K is called a *crown* [**7**], if K is an union of disjoint sets

$$K_0 = \{1, 2, \ldots, n\}, \quad K_1 = \{\overline{1}, \overline{2}, \ldots, \overline{n}\} \tag{4}$$

for some $2 \leq n < \infty$ such that the following relations of comparability, and only they (except the relation of equality), are valid:

$$\forall \alpha \ (1 \leq \alpha \leq n) \quad \alpha \vartriangleleft \overline{\alpha}, \quad \alpha \vartriangleleft \overline{\alpha + 1},$$

where the indices are taken modulo n, i. e. $n + 1 = 1$ and $\overline{n + 1} = \overline{1}$. All elements in K_0 and K_1 are assumed to be distinct, i.e. $K_0 \cup K_1$ has $2n$ elements. The diagram of the crown is as follows:



https://doi.org/10.1017/S0017089515000166 Published online by Cambridge University Press

We refer to n as the *degree* of the crown. Without loss of generality, for any I-spectrum $M : I \to \mathrm{Set}$, the sets M_i will be assumed to be pairwise disjoint and the symbol ε_i, identifying any $a \in M_i$ with its image $\varepsilon_i(a) \in \coprod M_i$, will be omitted.

The following statement shows the existence of non-faithful posets:

PROPOSITION 2. *No crown is faithful.*

Proof. For each $k \in K \setminus \{\bar{1}\}$, we take a singleton $\{a_k\}$ as M_k. In addition, we set $M_{\bar{1}} = \{a_{\bar{1}}, b\}$. Thus, the bijections φ_{kl} for $(k, l) \neq (1, \bar{1}), (n, \bar{1})$ are determined uniquely. Further, let

$$\varphi_{1\bar{1}}(a_1) = a_{\bar{1}}, \quad \varphi_{n\bar{1}}(a_n) = b.$$

Clearly, the collections $\{M_k\}$, $\{\varphi_{kl}\}$ form a K-spectrum. All pairs

$$(a_1, a_{\bar{1}}), (a_1, a_{\bar{2}}), (a_2, a_{\bar{2}}), (a_2, a_{\bar{3}}), \dots, (a_n, a_{\bar{n}}), (a_n, b)$$

are contained in μ. By the transitivity $(a_{\bar{1}}, b) \in \nu$. This means that the considered spectrum is non-faithful. □

COROLLARY 1. *No crown is linearly faithful.*

Proof. This follows using Proposition 1. □

However, adding to the crown some additional links may make it faithful, as it is shown in the following statement.

REMARK 2. The poset I, obtained from the crown $K = \{1, 2; \bar{1}, \bar{2}\}$ by adding the link $\bar{1} \lhd \bar{2}$, is faithful. This follows from the fact that I is a directed set ([**2**], Section 7, no 6, Remark 1).

4. Pure subcrowns. Let I be a poset and $J \subseteq I$ a subset. Then, the partial order $\unlhd$ of I induces a partial order $\unlhd_J = \unlhd \cap (J \times J)$ on J. Then, we shall say that J is a *subposet* of I. A *subcrown* of I is a subposet of I which is a crown with respect to the induced partial order.

A subcrown K of I of degree $n \geq 4$ will be called *pure* if

$$(\forall i, j \in K) \;\; U(i, j) \neq \varnothing \Longrightarrow K \cap U(i, j) \neq \varnothing.$$

It is easy to see that this condition is equivalent to the following:

$$(\forall i, j \in K_0) \;\; \mathrm{dist}(i, j) \geq 4 \implies U(i, j) = \varnothing, \tag{5}$$

where $\mathrm{dist}(i, j)$ is the length of the shortest path between i and j in the undirected graph corresponding to the subcrown.

For $n = 3$, we shall strengthen the definition of a pure subcrown. Namely, a subcrown $K \subseteq I$ of degree 3 is called *pure* if $U(K_0) = \varnothing$. We mention by the way that any crown of degree 3 obviously satisfies (5).

For the case $n = 2$ we also need a special definition of purity. Let J be a subset of I and $i, k \in J$. We shall say i and k are J-*connected*, if there are $j_1, \dots, j_m \in J$, such that

$$i \mathrel{\substack{\rhd \\ \lhd}} j_1 \mathrel{\substack{\rhd \\ \lhd}} \dots \mathrel{\substack{\rhd \\ \lhd}} j_m \mathrel{\substack{\rhd \\ \lhd}} k,$$

https://doi.org/10.1017/S0017089515000166 Published online by Cambridge University Press

where every two neighbouring elements are connected by a relation $\trianglelefteq$ or $\trianglerighteq$. If there is no such a sequence, we say that i and k are J-*disconnected*.

A subcrown $K = \{1, 2; \overline{1}, \overline{2}\} \subseteq I$ of degree 2 is called *pure* if the elements $\overline{1}$ and $\overline{2}$ are $U(1, 2)$-disconnected.

THEOREM 1. *If I is non-faithful, then I contains a pure subcrown.*

Proof. Let $M_I = (\{M_i\}, \{\varphi_{ij}\})$ be a non-faithful I-spectrum.

(1) We show first the existence of a subcrown. There exists an index, say $1 \in I$, and elements $a, b \in M_1$, $a \neq b$ with $\psi_1(a) = \psi_1(b)$. It follows that $(a, b) \in \nu$, i. e. there are non-necessarily distinct indices $i_2, \ldots, i_m \in I$ and elements $c_2 \in M_{i_2}, \ldots, c_m \in M_{i_m}$ such that

$$(a, c_2), (c_2, c_3), \ldots, (c_{m-1}, c_m), (c_m, b) \in \tilde{\mu}, \tag{6}$$

where $\tilde{\mu}$ is the symmetric closure of μ (i. e. $\tilde{\mu} = \mu \cup \mu^{-1}$). It will be convenient to write $2 = i_2, \ldots, m = i_m$ and $i_1 = i_{m+1} = 1$, $c_1 = a$, $c_{m+1} = b$. Then, from (6) we obtain the following chain

$$1 \overset{\triangleright}{\underset{\triangleleft}{}} 2 \overset{\triangleright}{\underset{\triangleleft}{}} \ldots \overset{\triangleright}{\underset{\triangleleft}{}} m \overset{\triangleright}{\underset{\triangleleft}{}} 1, \tag{7}$$

and for each $i = 2, \ldots, m-1$ one has $c_{i+1} = \varphi_{i,i+1}(c_i)$ or $c_i = \varphi_{i+1,i}(c_{i+1})$. This will be denoted by $c_i \to c_{i+1}$ or $c_i \leftarrow c_{i+1}$, respectively.

We choose M_1 and $a, b \in M_1$ ($a \neq b$, $\psi_1(a) = \psi_1(b)$) for which the sequence (6) has the smallest length.

First, note that in a minimal chain the signs $\triangleleft$ and $\triangleright$ alternate. Secondly, for the pairs $(1, 2)$ and $(m, 1)$ of (7) the signs are different. Indeed, if, for example, $1 \triangleleft 2$ and $m \triangleleft 1$, then we obtain a chain

$$(c_2, c_3), \ldots, (c_{m-1}, c_m), (c_m, c_2'),$$

where $c_2' = \varphi_{12}(b) \neq \varphi_{12}(a) = c_2$ (since $a \neq b$). This chain is shorter than (6), a contradiction.

Consequently, the chain of indices (7) has one of the two forms:

$$1 \triangleleft 2 \triangleright 3 \triangleleft \ldots \triangleleft m \triangleright 1$$

or

$$1 \triangleright 2 \triangleleft 3 \triangleright \ldots \triangleright m \triangleleft 1. \tag{8}$$

In both cases m is even, and it is easy to see that $m \neq 2$.

Observe that without loss of generality, we may assume that our chain is of form (8). Indeed, suppose that the minimal sequence of elements is

$$a \to c_2 \leftarrow c_3 \to \ldots \to c_m \leftarrow b. \tag{9}$$

In this case, $1 \triangleleft 2$. Denote $d = \varphi_{12}(b)$. Then, $c_2 \neq d$ (otherwise, $a = b$ in view of the injectivity) and we obtain a chain of form (8) with the same length as (9):

$$c_2 \leftarrow c_3 \to \ldots \to c_m \leftarrow b \to d.$$

https://doi.org/10.1017/S0017089515000166 Published online by Cambridge University Press

A priori, some elements of the sequence (8) may coincide. We shall show that for a minimal chain this is impossible, and, moreover, that the elements in (8) form a subcrown.

Suppose $i \trianglelefteq j$ for some $2 \leq i, j \leq m$, $|i-j| > 1$, i. e. there exists $\varphi_{ij} : M_i \to M_j$. Assume $i < j$. Then, if $\varphi_{ij}(c_i) = c_j$, we obtain a shorter chain

$$a \leftarrow c_2 \rightarrow \ldots \rightleftarrows c_i \rightarrow c_j \rightleftarrows \ldots \leftarrow c_m \rightarrow b$$

(here the symbol $\rightleftarrows$ denotes one of the two arrows $\leftarrow$ or $\rightarrow$). If $\varphi_{ij}(c_i) \neq c_j$, then $(c_i, c_j) \in \nu$ and again we get a shorter chain

$$\varphi_{ij}(c_i) \leftarrow c_i \rightleftarrows \ldots \rightleftarrows c_j.$$

The case $i > j$ is symmetric.

All these arguments remain valid if we take a instead of c_i (or of c_j), noting that in this case $i = 1$ and $3 \leq j \leq m-1$ (or $j = 1$ and $3 \leq i \leq m-1$). This shows that (8) gives a subcrown.

(2) We show next that the obtained subcrown K is pure for $m \geq 8$ (i. e. when the degree of the crown ≥ 4). It suffices to verify the condition (5). Take arbitrary $i, j \in K_0$ with $\mathrm{dist}(i, j) \geq 4$ and suppose that there exists $k \in I$ such that $i \trianglelefteq k$, $j \trianglelefteq k$. Assume $i < j$. Then, we have

$$\ldots \rightarrow c_{i-1} \leftarrow c_i \rightarrow c_{i+1} \leftarrow \ldots \rightarrow c_{j-1} \leftarrow c_j \rightarrow c_{j+1} \leftarrow \ldots.$$

If $\varphi_{ik}(c_i) = \varphi_{jk}(c_j) = d$, then the original chain can be shortened:

$$\ldots \rightarrow c_{i-1} \leftarrow c_i \rightarrow d \leftarrow c_j \rightarrow c_{j+1} \leftarrow \ldots$$

If $\varphi_{ik}(c_i) \neq \varphi_{jk}(c_j)$, we get a new shorter chain:

$$\varphi_{ik}(c_i) \leftarrow c_i \rightarrow c_{i+1} \leftarrow \ldots \rightarrow c_{j-1} \leftarrow c_j \rightarrow \varphi_{jk}(c_j).$$

The case $i > j$ is similar.

(3) We check now the purity of K for $m = 6$. Consider the chain (8). Suppose that there is an element $i \in I$ such that $2 \triangleleft i$, $4 \triangleleft i$, $6 \triangleleft i$. Write

$$\varphi_{2i}(c_2) = x, \quad \varphi_{4i}(c_4) = y, \quad \varphi_{6i}(c_6) = z.$$

If $x \neq y$, we obtain a shorter chain

$$x \leftarrow c_2 \rightarrow c_3 \leftarrow c_4 \rightarrow y.$$

Therefore, $x = y$ and, similarly, $y = z$. But then we get again the shorter chain

$$a \leftarrow c_2 \rightarrow x \leftarrow c_6 \rightarrow b,$$

contradicting our assumption.

(4) It remains to consider the case $m = 4$. Suppose that our crown $K = \{2, 4; 1, 3\}$ has form (8):

$$1 \triangleright 2 \triangleleft 3 \triangleright 4 \triangleleft 1$$

with the corresponding sequence of elements

$$a \leftarrow c_2 \rightarrow c_3 \leftarrow c_4 \rightarrow b,$$

where $a \neq b$.

Assume that the elements 1 and 3 are $U(2,4)$-connected. Write $j_0 = 1$ and $j_{l+1} = 3$, and let $j_1, \dots, j_l \in U(2,4)$ be such that

$$j_0 \overset{\triangleright}{\underset{\triangleleft}{}} j_1 \overset{\triangleright}{\underset{\triangleleft}{}} \dots \overset{\triangleright}{\underset{\triangleleft}{}} j_l \overset{\triangleright}{\underset{\triangleleft}{}} j_{l+1}.$$

Consider the "intermediate" subposets $K^\alpha = \{2, 4; j_\alpha, j_{\alpha+1}\}$, $0 \leq \alpha \leq l$. By Remark 2, they are faithful. Therefore, $(\varphi_{2j_l}(c_2), \varphi_{4j_l}(c_4)) \in \nu|_{K^l}$ implies $\varphi_{2j_l}(c_2) = \varphi_{4j_l}(c_4)$. The latter equality, in turn, implies $\varphi_{2j_{l-1}}(c_2) = \varphi_{4j_{l-1}}(c_4)$ and so on. At the last step, we get $a = \varphi_{21}(c_2) = \varphi_{41}(c_4) = b$, contradicting the assumption $a \neq b$.

□

5. Filters and faithfulness. For the converse of Theorem 1, we need additional notions.

Let I be a poset. Denote by $\min I$ the set of minimal elements of I. For $a \in I$, let $F_a = \{x \in I \mid x \trianglerighteq a\}$ (the principal filter generated by a).

Consider the following conditions on I:

(1) $|\min I| < \infty$. In this case, we shall use positive integers to denote the minimal elements, i. e. we write $\min I = \{1, 2, \dots, n\}$.

(2) $\bigcup\limits_{i \in \min I} F_i = I$.

(3) For all $i, j \in \min I$, $i \neq j$,

$$F_i \cap F_j \begin{cases} \neq \varnothing, & \text{if } |i - j| \leq 1, \\ = \varnothing & \text{otherwise} \end{cases}$$

(as in Section 3 we work with i, j modulo n). In particular, if $n \geq 4$ then $F_i \cap F_j \cap F_k = \varnothing$ for pairwise different $i, j, k \in \min I$.

For $n = 3$, the last equality may not hold, so we add one more condition:

(4) If $n = 3$, then $F_1 \cap F_2 \cap F_3 = \varnothing$.

Proposition 3. *If a poset I satisfies conditions (1)–(4) and $n \geq 3$, then I is non-faithful.*

Proof. We construct an I-spectrum $M_I = (\{M_x | x \in I\}, \{\varphi_{xy} | x \trianglelefteq y\})$ as follows.

Write $G_i = F_{i-1} \cap F_i$ (in particular, $G_1 = F_n \cap F_1$). Note that by conditions (3) and (4), $G_i \cap G_j \neq \varnothing$ implies $i = j$. We put

$$M_x = \begin{cases} \{a_x\} & \text{if } x \notin G_1, \\ \{a_x, b_x\} & \text{if } x \in G_1. \end{cases}$$

We assume that $a_x \neq b_x$ and that the sets M_x are disjoint.

The maps φ_{xy} $(x \trianglelefteq y)$ are defined uniquely, if $|M_y| = 1$. For $|M_x| = |M_y| = 2$, $x \trianglelefteq y$, we set

$$\varphi_{xy}(a_x) = a_y, \quad \varphi_{xy}(b_x) = b_y. \tag{10}$$

https://doi.org/10.1017/S0017089515000166 Published online by Cambridge University Press

Next, we define

$$\varphi_{xz}(a_x) = a_z, \quad \varphi_{yz}(a_y) = b_z \tag{11}$$

for any $x \in F_1 \setminus G_1,\ y \in F_n \setminus G_1,\ z \in G_1$ such that $x \lhd z$ and $y \lhd z$. Since $y \unrhd x \in G_1$ implies $y \in G_1$, it follows by conditions (2)–(4) that all φ_{xy} are completely defined. Evidently, each φ_{xy} is injective.

To prove that M_I is an I-spectrum, we have to verify the condition

$$(x \unlhd y \unlhd z) \Longrightarrow \varphi_{yz}\varphi_{xy} = \varphi_{xz}. \tag{12}$$

If either $x, y, z \notin G_1$ or $x, y, z \in G_1$, then (12) is evident. Therefore, we can assume that $z \in G_1$ (otherwise $x, y \notin G_1$) and $x \notin G_1$ (otherwise $y, z \in G_1$). Then, from condition (3) we have $x \in F_n \setminus G_1$ or $x \in F_1 \setminus G_1$. In the first case, one have either $y \in F_n \setminus G_1$ or $y \in G_1$, and (12) is true. The second case is treated the same way.

Choosing an arbitrary element $\overline{k} \in G_k$ for each $k \le n$, we get a crown

$$K = \{1, \dots, n; \overline{1}, \dots, \overline{n}\}.$$

Now the non-faithfulness of M_I follows from the fact that its restriction to K coincides with the spectrum constructed in the proof of Proposition 2.

□

6. The converse of Theorem 1. We begin with the following statement.

LEMMA 1. *Each filter in a faithful poset is faithful.*

Proof. Let F be a filter in a faithful poset I, and let

$$M_F = (\{M_x | x \in F\}, \{\varphi_{xy} | x \unlhd y\})$$

be an F-spectrum. We extend it to I by putting $M_x = \varnothing$ for $x \in I \setminus F$. The maps φ_{xy} for $x \in I \setminus F$, $x \unlhd y$, are determined automatically and it is easy to see that we obtained an I-spectrum M_I which must be faithful. Therefore, M_F is also faithful. □

THEOREM 2. *If a poset I contains a pure subcrown, then I is non-faithful.*

Proof. Let K be a pure subcrown in I of form (4) (see Section 3). By Lemma 1, replacing I by the filter generated by $K_0 = \{1, 2, \dots, n\}$, we may assume that $\bigcup_{i \in K_0} F_i = I$. In particular, $\min I = K_0$.

First, suppose that $n \ge 3$. We have that I satisfies conditions (1) and (2) from Section 5. Furthermore, $G_k \ne \varnothing$ for any $k \in K_0$, since it contains the element $\overline{k}$ of the crown. If $F_i \cap F_j \ne \varnothing$, then it follows from the purity that $|i - j| \le 1$, i. e. (3) holds. Finally, condition (4) follows from the definition of purity for $n = 3$. Then by Proposition 3, we conclude that I is non-faithful.

It remains to consider the case $n = 2$. Let $K = \{1, 2; \overline{1}, \overline{2}\}$ be a pure subcrown in I and $I = F_1 \cup F_2$, where F_1 and F_2 are the principal filters generated by 1 and 2. Recall that $\min I = \{1, 2\}$. Write $G = F_1 \cap F_2 = U(1, 2)$. Denote by H the set of elements of G which are G-connected with $\overline{1}$. We note some obvious properties of H:

H is a filter in I;
H is an ideal in G;
$\overline{2} \notin H$, as K is pure.

Similarly as we did in the proof of Proposition 3, set

$$M_x = \begin{cases} \{a_x\}, & \text{if } x \notin H, \\ \{a_x, b_x\}, & \text{if } x \in H. \end{cases}$$

The maps φ_{xy} ($x \trianglelefteq y$) are uniquely defined, if $y \notin H$. For $x, y \in H$, $x \trianglelefteq y$, they are given by the equalities (10). For $x \in F_1 \setminus G$, $y \in F_2 \setminus G$, $z \in H$ such that $x \triangleleft z$ and $y \triangleleft z$ the maps φ_{xz} and φ_{yz} are defined by (11). Since H is a filter in I and an ideal in G, all φ_{xy} are defined and they are injective.

We need to check (12). If either $x, y, z \notin H$ or $x, y, z \in H$, then (12) is obvious. Therefore, we can assume that $z \in H$ (otherwise $x, y \notin H$) and $x \notin H$ (otherwise $y, z \in H$). Further, if $y \in G$, then $y \in H$ as H is an ideal in G, and (12) is readily verified. Assume $y \notin G$.

Suppose $y \in F_1 \setminus G$. Then, $x \notin F_2$, as $y \notin G$. Hence, $x \in F_1 \setminus G$ and (12) is true. The case $y \in F_2 \setminus G$ is symmetric. It remains, as in the Proposition 3, to refer to the proof of Proposition 2. □

Applying Proposition 1, we obtain the following:

COROLLARY 2. *If a poset I contains a pure subcrown, then I is not linearly faithful.*

ACKNOWLEDGEMENTS. We thank the referee for the valuable remarks which helped to improve the article. M.D. was partially supported by FAPESP of Brazil and partially by CNPq of Brasil. B.N. was supported by FAPESP of Brasil.

REFERENCES

1. I. Assem, D. Castonguay, E. N. Marcos and S. Trepode, Quotients of incidence algebras and the Euler characteristic, *Commun. Algebra* **35**(4) (2007), 1075–1086.

2. N. Bourbaki, *Éléments de mathématique. Théorie des ensembles* (Paris, Hermann, 1970).

3. I. Bucur and A. Deleanu, *Introduction to the theory of categories and functors* (Wiley Interscience, N.Y., 1968).

4. J. Gibbons, Towards a colimit-based semantics for visual programming, in *Coordination models and languages. Proceedings of the 5th International Conference, COORDINATION 2002, York, GB, April 8–11* (Arbab, F. et al., Editors) Lect. Notes Comput. Sci., vol. 2315 (Springer, Berlin, 2002), 166–173.

5. C. Cheng and B. Mitchell, DCC posets of cohomological dimension one, *J. Pure Appl. Algebra* **13** (1978), 125–137.

6. P. Dräxler, Completely separating algebras, *J. Algebra* **165**(2) (1994), 550–565.

7. D. Duffus, I. Rival, Retracts of partially ordered sets, *J. Austral. Math. Soc.* (Series A) **27** (1979), 495–506.

8. R. Exel, Noncommutative Cartan subalgebras of C*-algebras, *New York J. Math.* **17** (2011), 331–382.

9. J. A. Goguen, Categorical foundations for general systems theory, in *Advances in cybernetics and system research* (Transcripta Books, London, 1973), 121–130.

10. J. A. Goguen, A categorical manifesto, *Math. Struct. Comput. Sci.* **1**(1) (1991), 49–67.

11. K. A. Hardie and P. J. Witbooi, Crown multiplications and a higher order Hopf construction, *Topology Appl.* **154**(10) (2007), 2073–2080.

12. S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology* (Princeton University Press, Princeton, New Jersey, 1952).

13. C. Oriat, Detecting equivalence of modular specifications with categorical diagrams, *Theor. Comput. Sci.* **247**(1–2) (2000), 141–190.

14. D. R. Smith, Composition by colimit and formal software development, in *Algebra, meaning and computation. Essays dedicated to Joseph A. Goguen on the occasion of his 65th birthday* (Futatsugi, K. et al., Editors) Lecture Notes in Computer Science, vol. 4060 (Springer, Berlin, 2006), 317–332.