HARMONIC p-TENSORS ON NORMAL HYPERBOLIC RIEMANNIAN SPACES

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Introduction. The subject of this paper is the study of boundary value theorems for harmonic p-tensors on a Riemannian space with an indefinite metric of the normal hyperbolic signature. The p-tensors or p-vectors $\phi_{i_1 \dots i_p}$ are alternating covariant tensors of rank p, which are closely related to differential forms ϕ of degree p, $p \leqslant m$, on an m-dimensional manifold.

On a Riemannian space with positive definite metric, the harmonic p-tensors have been studied by Hodge, de Rham, Kodaira, and others, and a theory which generalizes the classical potential theory, and possesses in addition certain new features, has been developed.

The present paper continues a program of extending the boundary value theorems of the theory of partial differential equations to p-tensors. In the positive definite (elliptic) case, boundary value theorems have been given [**3**] for the Beltrami-Laplace equation

$$\Delta\phi = 0$$

and for the harmonic field equations

$$d\phi = 0, \quad \delta\phi = 0.$$

These equations are now investigated under an indefinite metric.

The principal result is the solution of the Cauchy problem for the Beltrami-Laplace equation. After a preliminary section, the auxiliary conditions for the Cauchy problem are formulated, and uniqueness under these conditions is established. To construct the solution, I use the method of Riesz potentials. For the sake of brevity, reference to Riesz' paper [**9**] has been made wherever possible. The solution of the Beltrami-Laplace equation thus obtained is then applied to the construction of solutions of the harmonic field equations. In a concluding section, some special cases, in particular the electromagnetic field equations, are examined, and the main theorems are compared with the results which hold in the elliptic case.

1. Preliminaries. Let M be an m-dimensional orientable Riemannian space, with metric

$$ds^2 = g_{ij}\, dx^i dx^j$$

having the Lorentz signature. That is, on being transformed to a sum of squares,

Received October 11, 1951. This paper was written in the course of work sponsored by the Office of Naval Research.

https://doi.org/10.4153/CJM-1953-008-8 Published online by Cambridge University Press

ds^2 contains one positive and m-1 negative terms. For convenience we shall refer to displacements as timelike if $ds^2 > 0$ and spacelike if $ds^2 < 0$. We shall also distinguish the timelike coordinate, say x^m, by the letter t in certain cases. This metric is also known as a normal hyperbolic metric (Hadamard), the corresponding Laplace equation being of the normal hyperbolic type. During most of this paper we assume that the metric is analytic, but this restriction can be removed.

Throughout we shall use such terms as continuous, compact, convergent, on the understanding that they refer to a suitable positive definite distance, such as the Euclidean distance.

We introduce on M the skew symmetric covariant tensors $\phi_{i_1 \ldots i_p}$; corresponding to these are differential forms of degree p:

$$(1.1) \quad \phi = \phi_p = \sum_{i_1 < \ldots < i_p} \phi_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} = \phi_{(i_1 \ldots i_p)} dx^{i_1} \wedge \ldots \wedge dx^{i_p}.$$

The bracket enclosing the group of indices shall mean that the indices inside it are arranged in strictly increasing order. The differentials dx^i are multiplied together by the exterior multiplication indicated by the sign $\wedge$; hence these differentials anticommute. We now define the differential operator d;

$$(1.2) \qquad d\phi = (d\phi_{(i_1 \ldots i_p)})\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

Thus $d\phi$ is a differential form of degree $p + 1$. Let

$$(1.3) \qquad \Gamma_{i_1 \ldots i_p;\, j_1 \ldots j_p} = \begin{vmatrix} g_{i_1 j_1} & \ldots & g_{i_1 j_p} \\ \cdot & & \cdot \\ g_{i_p j_1} & \ldots & g_{i_p j_p} \end{vmatrix},$$

then

$$\Gamma_{i_1 \ldots i_p}{}^{j_1 \ldots j_p}$$

is just the Kronecker symbol often denoted by

$$\delta_{i_1 \ldots i_p}{}^{j_1 \ldots j_p}.$$

Also let

$$(1.4) \qquad e_{i_1 i_2 \ldots i_n} = \Gamma_{i_1 \ldots i_n}{}^{12 \ldots n} \left|\Gamma_{12 \ldots n,\, 12 \ldots n}\right|^{\frac{1}{2}}.$$

We denote covariant derivatives of a p-tensor

$$\phi_{i_1 \ldots i_p} \text{ by } D_i\, \phi_{i_1 \ldots i_p},$$

and set $D^i = g^{ij} D_j$. Thus

$$(1.5) \qquad D_i \phi_{i_1 \ldots i_p} = \frac{\partial \phi_{i_1 \ldots i_p}}{\partial x^i} - \sum_{k=1}^{p} \left\{ {h \atop i_k\, i} \right\} \phi_{i_1 \ldots i_{k-1} h\, i_{k+1} \ldots i_p}.$$

In the notation of [**8**a],

$$(1.6) \qquad \begin{aligned} (d\phi)_{i_1 \ldots i_{p+1}} &= \Gamma_{i_1 \ldots i_{p+1}}{}^{j(j_1 \ldots j_p)} D_j \phi_{(j_1 \ldots j_p)} \\ &= \Gamma_{i_1 \ldots i_{p+1}}{}^{j(j_1 \ldots j_p)} \frac{\partial}{\partial x^j} \phi_{(j_1 \ldots j_p)}. \end{aligned}$$

https://doi.org/10.4153/CJM-1953-008-8 Published online by Cambridge University Press

The operator $d.d$ is identically zero. If $d\phi = 0$, ϕ is said to be closed; if $\phi = d\chi$, ϕ is said to be derived, and is therefore closed. The converse statement that if ϕ is closed then ϕ is derived, is true "locally," but not in the large.

The dual $*\phi$ of a p-form ϕ is an $(m - p)$-form whose components are defined by the formula

$$(1.7) \qquad (*\phi)_{j_1 \ldots j_{m-p}} = e_{(i_1 \ldots i_p) j_1 \ldots j_{m-p}} \phi^{(i_1 \ldots i_p)}$$

If p is the degree of ϕ, then

$$**\phi = (-1)^{mp+p}\phi.$$

Hence, save for a sign, the $*$ operation is its own inverse. It will be seen that ϕ and $*\phi$ are "perpendicular" forms at each point.

The operation of dual derivation is given by δ, where

$$(1.8) \qquad \begin{aligned} (\delta\phi)_{i_1 \ldots i_{p-1}} &= (-1)^{mp+m+1}(*d*\phi)_{i_1 \ldots i_{p-1}} \\ &= -\Gamma_{i i_1 \ldots i_{p-1}}{}^{(j_1 \ldots j_p)} D^i \phi_{(j_1 \ldots j_p)} \end{aligned}$$

It follows from the preceding remarks that $\delta.\delta = 0$. If $\delta\phi = 0$, ϕ is said to be coclosed, and if $\phi = \delta\chi$, ϕ is coderived. A p-tensor ϕ which is both closed and coclosed will be known as a harmonic field [**6**].

The Beltrami-Laplace (B.L.) operator Δ for p-tensors is given by

$$(1.9) \qquad -\Delta = d\delta + \delta d,$$

and in expanded form may be written

$$(1.10) \quad (\Delta\phi)_{i_1 \ldots i_p} = D^i D_i \phi_{i_1 \ldots i_p} - \sum_{n=1}^{p} \Gamma_{k\ i_1 \ldots i_p}{}^{i(j_1 \ldots j_p)}\, g^{kj} R^h{}_{j_n j i} \phi_{j_1 \ldots j_{n-1} h j_{n+1} \ldots j_p}$$

where $R^h{}_{ijk}$ is the Riemannian curvature tensor and summation over the indices i, j, h, k and $(j_1 \ldots j_p)$ is understood. If ϕ is a scalar, then $\Delta\phi$ reduces to the usual Laplacian. A p-tensor ϕ whch satisfies the B.L. equation

$$(1.11) \qquad \Delta\phi = 0$$

is said to be a harmonic form.

From (10) it is clear that the B.L. equation is of the normal hyperbolic type under our Lorentzian metric. In fact, the equation (1.11) stands for a system of $\binom{m}{p}$ equations, one for each component of ϕ. Note that each of these equations has the same principal part, namely,

$$g^{ij} \frac{\partial^2 \phi_{i_1 i_2 \ldots i_p}}{\partial x^i\, \partial x^j}.$$

Actually all terms containing first or second derivatives are of the same form in each component equation. Furthermore the form of the second order (principal) terms is independent of p, so the theory of the characteristics will be carried over unchanged from the scalar case. The characteristic surfaces of each component equation being the same, we may speak of a characteristic surface of (11). These surfaces are given by

$$C(x^1 \dots x^m) = C(x) = 0,$$

where

$$g^{ij} \frac{\partial C}{\partial x^i} \frac{\partial C}{\partial x^j} = 0. \tag{1.12}$$

Let O be an arbitrary point of M which we select as origin of coordinates. The geodesics of zero length through O form a conoid [**5**; **8**] which is a characteristic surface (1.12).

Let S be a sufficiently differentiable spacelike "initial" surface. For convenience we assume that S is compact. We construct a system of geodesic normal coordinates in which S is the hyperplane $x^m \equiv t \equiv 0$. Since the normals to S are timelike, we can write

$$ds^2 = dt^2 - g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \tag{1.13}$$

where the Greek indices range from 1 to $m - 1$. In this system [**10**], $g_{im} = \delta_m{}^i$, $D_m \equiv D^m$.

Associated with S we define a region R as follows. R shall consist of those points P on the positive side ($t \geqslant 0$) of S, such that the retrograde characteristic cones (null cones) C_P with vertex P, together with that part of S intercepted by C_P, bound a simply-connected region of M. Denoting this region by $D_S{}^P$, following Riesz, we see that if $P \in R$, every point of $D_S{}^P$ also belongs to R. We may refer to $R = R_S$ as the region of exclusive dependence upon S, in the sense that, as will be shown, solutions of the B.L. equation with data assigned on S are determined throughout R.

Thus defined, R_S may contain non-bounding p-cycles. Any p-cycle of R is however homologous to a p-cycle of S, since a continuous cylindrical $(p + 1)$-dimensional surface can be constructed on the cycle, lying in R and joining the cycle to a (homologous) p-cycle which is its intersection with S. An analogous remark holds for the relative p-cycles of R (mod S): these can all be deformed into S and are therefore zero.

A p-form ϕ induces on any surface S (supposed given by $x^m = 0$) a p-form $t\phi$, the components of which are precisely those components of ϕ which are not multiplied by dx^m. The form $t\phi$ is known as the tangential boundary component of ϕ on S. The residual part of ϕ, which contains the factor dx^m, is known as the normal boundary component and is denoted by $n\phi$. Thus

$$\phi = t\phi + n\phi = t\phi + \phi_1 \wedge dx^m. \tag{1.14}$$

If $p = 0$, $t\phi \equiv \phi$; if $p = m$, $t\phi \equiv 0$. On S, and only there, this division into tangential and normal components has an invariant meaning. From the properties of the dual operator (7) it follows that the commutation rules

$$*t = n*, \quad *n = t* \tag{1.15}$$

hold. Also we note that if ϕ is assigned on S (both $t\phi$ and $n\phi$), then $td\phi$ and $n\delta\phi$ are thereby determined. However, $nd\phi$ and $t\delta\phi$ can be assigned independently,

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since each component of these forms contains a normal derivative of a component of ϕ. Note especially that $nd\phi$ and $t\delta\phi$ have $\binom{m-1}{p}$ and $\binom{m-1}{p-1}$ components respectively, making a total of $\binom{m}{p}$ components. Finally, we easily verify that

$$td\phi = d_S\, t\phi, \tag{1.16}$$

where d_S is the differential operator (6) in the surface S.

The surface S will be the carrier of the "initial" data for the Cauchy problem for the p-tensor B.L. equation. Since this equation is of the second order, it is to be expected that values of the components of ϕ and of their normal derivatives will be assigned on S. In order to express the data on the surface S in an invariant form, we shall use the following lemma.

LEMMA I. *Let S be a surface whose equation is $x^m = 0$ in a sufficiently differentiable system of coordinates in M, and let ϕ be a p-form defined in a neighbourhood of S. Then the specification of any one of the following sets of data is equivalent*:

$$\text{(a) } \phi,\, nd\phi,\, t\delta\phi;\quad \text{(b) } \phi,\, D_m\phi,\quad \text{(c) } \phi,\, \frac{\partial\phi}{\partial x^m}. \tag{1.17}$$

The vanishing of one implies the vanishing of the other two.

Proof of the lemma will be given in cyclic order. First we show that knowledge of (a) enables us to calculate (b). If $i_1 \ldots i_p$ are all less than m, then from (6) follows the value of

$$D_m\phi_{i_1\ldots i_p}$$

in terms of

$$(d\phi)_{i_1\ldots i_p m}$$

and covariant derivatives of ϕ along the surface. Similarly, if $i_p = m$, say

$$D_m\phi_{i_1\ldots i_{p-1}m}$$

is given by (8) in terms of data included in (a). Next, (b) clearly implies (c), since all components of ϕ are known on S. Finally, (c) implies that $d\phi$ is known on S, by (6). Since $*\phi$ and $D_m*\phi$ are obtainable from (c), $\delta\phi$ is also known on S. This proves the equivalence. The last statement of the lemma is evident.

Let C be a $(p + 1)$-chain of M with real coefficients, and let its boundary be denoted by bC. Then Stokes's formula

$$\int_C d\phi = \int_{bC} \phi \tag{1.18}$$

holds. Hence, if ϕ is closed, its integral over a bounding cycle C vanishes. It follows that a closed p-form ϕ has periods $\int_z \phi$ on p-cycles Z, which depend only on the homology class of Z. In a manifold with boundary [**2**; **8b**], a closed p-form ϕ is derived if and only if its periods (on absolute cycles) vanish.

If ϕ and ψ are two forms, the sum of whose degrees does not exceed m, we have

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$$(1.19) \qquad d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi,$$

where p is the degree of ϕ.

Consider a subregion of M, which we may take to be the region R. Let α and β be two forms of degree p defined in R, and let

$$(1.20) \qquad (\alpha, \beta)_R = \int_R \alpha \wedge *\beta = (\beta, \alpha)_R$$

denote the scalar product over R. We remark that in the elliptic case $(ds^2 > 0)$, the scalar square $(\alpha, \alpha)_R$ is positive definite; however this no longer holds for our metric. However, for any two forms ϕ and ψ of degrees p and $(p+1)$ respectively, we have the formula of Green:

$$(1.21) \qquad (d\phi, \psi)_R - (\phi, \delta\psi)_R = \int_{bR} \phi \wedge *\psi,$$

which follows from (18), (19), and (20). The metric being indefinite, we should change the sign of the surface integrals over any timelike surface. However, for all applications which we have in view, the surface bR will consist of spacelike surfaces and null cones, and the integrand will be made to vanish on the latter; so the formula will apply as written. From (21) we obtain by formal transformation the extended formula of Green, namely,

$$(1.22) \qquad -(\Delta\phi, \psi)_R + (\phi, \Delta\psi)_R$$
$$= \int_{bR} (\phi \wedge *d\psi - \psi \wedge *d\phi + \delta\phi \wedge *\psi - \delta\psi \wedge *\phi),$$

where ϕ and ψ are now of equal degree. Regarding the right-hand side of (22) as a linear functional of the form ϕ, ψ being held fixed, we observe that the successive terms contain $t\phi$, $nd\phi$, $t\delta\phi$, and $n\phi$; and that each of these expressions can be assigned independently of the other three. All four together are equivalent to any one of (17).

2. A uniqueness property of the B.L. equation. Let $R = R_S$ be the region of dependence associated with an initial surface S as described in the preceding section. In terms of the geodesic normal coordinate system (13), we may write the partial differential equation (11) in the form

$$(2.1) \qquad L(\phi_{i_1 \ldots i_p}) \equiv \frac{\partial^2 \phi_{i_1 \ldots i_p}}{\partial t^2} - g^{\alpha\beta} \frac{\partial^2 \phi_{i_1 \ldots i_p}}{\partial x^\alpha \partial x^\beta}$$
$$+ a_{i_1 \ldots i_p}{}^{k(j_1 \ldots j_p)} \frac{\partial}{\partial x^k} \phi_{(j_1 \ldots j_p)} + b_{i_1 \ldots i_p}{}^{(j_1 \ldots j_p)} \phi_{(j_1 \ldots j_p)} = 0,$$

where $t = x^m$, $i \leqslant \alpha, \beta \leqslant m - 1$; and

$$a_{i_1 \ldots i_p}{}^{k(j_1 \ldots j_p)}, \quad b_{i_1 \ldots i_p}{}^{(j_1 \ldots j_p)}$$

are polynomial functions of the components of the metric tensor and their first derivatives, which are skew-symmetric with respect to the indices $i_1 \ldots i_p$;

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$j_1 \ldots j_p$. In this coordinate system the Cauchy problem consists of solving (2.1) (in general with a non-homogeneous right-hand term) given the values of the

$$\phi_{i_1 \ldots i_p} \text{ and } \frac{\partial}{\partial t} \phi_{i_1 \ldots i_p}$$

on S.

LEMMA II. *The solution of the Cauchy problem for* (2.1) *is unique in* R_s.

The argument which we use is of a standard type [**1**, II, p. 310]. Let P be a point of R with retrograde null cone C_P, and suppose ϕ and $\partial\phi/\partial t$ are assigned the value zero on the section S_P of S enclosed by C_P. We have to prove that $\phi(P)$ is zero. Let ϕ_i denote a typical component of ϕ. For $Q \in R$, we have

$$\phi_i(Q) = \int_0^{t(Q)} \frac{\partial \phi_i}{\partial t}\, dt,$$

so that by the Schwarz inequality

$$\phi_i^2(Q) \leqslant t(Q) \int_0^{t(Q)} \left(\frac{\partial \phi_i}{\partial t}\right)^2 dt = t \int_0^t \phi_{it}^2 dt,$$

where $t = t(Q)$ and the integration is taken along a parametric line of the coordinate system.

The subscripts α and t will denote partial derivatives with respect to x^α and t, respectively. Let D_h be the region enclosed by C_P, and the planes $t = 0$, $t = h$, and $S_P(h)$ that part of the plane $t = h$ cut off by C_P. We have

$$\begin{aligned} \int_{S_P(h)} \phi_i^2(Q)\, dA &\leqslant h \int_{S_P(h)} dA \int_0^h \phi_{it}^2\, dt \\ &\leqslant h \int_{D_h} \phi_{it}^2\, dV \leqslant h \int_0^h E_i(h)\, dh \end{aligned} \tag{2.2}$$

where

$$E_i(h) = \int_{S_h} (\phi_{it}^2 + g^{\alpha\beta} \phi_{i\alpha} \phi_{i\beta})\, dA. \tag{2.3}$$

This follows since the metric form $g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ in the spacelike surface S_h is positive definite. Integrating (2.2) with respect to t from 0 to h, and noting that $E_i(h) \geqslant 0$, we have

$$\int_{D_h} \phi_i^2\, dV \leqslant h^2 \int_0^h E_i(t)\, dt. \tag{2.4}$$

We next consider the identity

$$\begin{aligned} 2\phi_{it} L(\phi_i) &= 2\phi_{it}\phi_{itt} - 2g^{\alpha\beta}\phi_{i\alpha\beta}\phi_{it} + B_i(\phi_j, \phi_{j\alpha}, \phi_{jt}) \\ &= (\phi_{it})_t^2 + (g^{\alpha\beta}\phi_{i\alpha}\phi_{i\beta})_t - 2(g^{\alpha\beta}\phi_{i\alpha}\phi_{it})_\beta + B_i(\phi_j, \phi_{j\alpha}, \phi_{jt}), \end{aligned} \tag{2.5}$$

where B_i denotes an expression linear in the components of ϕ and their first derivatives, multiplied by ϕ_{it}. Let α_ν and t_ν be the direction numbers ds/dx^α, ds/dt of the conormal ν to a surface. On the surface C_P we have $ds^2 = 0$, whence

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$$(2.6) \qquad t_\nu^2 = g^{\alpha\beta}\alpha_\nu\beta_\nu \qquad \text{on } C_P.$$

We integrate (2.5) over D_h and apply the divergence theorem. Since $L(\phi_i)$ is zero we have

$$0 = \int_{bD_h}\left\{(\phi_{it})^2 t_\nu + (g^{\alpha\beta}\phi_{i\alpha}\phi_{i\beta})t_\nu - 2(g^{\alpha\beta}\phi_{i\alpha}\phi_{it})\beta_\nu\right\} dA + \int_{D_h} B_i\, dV.$$

On S_P ($t = 0$), ϕ and its derivatives are to be zero; on S_h we have $t_\nu = 1, \alpha_\nu = 0$. Using (2.6) we then find

$$\int_{C_P(h)} \frac{1}{t_\nu}\left\{g^{\alpha\beta}\alpha_\nu\beta_\nu\phi_{it}^2 - 2t_\nu g^{\alpha\beta}\beta_\nu\phi_{i\alpha}\phi_{it} + g^{\alpha\beta}t_\nu^2\phi_{i\alpha}\phi_{i\beta}\right\} dA$$
$$+ \int_{S_h}\left\{\phi_{it}^2 + g^{\alpha\beta}\phi_{i\alpha}\phi_{i\beta}\right\} dA$$
$$= \int_{C_P(h)} \frac{1}{t_\nu}\sum_{\alpha\beta} g^{\alpha\beta}(t_\nu\phi_{i\alpha} - \alpha_\nu\phi_{it})(t_\nu\phi_{i\beta} - \beta_\nu\phi_{it})\, dA + E_i(h)$$
$$= -\int_{D_h} B_i\, dV = R_i,$$

denoting the value of the last integral by R_i. Here we have denoted by $C_P(h)$ that portion of C_P which lies between the planes $t = 0$, $t = h$. On C_P, t_ν is positive. Recalling that the metric form $g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ is positive definite, we conclude that

$$(2.7) \qquad E_i(h) \leqslant R_i.$$

Adding together these relations for all components ϕ_i, we find

$$(2.8) \qquad \Sigma(h) = \sum_i E_i(h) \leqslant \sum_i R_i = R.$$

Now the quantity R defined by (2.8) is an integral over D_h of terms $\phi_{i\alpha}\,\phi_{jt}$ or $\phi_j\,\phi_{it}$, each of which is less than half the sum of the squares of the two factors. The coefficients of these terms in the integral R are bounded. Hence, in view of (2.4) we have an inequality

$$(2.9) \qquad R \leqslant K\int_0^h \Sigma(h)\, dh,$$

for some positive constant K. From (2.8) and (2.9) follows

$$\Sigma(h) \leqslant K\int_0^h \Sigma(t)\, dt \leqslant K\int_0^k \Sigma(t)\, dt$$

for every $k \geqslant h$. Integrating between 0 and h,

$$(2.10) \qquad \int_0^h \Sigma(h)\, dh \leqslant Kh\int_0^h \Sigma(h)\, dh.$$

But (2.10) is manifestly false for $h < 1/K$, unless $\Sigma(h) \equiv 0$, $(0 \leqslant h \leqslant 1/K)$. Hence $\Sigma(h)$ must vanish in this range, so that all derivatives of components of ϕ vanish for $t < 1/K$. Hence ϕ itself is zero in this interval of values of t. Since

K can be chosen to depend only on S and P and the metric, we can repeat the above process starting from the surface $S_{1/K}$, and so on. Hence finally we obtain $\phi(P) = 0$. This completes the uniqueness proof.

3. The Riesz kernel. A detailed exposition of the solution of the Cauchy problem for normal hyperbolic equations in Riemannian spaces has been given by M. Riesz [**9**]. We shall construct the solution for the p-tensor B.L. equation using Riesz's method. For a complete discussion of matters upon which we only touch, the reader is referred to Riesz's work.

It is required to construct a double p-tensor kernel

$$V^\alpha(x, y) = V_p^\alpha(x, y) = \left\{ V_p^\alpha(x, y)_{i_1 \ldots i_p;\, j_1 \ldots j_p} \right\} \tag{3.1}$$

having the properties:

(a) $\Delta_x V^{\alpha+2} = V^\alpha$,

(b) $V^\alpha(x, y) = 0(s^{\alpha - m})$,

where s is the geodesic distance from x to y,

(c) $V^\alpha(x, y) = V^\alpha(y, x)$,

(d) $\int_{D_y^x} V^\alpha(x, y) \wedge *V^\beta(z, y) = V^{\alpha+\beta}(x, y)$,

where D_y^x is the double conoid enclosed by the retrograde null cone of one argument point and the direct cone of the other,

(e) $V^m(x, y)|_{y=x} = \sum_{i_1 < \ldots < i_p} \frac{1}{2^{m-1}\pi^{\frac{1}{2}(m-1)}} dx^{i_1} \wedge \ldots \wedge dx^{i_p} dy^{i_1} \wedge \ldots \wedge dy^{i_p}$.

We proceed to construct V^α by means of a series expansion in powers of s. Properties (c) and (d) will be established later.

Certain facts from the theory of geodesics, which we now set forth, will be required. Riesz gives a detailed account of these matters. Choosing the point x_0 as origin, we denote by $x^1 \ldots x^m$ a coordinate system in M. On a family of curves C, let $p_1 \ldots p_m$ be the conjugate variables defined by

$$p_i = g_{ik}\dot{x}^k, \quad \dot{x}^k = \frac{dx^k}{d\sigma}, \tag{3.2}$$

where σ is a parameter along C. We introduce the Hamiltonian function $H(x_0, x)$, where

$$2H(x_0, x) = g^{ik} p_i p_k, \tag{3.3}$$

and note that the differential equations of the geodesics can be written

$$\dot{x}^i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}. \tag{3.4}$$

The geodesics passing through x_0 are determined by the quantities $\dot{x}_0^i$. As we

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shall need to consider only timelike geodesics ($ds^2 > 0$), we may take s as proportional to the parameter σ on the curve. If we set $\sigma(x) = 1$, then since H is constant along a geodesic we have

(3.5)
$$s = s(x_0, x) = \int_0^1 \sqrt{g_{ik}\dot{x}^i\dot{x}^k}\, d\sigma = \int_0^1 \sqrt{g^{ik}p_ip_k}\, d\sigma = \int_0^1 \sqrt{2H}\, d\sigma = \sqrt{2H};$$

letting $P = s^2$ we find $P = 2H$.

We now select normal coordinates ξ^k which determine [9, p. 173] the point x:

(3.6)
$$\xi^k = \sigma_x \dot{x}_0^k = s\, \xi_0^k.$$

Taking S as parameter, we have $\xi^k = \xi_0^k$ along the geodesic arc.

A number of useful identities may now be derived. Since H is constant along a geodesic, we have

(3.7)
$$g^{ik}(0)\eta_i\eta_k = g^{ik}(x)\eta_i\eta_k$$

where the η_i are variables conjugate to the ξ^k. From (3.7) follows in view of (3.2) the relation.

(3.8)
$$g_{ik}(0)\xi^i\xi^k = g_{ik}(0)s^2(0, \xi)\xi_0^i\xi_0^k = g_{ik}(\xi)s^2(0, \xi)\xi^i\xi^k = g_{ik}(\xi)\xi^i\xi^k;$$

hence from (3.5)

(3.9)
$$P(0, \xi) = s^2(0, \xi) = g_{ik}(0)\xi^i\xi^k = g_{ik}(\xi)\xi^i\xi^k.$$

Thus, choosing Riemannian coordinates at x_0 such that $g_{mm} = 1$, $g_{kk} = -1$, $k \neq m$, $g^{ik} = 0$, $i \neq k$, we find for P the Lorentz distance

(3.10)
$$P = s^2 = (\xi^m)^2 - (\xi^{m-1})^2 - \ldots - (\xi^1)^2.$$

Next, denoting $\partial P/\partial \xi^i$ by P_i, we have

$$P_i = 2g_{ij}\xi^j, \quad P^i = 2\xi^i,$$

whence

(3.11)
$$P^iP_i = 4g_{ij}\xi^i\xi^j = 4P.$$

This partial differential equation for P shows, that, as has been mentioned, the geodesics of zero length through x_0 form a characteristic surface for the B.L. equation. In fact, the geodesic lines are just the bicharacteristics.

Again, let V be a differentiable function of position; then

(3.12)
$$P^i \frac{\partial U}{\partial \xi^i} = 2\xi^i \frac{\partial U}{\partial \xi^i} = 2s\xi^i \frac{\partial U}{\partial \xi^i} = 2s \frac{d\xi^i}{ds} \frac{\partial U}{\partial \xi^i} = 2s \frac{dU}{ds}.$$

Finally, we note that

(3.13)
$$g^{ij} \frac{\partial^2 P}{\partial \xi^i \partial \xi^j} = 2g^{ij}g_{ij} + 2g^{ij}\xi^k \frac{\partial g_{jk}}{\partial \xi^i} = 2m + 0(s^2).$$

These formulae will now be applied to the construction of the kernel. We shall need to calculate the Laplacian of a p-tensor ϕ multiplied by a power of the geodesic distance. Let $F(P)$ be a scalar function of P. Then we have

$$\begin{aligned}\Delta(F(P)\phi) &= D^iD_i(F(P)\phi) + BF(P)\phi \\ &= F(P)\Delta\phi + (D^iD_iF(P) + 2D^iF(P)D_i)\phi,\end{aligned} \tag{3.14}$$

where B denotes a suitable multiplying matrix. We find

$$\begin{aligned}D^iD_iF(P) &= D^i(F'(P)P_i) \\ &= g^{ik}[F''(P)P_iP_k + F'(P)P_{ik} - \{_k{}^\sigma{}_i\}F'(P)P_\sigma] \\ &= 4PF''(P) + 2mF'(P) + F'(P)sA_1,\end{aligned} \tag{3.15}$$

where A_1 is a suitable bounded multiplying matrix. For the last term in (3.14) we find

$$\begin{aligned}&D^iF(P)D_i\phi_{(j_1\ldots j_p)} \\ &\quad= F'(P)\left[P^i\frac{\partial}{\partial\xi^i}\phi_{(j_1\ldots j_p)} - P^i\sum_{n=1}^{p}\{_{i_n}{}^h{}_i\}\phi_{j_1\ldots j_{n-1}h\, j_{n+1}\ldots j_p}\right] \\ &\quad= F'(P)\left[2s\frac{d\phi_{(j_1\ldots j_p)}}{ds} + sA_{2(j_1\ldots j_p)}\phi\right]\end{aligned} \tag{3.16}$$

where A_2 is a second matrix with bounded elements near x_0.

In the preceding equations, ϕ is to be understood as a vector with $\binom{m}{p}$ components, while A_1, A_2 and B are square matrices of order $\binom{m}{p}$. Continuing with this notation, we have

$$\begin{aligned}\Delta(F(P)\phi) = F(P)\Delta\phi &+ [4PF''(P) + 2mF'(P)]\phi \\ &+ sF'(P)A\phi + 4F'(P)s\frac{d\phi}{ds},\end{aligned} \tag{3.17}$$

where $A = A_1 + A_2$ is also bounded, the bound depending solely upon the bounds of components of the metric tensor and their first derivatives.

Following Riesz, we set

$$V^\alpha(0,\xi) = \sum_{k=0}^{\infty}\frac{s^{\alpha-m+2k}V_k}{H_m(\alpha,k)} = \frac{s^{\alpha-m}}{K_m(\alpha)}\sum_{k=0}^{\infty}\frac{s^{2k}V_k(0,\xi)}{L_m(\alpha+2k)}, \tag{3.18}$$

and determine the double (matrix) forms V_k successively so that (a) shall hold. Since $P = s^2$, we have

$$\begin{aligned}\Delta V^{\alpha+2} &= \sum_{k=0}^{\infty}\frac{\Delta s^{\alpha+m+2k+2}V_k}{K_m(\alpha+2)L_m(\alpha+2k)} \\ &= \sum_{k=0}^{\infty}\frac{s^{\alpha-m+2k}}{H_m(\alpha+2k)}\Big(s^2\Delta V_k + (\alpha-m+2k+2)(\alpha+2k)V_k \\ &\qquad + 2(\alpha-m+2k+2)sAV_k + 2(\alpha-m+2k+2)s\frac{dV_k}{ds}\Big)\end{aligned} \tag{3.19}$$

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$$= \frac{s^{\alpha-m}}{K_m(\alpha+2)} \sum_{k=0}^{\infty} s^{2k} \left(\frac{\Delta V_{k-1}}{L_m(\alpha+2k)} + \frac{\alpha-m+2k+2}{L_m(\alpha+2k+2)} \left[2s\frac{dV_k}{ds} + (\alpha+2k)V_k + 2sAV_k \right] \right).$$

We now choose

$$L_m(\alpha+2) = (\alpha+2-m)\,L_m(\alpha), \tag{3.20}$$

and determine the V_k by the recurrent system of differential equations

$$2s\frac{dV_k}{ds} + (2k+2sA)V_k + \Delta V_{k-1} = 0, \tag{3.21}$$

with

$$V_{-1} \equiv 0, \quad V_0(0,0) = I, \tag{3.22}$$

where I is the unit matrix of order $\binom{m}{p}$. The expression (3.19) now becomes

$$\Delta_\xi V^{\alpha+2} = \frac{\alpha}{K_m(\alpha+2)} \sum_{k=0}^{\infty} \frac{s^{\alpha-m+2k}V_k}{L_m(\alpha+2k)}$$

which is equal to V^α if we choose

$$K_m(\alpha+2) = \alpha K_m(\alpha). \tag{3.23}$$

From (3.20) and (3.23) we have

$$K_m(\alpha) = K2^{\frac{1}{2}\alpha}\,\Gamma\left(\frac{\alpha}{2}\right); \quad L_m(\alpha) = L2^{\frac{1}{2}\alpha}\,\Gamma\left(\frac{\alpha+2-m}{2}\right),$$

and we choose K, L so that (e) is satisfied:

$$H_m(\alpha,k) = \pi^{\frac{1}{2}(m-2)}\,2^{\alpha+k-1}\,\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+2k+2-m}{2}\right). \tag{3.24}$$

In these details we follow the scalar treatment.

We have now to prove the convergence of the series for V^α. The relation (3.21) can be written

$$\frac{d}{ds}(s^k\,V_k) + A\,s^k\,V_k = -\tfrac{1}{2}\Delta V_{k-1}\,s^{k-1}. \tag{3.25}$$

To solve (3.25) consider the adjoint system

$$\frac{d}{ds}Y = A'Y, \tag{3.26}$$

and let Y be a (nonsingular) matrix solution of (3.26) with $|Y(0)| = 1$. It follows from the Lyapunov relation

$$Y(s) = Y(0)\exp\left[\int_0^s \mathrm{tr}\,A\,ds\right]$$

that $Y(s)$ is nonsingular, hence $Y^{-1}(s)$ exists and is bounded for $|s| < 1$. Since $s^k\,V_k$ vanishes for $s = 0$, we find as solution of (3.25)

(3.27) $$s^k V_k = -\tfrac{1}{2} Y^{-1}(s) \int_0^s Y(\sigma)\Delta V_{k-1}(\sigma)\sigma^{k-1} d\sigma.$$

With the vector and matrix norm

$$||X|| = \sqrt{\Sigma |X_i|^2}$$

(sum over all components), we have

(3.28) $$s^k ||V_k|| \leqslant \tfrac{1}{2} ||Y^{-1}(s)|| \int_0^s ||Y(\sigma)||\ ||\Delta V_{k-1}(\sigma)|| \sigma^{k-1} d\sigma.$$

Let us assume that for suitable constants τ and k and for $|s| < \epsilon$,

(3.29) $$||V_{k-1}|| < K^k \left(1 - \frac{\rho}{\tau}\right)^{-2k+1};\ \rho = \sum_j |\xi^j|;$$

Then [5; 6],

$$||\Delta V_{k-1}|| < K^k K_1 \left(1 - \frac{\rho}{\tau}\right)^{-2k-2},$$

and from (3.28) we have

(3.30) $$s^k ||V_k|| \leqslant \tfrac{1}{2} M K^k K_1 \int_0^s \frac{\sigma^{k-1} d\sigma}{(1 - \sigma\lambda/\tau)^{2k+2}} \leqslant s^k \frac{MK^k K_1}{2k}\left(1 - \frac{\rho}{\tau}\right)^{-2k-1},$$

where λ and M are independent of k. Since an estimate (3.29) holds for V_0 it holds for V_1 and hence for all V_k. It follows from (3.30) that the series $\Sigma s^{2k} V_k$ converges geometrically for s sufficiently small. From (3.24) it follows that the series for V^α converges for all values of α if $s \neq 0$, is sufficiently small; and uniformly in any right half plane of α. Hence V^α $(0, \xi)$ is defined as an analytic function of α, and regular for $R(\alpha) > m$ and $\xi \in R$ [cf **9**; **5**].

Exactly as in [**9**, p. 186] it follows that an estimate

(3.31) $$||V^\alpha(x, y)|| \leqslant C\, s^{R(\alpha)-m} \left|\left(\frac{\alpha}{e}\right)^{-\alpha}\right| |\alpha|^q,$$

holds, where $R(\alpha) > m$, and C, q are independent of α.

We remark that the present construction follows that of Riesz very closely; the only additional features being that it was necessary to show that the matrix A was bounded and to solve a system of recursive equations for the V_k. For the non-analytic case Riesz has given a method of construction for the kernel, and the same formal considerations apply to the present p-tensor kernel. Since the details are lengthy we omit them.

4. Construction of the solution. We continue the adaptation of Riesz' method for the p-tensor B.L. equation. We now define the Riesz fractional potential which plays the central role in the solution of the Cauchy problem. Let

(4.1) $$I^\alpha \phi(x) = (\phi(y), V^\alpha(x, y))_{D_s^x} = \int_{D_s^x} \phi(y) \wedge *V^\alpha(x, y),\ R(\alpha) > m - 2.$$

Here D_s^x is the conical region bounded by C_x and S, and ϕ is a p-form defined in R and assumed sufficiently differentiable for the prupose in hand. Riesz has proved certain relations in the scalar case, which as we now show, can be extended to the present p-tensor treatment. If ϕ is of the class $C[\frac{1}{2}(m-1)]$, then

$$I^0 \phi = \phi, \tag{4.2}$$

and in general,

$$\Delta I^\alpha \phi = I^{\alpha-2}\phi, \tag{4.3}$$

and

$$I^\alpha I^\beta \phi = I^{\alpha+\beta}\phi. \tag{4.4}$$

Riesz has also given a method of analytic continuation of $I^\alpha \phi$ into the left half-plane $R(\alpha) < m - 2$.

First we shall prove (4.2). A typical component of $I^\alpha \phi$ is

$$\begin{aligned}(I^\alpha \phi)_{i_1\ldots i_p} &= \int_{D_s^x} \sum_{k_1<\ldots<k_p} \phi_{k_1\ldots k_p}(y)\, V^\alpha_{i_1\ldots i_p;\, k_1\ldots k_p}\, dV_m \\ &= \sum_{k_1<\ldots<k_p} \int_{D_s^x} \phi_{k_1\ldots k_p}(y) \frac{s^{\alpha-m}}{H_m(\alpha, 0)} v^\alpha_{i_1\ldots i_p;\, k_1\ldots k_p}(x, y)\, dV_m,\end{aligned} \tag{4.5}$$

where $v^\alpha(x, y)$ is a double p-form regular on the cone D_s^x and such that

$$v^\alpha(x, x) = V_0(x, x) = I_p. \tag{4.6}$$

The recurrence relation (3.25) for V_0 can be written

$$\frac{dV_0}{ds} + A\, V_0 = 0,$$

whence it follows that the off-diagonal elements of V_0, hence also those of $v^\alpha(x, y)$, are $O(s)$. The typical component (4.5) then breaks up into a sum of integrals of the type studied by Riesz. We may assume normal coordinates lending to the Lorentz distance (3.10). Using this coordinate system, it has been shown [4, 9] that an expression of the form

$$\lim_{\alpha\to 0} \int_{D_s^x} \frac{s^{\alpha-m}}{H_m(\alpha, 0)} \phi(y)\, dV_m$$

depends only on the value of ϕ at the point x, and is in fact equal to $\phi(x)$. This result implies that as $\alpha \to 0$, all terms of (4.5) save the diagonal term tend to zero due to the factor $O(s)$, and that the remaining term is precisely $\phi_{i_1\ldots i_p}(x)$. This establishes (4.2).

We now introduce the formula of Green (1.22) for the region D_s^x, and set $\psi = V^{\alpha+2}(x, y)$. We obtain

$$\begin{aligned}&(\Delta\phi, V^{\alpha+2})_{D_s^x} - (\phi, \Delta V^{\alpha+2})_{D_s^x} \\ &\quad = \int_{bD_s^x} (V^{\alpha+2} \wedge *d\phi - \phi \wedge *dV^{\alpha+2} + \delta V^{\alpha+2} \wedge *\phi - \delta\phi \wedge * V^{\alpha+2}).\end{aligned}$$

Supposing that $R(\alpha) > m$, the surface integrals over the cone vanish because of the factor $s^{\alpha+2-m}$ contained in $V^{\alpha+2}$, and we are left with the surface integrals over S^x. Supposing further that $\Delta\phi = \rho$ and replacing $\Delta V^{\alpha+2}$ by V^α, we find

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$$(4.7)\quad I^{\alpha}\phi = I^{\alpha+2}\rho - \int_{s^x}(V^{\alpha+2}\wedge *d\phi - \phi\wedge *dV^{\alpha+2} + \delta V^{\alpha+2}\wedge *\phi - \delta\phi\wedge *V^{\alpha+2}).$$

To facilitate the study of (4.7), we write

$$(4.8)\quad \begin{aligned} J_1(\phi, \alpha) &= -\int_{s^x} V^{\alpha}\wedge *d\phi \\ J_2(\phi, \alpha) &= +\int_{s^x} \phi\wedge *dV^{\alpha}, \\ J_3(\phi, \alpha) &= -\int_{s^x} \delta V^{\alpha}\wedge *\phi, \\ J_4(\phi, \alpha) &= +\int_{s^x} \delta\phi\wedge *V^{\alpha}; \end{aligned}$$

then (4.7) may be written

$$(4.9)\quad I^{\alpha}\phi = I^{\alpha+2}\rho + \sum_{\mu=1}^{4} J_{\mu}(\phi, \alpha + 2).$$

We now formulate our theorem.

THEOREM I. *Let S be a spacelike surface of the class $C[\frac{1}{2}(m + 3)]$, and let R be its region of exclusive dependence. Let ξ be a p-form of the class $C[\frac{1}{2}(m + 3)]$ defined in a neighbourhood of S (or let $t\xi$, $n\xi$, $nd\xi$, $t\delta\xi$ be given on S); and let ρ be a p-form of class $C[\frac{1}{2}(m + 4)]$ in R. Then there exists in R a unique p-form ϕ such that*

$$(4.10)\quad \Delta\phi = \rho$$

in R, and

$$(4.11)\quad t\phi = t\xi,\quad n\phi = n\xi,\quad nd\phi = nd\xi,\quad t\delta\phi = t\delta\xi$$

on S. This form is given sufficiently close to S by

$$(4.12)\quad \phi = I^{2}\rho + \sum_{\mu=1}^{4} J_{\mu}(\xi, 2).$$

The main steps of the proof are as follows. It must first be shown that (4.9) can be continued analytically to $\alpha = 0$, so that (4.12) has a meaning. Then the conditions (4.10) and (4.11) are to be verified. The uniqueness has been established in Lemma II, so the proof will then be complete.

The analytical continuation has been carried through by Riesz [**9**] and by Fremberg [**4**, pp. 19-53]; since the essentials of the matter are in no way altered in the present case we shall not repeat the details. We remark that the continuation is possible if the forms ρ, ξ and the surface S have the differentiability properties stated in the theorem. It also follows that (4.12) is then continuous together with its first and second derivatives.

At $\alpha = 0$, the function $H_m(\alpha, 0)$ has a simple pole. Fremberg has also proved that the surface potentials can be continued to $\alpha = 0$, leading to finite expressions apart from the factor $1/H_m(\alpha, 0)$. It follows that the expressions J_{μ} vanish for $\alpha = 0$:

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$$(4.13) \qquad J_\mu(\xi, 0) = 0.$$

We must now establish a number of relations involving the B.L. operator and the integral I^α. First we make the important observation that in calculating derivatives with respect to x of I^α and the J_μ, we may disregard the variation at the limits of integration [9, pp. 67-70]. The justification of this is that for $R(\alpha) > m + k$, the kernel V^α is zero, together with its derivatives up to order k, on the cone D_s. It follows by analytic continuation that for all α we may calculate derivatives by formal differentiation under the integral sign. In view of condition (a) of §3, it therefore follows that (4.3) *holds whenever it has meaning*. In particular, if the continuation is valid to $\alpha = -2$, we have

$$(4.14) \qquad \Delta\phi = I^{-2}\phi,$$

in view of (4.2) and (4.3). Similarly, the relations

$$(4.15) \qquad \Delta J_\mu(\phi, \alpha) = J_\mu(\phi, \alpha - 2), \qquad \mu = 1, 2, 3, 4$$

hold. From (4.13) we see that

$$(4.16) \qquad \Delta J_\mu(\xi, 2) = 0, \qquad \mu = 1, 2, 3, 4.$$

The relation (4.4) follows as in the scalar case from the formula (d) of §3 which may be written

$$(4.17) \qquad (V^\alpha, V^\beta)_{D_y^x} = \int_{D_y^x} V^\alpha(x, z) \wedge *V^\beta(y, z) = V^{\alpha+\beta}(x, y),$$

and which we prove in essentially the same way. We apply Green's theorem to D_y^x for $R(\alpha) > m$, $R(\beta) > m$, and, noting that the surface terms vanish, we have

$$\begin{aligned}(V^{\alpha+2n}(x, z), V^\beta(y, z))_{D_y^x} &= (V^{\alpha+2n}(x, z), \Delta_z^n V^{\beta+2n}(y, z))_{D_y^x} \\ &= (\Delta_z^n V^{\alpha+2n}(x, z), V^{\beta+2n}(y, z))_{D_y^x} = (V^\alpha(x, z), V^{\beta+2n}(y, z))_{D_y^x},\end{aligned}$$

$n = 0, 1, 2, \ldots$, and, letting $\alpha, \beta \to 0$ we find [9, p. 196]

$$V^{2n}(x, y) = V^{2n}(y, x).$$

It follows that the coefficient forms V_k are symmetric, hence

$$(4.18) \qquad V^\alpha(x, y) = V^\alpha(y, x),$$

so that condition (c) of §3 is satisfied.

Consider the expression

$$(V^\alpha, V^\beta)_{D_y^x} - V^{\alpha+\beta}; \qquad R(\alpha) > m, R(\beta) > m.$$

A typical component $g(\alpha)$ of this double form is an analytic function of α which satisfies the estimate

$$(4.19) \qquad |g(\alpha)| \leqslant \left| C\left(\frac{\alpha}{e}\right)^{-\alpha} \alpha^\theta s^\alpha \right|$$

for $R(\alpha) > m$. Also $g(\alpha)$ is zero for $\alpha = 2n$ ($n = 0, 1, 2 \ldots$) by the preceding formulae. We may now apply the following theorem [7, Abs. III, problem 298, pp. 142, 327]: If two analytic functions $g(z)$, $h(z)$ are regular for $R(z) > a$, and $h(z) \not\equiv 0$ for $R(z) > a$, and

$$|g(z)| < |h(z)|,$$

then, denoting by z_n the zeros of $g(z)$ with $R(z_n) > a$, $|z_n| > 1$, the divergence of the series

$$\sum_n R(1/z_n)$$

implies that $g(z)$ is identically zero. Taking $h(\alpha)$ equal to the expression in the absolute bars on the right-hand side of (4.19), and noting that the series for the zeros of $g(\alpha)$ is the divergent harmonic series, we conclude that $g(\alpha) \equiv 0$. Hence (4.17) holds for $R(\alpha) > m$, $R(\beta) > m$, and therefore in general.

Using (4.17) we find by inverting a certain double integral that

$$I^\alpha J_\mu(\phi, \beta) = J_\mu(\phi, \alpha + \beta), \qquad \mu = 1, 2, 3, 4; \tag{4.20}$$

for $R(\alpha)$, $R(\beta)$ sufficiently large, and therefore in general.

The kernel V^α has certain additional properties which follow from the fact that if an operator T commutes with Δ it commutes with I^α. We first establish this assertion. Let T be a (linear) operator such that $T\Delta = \Delta T$. We wish to prove that

$$TI^\alpha\phi = I^\alpha T\phi. \tag{4.21}$$

Now (4.21) is certainly true for $\alpha = 0$, by (4.2). Using (4.3), we can prove that it holds also for $\alpha = 2n$ ($n = 0, 1, 2, \ldots$). For,

$$TI^2\phi = I^2\Delta TI^2\phi = I^2T\Delta I^2\phi = I^2T\phi,$$

since T and Δ commute, and Δ is inverse to I^2. By iteration of this, we find that (4.20) holds for $\alpha = 2n$. Consider now any typical component of

$$TI^\alpha\phi - I^\alpha T\phi, \tag{4.22}$$

this component is an analytic function of α which admits the estimate (4.19) and vanishes for $\alpha = 2n$. By the theorem we have just quoted above, we must have (4.22) identically zero. This proves that T commutes with I^α.

Now Δ commutes with $*$, d, and δ. Applying our remark to the $*$ operator, we have

$$\underset{x}{*}(\phi(y), V_p^\alpha(x, y))_{D^x} = (*\phi(y), V_{m-p}^\alpha(x, y))_{D^x} = (-)^{mp+p}(\phi(y), \underset{y}{*}V_{m-p}^\alpha)_{D^x}$$

for all ϕ. Taking the dual with respect to x, we find

$$V_p^\alpha(x, y) = \underset{xy}{**}V_{m-p}^\alpha(x, y). \tag{4.23}$$

Similarly, from $dI^\alpha = I^\alpha d$ we find that

$$\underset{x}{d}\, V_p^\alpha(x, y) = \underset{y}{\delta}\, V_{p+1}^\alpha(x, y). \tag{4.24}$$

The relation $\delta I^\alpha = I^\alpha \delta$ yields nothing additional. Note that (4.23) and (4.24) are essentially the same as relations satisfied by de Rham's Green's form [**8a**] and for the same reason.

From these relations it follows that the "single layer" potentials J_1 and J_4 are dual in the sense that

$$J_1(*\phi, \alpha) = *J_4(\phi, \alpha). \tag{4.25}$$

Similarly, the "double layer" potentials J_2 and J_3 are related by the equation

$$J_2(*\phi, \alpha) = *J_3(\phi, \alpha). \tag{4.26}$$

Since derivatives may be calculated formally for the potentials J_μ, it follows from (4.8) and (4.24) that

$$\delta J_2(\phi, \alpha) = 0, \; dJ_3(\phi, \alpha) = 0. \tag{4.27}$$

In order to show that our solution (4.12) satisfies the conditions (4.11), we must examine the potentials I^2 and $J_\mu(\xi, 2)$ as x tends to the surface S. First let us consider the volume potential I^2. Just as in the scalar case it follows that $I^2\rho \rightarrow 0$ since the conoid D^x_s becomes infinitesimal as x tends to S. The first derivatives of $I^2\rho$ also tend to zero.

As for the surface potentials, it will be sufficient, in view of (4.25) and (4.26), to consider J_1 and J_2. We shall reduce these expressions to integrals of the type studied by Fremberg, and apply his results. We may assume that S has the Lorentz form (3.10); it follows that as x tends to S, s becomes small of the same order as ξ^m. We may discard all terms of V^α except the first, noting that these terms, together with their first derivatives, become negligible in comparison with the first term.

Considering first J_1, we have

$$J_1(\xi, \alpha) \sim -\int_{S^x} \frac{s^{\alpha-m}}{H_m(\alpha)} I_p \wedge *d\xi,$$

whence, on S,

$$(J_1)_{i_1 \dots i_p} = -\int_{S^x} \frac{s^{\alpha-m}}{H_m(\alpha)} (I_{i_1 \dots i_p} \wedge *d\xi)_{1 \dots m-1} dx^1 \wedge \dots \wedge dx^{m-1}.$$

In the integrand we have

$$(I_{i_1 \dots i_p} \wedge *d\xi)_{1 \dots (m-1)} = (-1)^p (d\xi)_{i_1 \dots i_p m}.$$

Hence the components of tJ, are simple layer potentials of Fremberg's type, so that [**4**, p. 44],

$$tJ_1 = 0, \; \frac{d}{dn}(tJ_1)_{i_1 \dots i_p} = (-1)^p (d\xi)_{i_1 \dots i_p m}.$$

Here n denotes the inward normal to the surface of the conoid. The integrand is zero if, say, $i_p = m$, so that

$$nJ_1 = 0, \; \frac{d}{dn}(nJ_1) = 0.$$

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We now calculate

$$(ndJ)_{i_1\ldots i_pm} = \Gamma_{i_1\ldots i_pm}{}^{j(j_1\ldots j_p)}\partial_j J_{1(j_1\ldots j_p)}$$
$$= \Gamma_{i_1\ldots i_pm}{}^{m(j_1\ldots j_p)}\partial_m J_{1(j_1\ldots j_p)} \text{ since } nJ_1 = 0$$
$$= (-1)^p \frac{\partial}{\partial x^m} J_{1(i_1\ldots i_p)}$$
$$= (d\xi)_{i_1\ldots i_pm}.$$

Finally, we have

$$(t\delta J_1)_{i_1\ldots i_{p-1}} = -\Gamma_{ii_1\ldots i_{p-1}}{}^{(j_1\ldots j_p)}D^i J_{1(j_1\ldots j_p)}.$$

If the summation index i is less than m, then none of the j_μ are equal to m, hence we get a tangential derivative of a component of tJ_1, which yields zero. If $i = m$, we get the normal derivative of a component of nJ_1, which is again zero. Altogether, we have

(4.28) $$tJ_1 = 0,\ nJ_1 = 0,\ ndJ_1 = nd\xi,\ t\delta J_1 = 0.$$

A similar analysis holds for J_2. We find

$$J_2(\xi, \alpha) \sim \int_{S^z} \xi \wedge *d\frac{s^{\alpha-m}}{H_m(\alpha)} I_p$$
$$= \int_{S^z}\left(\xi \wedge *I_p\, d\frac{s^{\alpha-m}}{H_m(\alpha)}\right)_{1\ldots(m-1)} dx^1 \ldots dx^{m-1}.$$

Taking a typical normal component of J_2, we find that the integrand is of the form

$$\xi_{i_1\ldots i_{p-1}i}\frac{\partial}{\partial x^j}\frac{s^{\alpha-m}}{H_m(\alpha)}, \qquad i, i_1\ldots i_{p-1} < m,$$

which shows that components of nJ_2 are tangential derivatives of single layer potentials. As we have seen above, these are zero. Hence $nJ_2 = 0$. For tJ_2 we find

$$\left(\xi \wedge *\left\{I_{pi_1\ldots i_p} \wedge d\frac{s^{\alpha-m}}{H_m(\alpha)}\right\}\right)_{1\ldots(m-1)} = \xi_{i_1\ldots i_p}\frac{d}{dn}\frac{s^{\alpha-m}}{H_m(\alpha)},$$

which leads to a double layer potential of Fremberg's type for $(tJ_2)_{i_1\ldots i_p}$. From his results [**4**, p. 48] we conclude that

$$tJ_2 = t\xi,\ \frac{d}{dn}tJ_2 = 0.$$

We note from (4.27) that $t\delta J_2$ is zero. Again,

$$(dJ_2)_{i_1\ldots i_pm} = \Gamma_{i_1\ldots i_pm}{}^{j(j_1\ldots j_p)}\partial_j J_{2(j_1\ldots j_p)}.$$

For $j = m$, we obtain a normal derivative of a component of tJ_2, which is zero. For $j \neq m$ we obtain a tangential derivative of a component of nJ_2, which also yields zero. Collecting our results, we have

(4.29) $$tJ_2 = t\xi,\ nJ_2 = 0,\ ndJ_2 = 0,\ t\delta J_2 = 0.$$

The corresponding results for J_3 and J_4 follow from (4.25) and (4.26).

The solution (4.12) may now be verified. From (4.14) and (4.16) we have

$$\Delta\phi = \Delta\{I^2\rho + \sum_\mu J_\mu(\xi, 2)\} = \rho,$$

so that (4.10) holds. To verify (4.11) we note that the volume potential and its first derivatives contribute nothing on S. Applying (4.28), (4.29) and their duals, the result (4.11) follows.

The proof can be used to show that the kernel $V^\alpha(x, y)$ is uniquely determined independently of the coordinate system in which it was constructed. The volume potential $I^2\rho$ is uniquely determined, hence so is V^2. It follows from (4.17) that V^{2n} is unique for $n = 1, 2, 3, \ldots$; and the uniqueness theorem for analytic functions quoted above shows that V^α is unique for all values of α. The kernel V^α is defined in a small region only, but repetition of the integration process shows that the solution can be extended throughout R.

5. The harmonic field equations. We consider the system of first order equations for the components of the p-tensor $\phi_{i_1\ldots i_p}$;

(5.1) $$d\phi = \rho,\ \delta\phi = \sigma$$

and the problem of solving this system under conditions sufficient to determine the solution uniquely. When ρ and σ are replaced by zero in (5.1) we have the equations of harmonic fields. The solution of our problem will utilize Theorem I.

Let S be a spacelike surface and R its region of dependence, as before. Obvious necessary conditions that (1) should possess a solution are that ρ be a derived form, and σ the dual of a derived form. Hence we impose the conditions

(5.2) $$d\rho = 0, \quad \int_{z_{p+1}} \rho = 0,$$
$$\delta\sigma = 0, \quad \int_{z_{n-p+1}} *\sigma = 0$$

for all cycles z_{p+1}, z_{n-p+1} of R.

Since (5.1) is a first order system, it is to be expected that the assignment of $t\phi$ and $n\phi$ on S will determine the solution. That this is indeed true follows from Lemma II. For, the homogeneous system

(5.3) $$d\phi = 0,\ \delta\phi = 0 \text{ in } R,\ \phi \equiv 0 \text{ on } S$$

has only $\phi = 0$ as solution. To show this, we observe that $\Delta\phi = 0$, $t\phi = 0$, $n\phi = 0$, $nd\phi = 0$ and $t\delta\phi = 0$. Hence $\phi = 0$ in R.

Returning to the system (5.1) we observe that if $\phi \equiv \xi$ on S, then ξ must satisfy the conditions

(5.4) $$td\xi = t\rho,\ n\delta\xi = n\sigma,$$

since the corresponding component equations contain only derivatives in directions lying in S. A p-form ξ defined on S which satisfies (5.4) will be called admissible with respect to (5.1), or, more briefly, admissible.

As a simpler example of the method which we shall use to solve (5.1), we consider the homogeneous equations

$$d\phi = 0, \ \delta\phi = 0 \tag{5.5}$$

with the boundary condition $\phi \equiv \xi \in C\,[\frac{1}{2}(m+1)]$ on S, where ξ is admissible with respect to (5.5). Let ϕ be the solution of $\Delta\phi = 0$, such that $t\phi = t\xi$, $n\phi = n\xi$, $nd\phi = 0$, $n\delta\phi = 0$ on S. Thus

$$\phi = J_2(\xi, 2) + J_3(\xi, 2). \tag{5.6}$$

It remains to be shown that ϕ is closed and coclosed. But $td\phi = td\xi = 0$, $nd\phi = 0$, $\Delta(d\phi) = d\,\Delta\phi = 0$, and $nd(d\phi) = 0$ since $d\,.\,d\phi$ is zero. Finally, we have

$$t\delta(d\phi) = -\,td\delta\phi = -\,\underset{s}{d}t\delta\phi = 0,$$

since $\delta\phi = 0$ on S. Hence, by Lemma II, $d\phi \equiv 0$. Reasoning exactly dual to the preceding shows that $\delta\phi = 0$ in R. Hence ϕ is the unique harmonic field which assumes the admissible value ξ on S.

Since the equations (5.1) are non-homogeneous, we shall need to treat them in a less direct fashion. We divide the problem into two parts. Consider first the problem of solving the system

$$d\phi = \rho, \ \delta\phi = 0 \tag{5.7}$$

where

$$t\phi = t\xi, \ n\phi = 0 \tag{5.8}$$

on S, and $t\xi$ is admissible with respect to (5.7). That is,

$$td\xi = t\rho, \tag{5.9}$$

on S.

Assume for the moment that a solution ϕ exists. Then we have from (5.7),

$$d*\phi = 0,$$

and

$$\int_{z_{n-p}} *\phi = \int_{z^s_{n-p}} *\phi = 0$$

from (5.8), where z_{n-p} is any (absolute) cycle of R, and z^s_{n-p} is a homologous cycle of S. The cycle z^s_{n-p} homologous to z_{n-p} always exists (see §1). It follows that $*\phi$ is a derived form in R:

$$\phi = \delta X. \tag{5.10}$$

We now obtain

$$d\delta X = \rho,$$

which suggests that we attempt to solve (5.7) by solving

$$\Delta X = -\rho \tag{5.11}$$

under conditions which ensure that $dX \equiv 0$. This is a kind of "gauge" condition. It is not difficult to write down the auxiliary conditions which X must satisfy on S. They are

$$tX = 0,\ nX = 0,\ ndX = 0,\ t\delta X = t\phi = t\xi. \tag{5.12}$$

These conditions ensure that (5.8) are satisfied. To prove that X is closed, note that

$$\Delta(dX) = d(\Delta X) = -d\rho = 0,$$

that $tdX = 0$, $ndX = 0$, and that $nd \,.\, dX$ is zero. Finally, we have

$$\begin{aligned} t\delta dX &= -td\delta X + t\rho = -td\delta X + td\xi \\ &= td(\xi - \delta X) = \underset{s}{d}\, t(\xi - \delta X) = 0 \end{aligned}$$

in view of (5.9) and (5.12). Hence $dX \equiv 0$ in R.

From Theorem I we have

$$X = -I^2\rho + \int_{S^x} \xi \wedge *V_{p+1}^{\alpha+2}\Big|_{\alpha=0},$$

whence from (5.10)

$$\begin{aligned} \phi &= -\delta I^2\rho + \delta\int_{S^x} \xi \wedge *V_{p+1}^{\alpha+2}\Big|_{\alpha=0} \\ &= -I^2\delta\rho + J_2(\xi, 2), \end{aligned} \tag{5.13}$$

in view of (4.8), (4.24) and the permissibility of differentiating under the integral sign.

The dual problem, in which

$$d\phi = 0,\ \delta\phi = \sigma, \tag{5.14}$$

and the boundary conditions are easily written down, can be solved in the same way, or by use of (4.23) and (4.26). The solution is

$$\phi = -I^2\, d\sigma + J_3(\xi, 2). \tag{5.15}$$

Summing up these results, we have

THEOREM II. *Let S be a spacelike surface of class $C[\frac{1}{2}(m+3)]$, and ρ, σ forms of class $C[\frac{1}{2}(m+3)]$, of degrees $(p+1)$ and $(p-1)$ respectively, which satisfy* (5.2). *Let $\xi \epsilon\, C[\frac{1}{2}(m+1)]$ be a p-form admissible with respect to* (5.1). *Then there exists a unique solution of* (5.1) *defined in R, given by*

$$\phi = -I^2(\delta\rho + d\sigma) + J_2(\xi, 2) + J_3(\xi, 2). \tag{5.16}$$

We note that if ρ and σ are harmonic fields, the first term of (5.16) vanishes, and the solution is formally the same as (5.6). However ξ must satisfy the conditions of admissibility (5.4).

6. Concluding remarks. We consider certain special cases of the two theorems which result when further restrictions are imposed on the manifold M or the metric. A comparison of the present results with the boundary value theorems for p-tensors in the elliptic case will also be made.

Under certain circumstances the solutions discussed in Theorems I and II may be valid in a wider region than has been described. To begin with, we have restricted ourselves to one "side" of the initial surface and it is clear that the solution is valid both for $t > 0$ and $t < 0$ in the appropriate regions R^+, R^-. The solution is also determined in the region R_1 associated with any surface S_1 which is spacelike and lies in R. Suppose for example that M is the product of an $(m - 1)$-dimensional closed manifold and a line $(-\infty < t < +\infty)$; with metric

$$ds^2 = dt^2 - g_{\alpha\beta}(t)\, dx^\alpha\, dx^\beta$$

where $g_{\alpha\beta}(t)$ gives rise to a positive definite line element depending on t as a parameter. Let S be the surface $t = 0$: the solution is then determined throughout the entire manifold. If the $g_{\alpha\beta}(t)$ are independent of t, the B.L. equation is separable:

$$\phi_{tt} = \Delta_s \phi$$

where Δ_s is the B.L. operator for S. Setting $\phi = e^{i\lambda t}u$, we would have

$$\Delta_s u + \lambda^2 u = 0$$

just as in the scalar case.

If the metric is of certain restricted types, the B.L. equation will separate into $\binom{m}{p}$ independent component equations. This is the case for the Lorentz metric

$$ds^2 = dt^2 - (dx^1)^2 - \ldots - (dx^{m-1})^2.$$

Each component of ϕ then satisfies a wave equation

$$\phi_{tt} = \sum_\alpha \phi_{x^\alpha x^\alpha}.$$

The Riesz p-tensor kernel reduces to the first term of the series:

$$V_p^\alpha(x, y) = \frac{s^{\alpha - m}}{H_m(\alpha)} \sum_{i_1 < \ldots < i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \cdot dy^{i_1} \wedge \ldots \wedge dy^{i_p},$$

for which the relations (4.23) and (4.24) are easily verified. This case can therefore be treated by scalar methods.

Of particular interest is the case $m = 4$, $p = 2$, since the equations

$$d\phi = \rho, \quad \delta\phi = 0$$

are then, in the Lorentz metric, equivalent to Maxwell's equations, with $\mu = \epsilon = 1$. In a general metric it would be natural to retain this invariant formulation. The two-tensor $\phi_{i_1 i_2}$ is the electromagnetic field tensor $F_{i_1 i_2}$ and the dual of the three-form ρ is the charge-current vector. Theorem II may therefore be interpreted as a characterization of electromagnetic fields in a general relativity metric. The conservation of charge current is expressed by the fact that ρ is a derived form, in other words that

$$\int_{z_3} \rho = 0$$

for all cycles z_3.

It is interesting to compare our results with the boundary value theorems for p-tensors in the elliptic case, and with the classical theory of differential equations ($p = 0$ or $p = m$). We consider first the B.L. equation. The result in the elliptic case is that the solution exists and is unique if $t\phi$ and $n\phi$ are given on the boundary. This result has, however, only been proved under certain topological restrictions [3]. In the hyperbolic case the initial surface does not bound the domain of determination of the solution, but the additional data $nd\phi$ and $t\delta\phi$ are needed to determine the solution uniquely. If $p = 0$, $n\phi$ is zero automatically, and so is $t\delta\phi$. Hence the data reduce to $t\phi = \phi$ and

$$nd\phi = \frac{\partial \phi}{\partial n} dn,$$

in the hyperbolic case, and to ϕ in the elliptic case. A similar remark holds if $p = m$.

For the harmonic field equations the elliptic boundary value problem is uniquely solvable given either the normal boundary component and the absolute periods, or the tangential boundary components together with the relative periods. In the hyperbolic case both $t\phi$ and $n\phi$ are to be prescribed on the (smaller) initial surface. The region R in which the hyperbolic solutions are defined has no relative cycles (modulo the initial surface) and the periods on all absolute cycles are fixed by the given data $t\phi$ and $n\phi$ on S since all absolute cycles of R are homologous to cycles of S. In the hyperbolic problem, therefore, period conditions are not needed explicitly.

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