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GORENSTEIN WITT RINGS

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Throughout R is a noetherian Witt ring. The basic example is the Witt ring WF of a field F of characteristic not 2 and $\dot{F}/\dot{F}^2$ finite. We study the structure of (noetherian) Witt rings which are also Gorenstein rings (i.e., have a finite injective resolution). The underlying motivation is the elementary type conjecture. The Gorenstein Witt rings of elementary type are group ring extensions of Witt rings of local type. We thus wish to compare the two classes of Witt rings: Gorenstein and group ring over local type. We show the two classes enjoy many of the same properties and are, in several cases, equal. However we cannot decide if the two classes are always equal.

In the first section we consider formally real Witt rings R (equivalently, $\dim R = 1$). Here the total quotient ring of R is R-injective if and only if R is reduced. Further, R is Gorenstein if and only if R is a group ring over $\mathbf{Z}$. This result appears to be somewhat deep. Both proofs we have found require the classification of reduced Witt rings. A typical consequence here is: for reduced R, every regular ideal I satisfies $(I^{-1})^{-1} = I$ if and only if R is a group ring over $\mathbf{Z}$.

In the second section we consider nonformally real R (equivalently, $\dim R = 0$). We have here a simple characterization of Gorenstein Witt rings: R is Gorenstein if and only if $|\text{ann } I_R| = 2$, where I_R is the fundamental ideal of R. A number of striking properties hold for Gorenstein Witt rings. For example, $\text{ann}(\text{ann } I) = I$ for any ideal I. Under an additional assumption on annihilators of ideals we can show that Gorenstein Witt rings are group ring extensions of Witt rings of local type. The proof of this involves a reduction step of independent interest: if $I_R^4 \neq 0$ and for all $1 \neq x \in G$, $R/\text{ann}\langle 1, -x\rangle$ is a group ring extension of a Witt ring of local type, then so is R.

Our results are, with one exception, for abstract Witt rings as defined by Marshall (cf. [**11**]). To the Witt ring R there is an associated group G of one dimensional forms and a mapping $q: G \times G \to B$, B a pointed set. I_R is the fundamental ideal of R and X_R is the set of orderings. The total quotient ring of R will be denoted K; thus $K = S^{-1}R$ where S is the set of non-zero-divisors of R. For ideals $I, J \subset R$ we set

$$[I:J] = \{r \in R| \quad rJ \subset I\}.$$

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D_n will denote a group of exponent 2 and order 2^n. The group ring extensions arising here are $R[\Delta_n]$ for some n. A Witt ring R is of local type if and only if $|I_R^2| = 2$. The structure of these rings is well known (cf. [**11**]).

1. One dimensional Witt rings. In this section R denotes a noetherian Witt ring with X_R non-empty. For $\alpha \in X_R$ and $n \in \mathbf{N}$ we set

$$P(\alpha, n) = \{r \in R|\ \mathrm{sgn}_\alpha\, r \equiv 0 \ (\mathrm{mod}\ n)\}$$

and write $P(\alpha)$ for $P(\alpha, 0)$. The prime ideals of R are I_R, $P(\alpha)$ and $P(\alpha, p)$ where $\alpha \in X_R$ and p is an odd prime (cf. [**10**]). The primary ideals of R are all ideals in I_R containing some 2^k, $P(\alpha)$ and $P(\alpha, p^i)$ where $i \geqq 1$ (cf. [**6**]).

We recall that an ideal $I \subset R$ is *irreducible* if it cannot be written as a finite intersection of ideals properly containing I. Irreducible ideals are primary but the converse need not hold. We first wish to determine some of the irreducible ideals.

LEMMA 1.1. *Let $Q \subset R$ be a primary ideal and $\alpha \in X_R$.*

(1) *If $P(\alpha) < Q$ then $Q = P(\alpha, p^i)$ for some odd prime p and $i \geqq 1$, or, $2^k \in Q$ for some k.*

(2) *If $P(\alpha, p^i) < Q$ then $Q = P(\alpha, p^j)$ for some $j < i$.*

Proof. (1) If Q is I_R-primary then $2^k \in Q$ for some k. Otherwise, since all $P(\beta)$ are minimal primes, $Q = P(\beta, p^i)$ for some $\beta \in X_R$, p an odd prime and $i \geqq 1$. For all $a \in G$ with $a <_\alpha 0$ we have

$$\mathrm{sgn}_\beta \langle 1, a\rangle \equiv 0 \ (\mathrm{mod}\ p^i).$$

Hence $a <_\beta 0$ and $\alpha = \beta$.

(2) Follows quickly from (1).

COROLLARY 1.2. *Let R be a noetherian Witt ring of dimension one. For all $\alpha \in X_R$, odd primes p and $i \geqq 1$, $P(\alpha)$ and $P(\alpha, p^i)$ are irreducible.*

Proof. We use primary decomposition. First suppose $P(\alpha)$ is reducible; say

$$P(\alpha) = \bigcap_{j=1}^{n} Q_j,$$

where each Q_j is a primary ideal properly containing $P(\alpha)$. Each Q_j contains some $n_j \in \mathbf{Z}\backslash\{0\}$ by (1.1). Then

$$\prod_{j=1}^{n} n_j \in \cap Q_j = P(\alpha),$$

which is impossible.

Next suppose $P(\alpha, p^i)$ is reducible; say

$$P(\alpha, p^i) = \bigcap_{j=1}^{n} Q_j,$$

where each Q_j is a primary ideal properly containing $P(\alpha, p^i)$. Each Q_j equals $P(\alpha, p^{m_j})$ for some $m_j < i$. If $m = \max\{m_j\}$ then

$$\cap Q_j = P(\alpha, p^m) \neq P(\alpha, p^i),$$

a contradiction.

COROLLARY 1.3. *Let R, α, p and i be as in* (1.2). *Set $P = P(\alpha, p)$ and $Q = P(\alpha, p^i)$ and let $k = R_p/P_p$. Then*

$$\dim_k[Q_p:P_p]/Q_p = 1.$$

Proof. Combine (1.2) and [**14**, Theorem 34].

Remarks. (1) Primary decomposition holds for some non-noetherian Witt rings (cf. [**6**]). Thus (1.2) and (1.3) hold more generally.

(2) There are reducible I_R-primary ideals. Indeed the intersection of two non-comparable I_R-primary ideals is again primary and clearly reducible.

As mentioned, K denotes the total quotient ring of R.

PROPOSITION 1.4. *Let R be a one-dimensional noetherian Witt ring. Then K is R-injective if and only if R is reduced.*

Proof. We apply [**1**, 6.1] and check if $(0) \subset R$ is unmixed with irreducible primary components. First suppose R is not reduced. Then I_R is maximal in the set of zero-divisors and $\text{ht}(I_R) = 1$. Thus (0) is not unmixed and K is not R-injective.

Next suppose R is reduced. The primes in the set of zero-divisors are the $P(\alpha)$, $\alpha \in X_R$ (cf. [**10**]). Hence (0) is unmixed. Further, $(0) = \cap_\alpha P(\alpha)$ is the primary decomposition of (0). Thus all the primary components of (0) are irreducible by (1.2). Hence K is R-injective.

COROLLARY 1.5. *If R is reduced and $I \subset R$ is an ideal then*

$$K \cdot \text{ann}_R(\text{ann}_R I) = K \cdot I.$$

Proof. $K \cdot \text{ann}_R(\text{ann}_R I) = \text{ann}_K(\text{ann}_R I) = KI$ by [**5**, 19.10].

For an R-submodule I of K we set

$$I^{-1} = \{x \in K | xI \subset R\}.$$

PROPOSITION 1.6. *Let R be reduced. R is Gorenstein if and only if* $[(2):I_R]$ *is generated by two elements.*

Proof. R is Gorenstein if and only if P^{-1} can be generated by two elements, for all maximal ideals P [**1**, 6.3]. Now

$$P(\alpha, p)^{-1} = n^{-1}P(\alpha, p^i)$$

for some n and i, which is generated by two elements [**7**, 2.3, 2.5]. Hence we need only check if $I_R^{-1} = 2^{-1}[(2):I_R]$ is generated by two elements.

LEMMA 1.7. *Let R be reduced. If $[(2):I_R]$ is generated by two elements then it is generated by 2 and any $\sigma \in [(2):I_R]\backslash(2)$.*

Proof. The result is clear if $R = \mathbf{Z}$. Suppose $R \neq \mathbf{Z}$ and note that then $I_R \neq (2)$ and so $[(2):I_R] \subset I_R$. Let $[(2):I_R] = (\varphi_1, \varphi_2)$ and say $2 = \varphi_1\alpha_1 + \varphi_2\alpha_2$. Then either α_i or $\alpha_i + \langle 1\rangle$ is in I_R so that $\varphi_i\alpha_i$ equals $2\psi_i$ or $2\psi_i + \varphi_i$ for some $\psi_i \in I_R$. Hence either φ_1, φ_2 or $\varphi_1 + \varphi_2$ lies in (2). Thus $[(2):I_R] = (2, \varphi_i)$ for $i = 1$ or 2. Let

$$\sigma \in [(2):I_R]\backslash(2).$$

Then $\sigma = 2\alpha + \varphi\beta$ with $\beta \notin I_R$. Writing $\beta = \langle 1\rangle + \beta_0$ and $\beta_0\varphi = 2\psi$, we obtain $\sigma = 2(\alpha + \psi) + \varphi$. Hence

$$[(2):I_R] = (2, \varphi) = (2, \sigma).$$

THEOREM 1.8. *Let R be a one-dimensional Witt ring. Then R is Gorenstein if and only if R is a group ring over $\mathbf{Z}$.*

Proof. The implication ($\leftarrow$) is by [**3**, Corollary 9]; it may also easily be checked by (1.6). For the implication ($\rightarrow$) suppose that $R = \mathbf{Z}[\Delta_n] \times \mathbf{Z}[\Delta_m]$ with Δ_n generated by $t_1, \dots, t_n$ and Δ_m generated by $s_1, \dots, s_m$. Then $\sigma_1 = \langle\langle t_1, \dots, t_n\rangle\rangle$ and $\sigma_2 = \langle\langle s_1, \dots, s_m\rangle\rangle$ are in $[(2):I_R]$ but $\sigma_1 - \sigma_2 \notin (2)$. Thus $[(2):I_R]$ is not generated by two elements (1.7). Thus R is not Gorenstein (1.6). An inductive argument, using Marshall's classification of reduced Witt rings [**11**, 6.23], completes the proof.

Remark. We sketch another proof that R Gorenstein implies R is a group ring over $\mathbf{Z}$. If R is reduced but not a group ring over $\mathbf{Z}$ then there exists an $a \in G$ with

$$D\langle 1, a\rangle = \{1, a, b, ab\}.$$

(This follows easily from Marshall's classification; I do not know a direct proof.) There is a well-defined R-module homomorphism $h:\langle 1, a\rangle R \rightarrow K/R$ by

$$h(r\langle 1, a\rangle) = 2^{-1}r\langle 1, -b\rangle + R.$$

But there does not exist an $x \in K/R$ with $h(y) = xy$ for all y. Thus K/R is not injective (Baer's criterion) and R is not Gorenstein.

There are many conditions on a ring equivalent to the Gorenstein property. We list some of the more interesting ones. Recall that for an

R-module M, if the natural map $M \to M^{**}$ is injective then M is *torsionless*, and if it is an isomophism then M is *reflexive*.

COROLLARY 1.9. *Let R be a reduced Witt ring. The following are equivalent*:

(1) *R is Gorenstein.*

(2) *Each regular element generates an ideal all of whose primary components are irreducible.*

(3) *All finitely generated torsionless R-modules are reflexive.*

(4) *All ideals are reflexive.*

(5) $I = (I^{-1})^{-1}$ *for all regular ideals.*

(6) *R is a group ring over $\mathbf{Z}$.*

(7) *Every character $\chi:G \to \{\pm 1\}$ with $\chi(-1) = -1$ is a signature of R.*

Proof. (1) through (5) are equivalent by [**1**, 6.2, 6.3], (1) $\leftrightarrow$ (6) is (1.8) and (6) $\leftrightarrow$ (7) is [**13**, Theorem 1].

COROLLARY 1.10. *Let R be a group ring over $\mathbf{Z}$ and let $I \subset R$ be an ideal. If $a \cdot \operatorname{ann}_R I \subset (b)$ for some regular $b \in R$ then there exists regular $d \in R$ and $c \in I$ with $ad\text{-}bc \in (bd)$.*

Proof. We have $ab^{-1} + R$ is in $\operatorname{ann}_{K/R}(\operatorname{ann}_R I) = (K/R) \cdot I$ by [**5**, 19.10]. Hence there exists regular $d \in R$ and $c \in I$ with

$$ab^{-1} \equiv cd^{-1} \pmod{R}.$$

2. Zero dimensional Witt rings. Throughout this section, R denotes a noetherian Witt ring with X_R empty; thus R is local of dimension zero.

THEOREM 2.1. *For R a local noetherian Witt ring, the following are equivalent*:

(1) *R is Gorenstein.*

(2) *R is an injective R-module.*

(3) $\operatorname{Ext}^i_R(M, R) = 0$ *for all $i > 0$ and for all finitely generated R-modules M.*

(4) $\operatorname{ann}(I_R) = \{0, p\}$, *for some $p \in R$.*

Proof. (1) $\to$ (2) is [**9**, Theorem 214] since R has grade 0. (2) $\to$ (3) is standard and (3) $\to$ (1) is [**1**, 4.5]. (1) $\leftrightarrow$ (4) is [**9**, Theorem 221].

Condition (4) shows that any group ring extension of a Witt ring of local type is Gorenstein. These are the only examples of elementary type and the only known examples.

Let P_nR denote the set of n-fold Pfister forms in R and let

$$PR = \bigcup_{n \geqq 0} P_nR.$$

For a form $\varphi \in R$ we let $D(\varphi)$ denote the value set and $G(\varphi)$ the set $\{x \in G \mid x\varphi = \varphi\}$.

LEMMA 2.2. *If* $X_R = \emptyset$ *and* $\varphi \in R$ *with* $G(\varphi) \neq G$ *then* $G(2\varphi) \neq G(\varphi)$.

Proof. Suppose $G(\varphi) = G(2\varphi)$. Then $D(2) \subset G(\varphi)$ and if $y \in D(2)$ then $\langle 1, y\rangle\varphi = 2\varphi$. Thus if $y \in D(2)$,

$$D\langle 1, y\rangle \subset G(\langle 1, y\rangle\varphi) = G(2\varphi) = G(\varphi).$$

Now

$$D(4) = D(2)(\cup\{D\langle 1, y\rangle \mid y \in D(2)\}).$$

So $D(4) \subset G(\varphi)$ also. Continuing the argument will show all $D(2^m)$ lie in $G(\varphi)$ and hence that $G(\varphi) = G$, a contradiction.

LEMMA 2.3. *Let* $X_R = \emptyset$ *and let* $0 \neq \sigma \in PR$. *Then there exists a* $\mu \in PR$ *with* $0 \neq \mu\sigma \in \operatorname{ann}(I_R)$.

Proof. An easy induction on $|\pm D(p)|$ shows there exists a non-zero $p \in PR$ with $\sigma | p$ and $\pm D(p) = G$. If $D(p) = G$ we are done. Otherwise, choose $y \notin D(p)$; Note $\langle 1, -y\rangle p \neq 0$. If $D(\langle 1, -y\rangle p) = G$ then we are again done. Otherwise, $D(\langle 1, -y\rangle p) = D(p)$ as $[G:D(p)] = 2$. Now $-y \in D(p)$ so $\langle 1, -y\rangle p = 2p$. Then $D(p) = G$ by (2.2).

We begin describing some of the properties of a local Gorenstein Witt ring.

PROPOSITION 2.4. *Let R be Gorenstein with* $\operatorname{ann}(I_R) = \{0, p\}$. *Then there exists an n such that*:

(1) $p \in P_nR$,

(2) *p is a multiple of all non-zero* $\sigma \in PR$,

(3) $I_R^n = \{0, p\}$ *and* $I_R^{n+1} = 0$, *and*

(4) *if* $\sigma \in P_{n-1}R\backslash\{0\}$ *then* $[G:D(\sigma)] = 2$.

Proof. (1), (2) and (3) follow quickly from (2.3). (3) implies R is n-local, hence, by [**12**, 2.2], $(n - 1)$-Hilbert. This implies (4) since no $\sigma \in P_{n-1}R\backslash\{0\}$ is universal.

Recall that

$$\nu(R) = \inf\{m | I_R^m = 0\}.$$

COROLLARY 2.5. *Let R be Gorenstein. If* $\nu(R) = 1$ *then* $R = \mathbf{Z}/2\mathbf{Z}$, *if* $\nu(R) = 2$ *then* $R = \mathbf{Z}/4\mathbf{Z}$ *or* $(\mathbf{Z}/2\mathbf{Z})[\Delta_1]$ *and if* $\nu(R) = 3$ *then R is of local type.*

Proof. This follows immediately from (2.4) (3).

PROPOSITION 2.6. *Let R be Gorenstein.*

(1) *If* $\sigma \in PR$ *then* $R/\operatorname{ann}\sigma$ *is a Gorenstein Witt ring.*

(2) *Suppose* $R = WF$ *for some field F. If* $x \in \dot{F}$ *then* $WF(\sqrt{x})$ *is a Gorenstein Witt ring.*

Proof. (1) Set $\bar{R} = R/\text{ann }\sigma$, $\text{ann}_R(I_R) = \{0, p\}$ and write $p = \sigma p_0$ with $p_0 \in PR$, using (2.4). Since $\bar{R}$ is a Witt ring [**11**, 4.24], it suffices to show

$$\text{ann}_{\bar{R}}(I_R + \text{ann }\sigma) = \{0, p_0 + \text{ann }\sigma\}$$

by (2.1). Let

$$\varphi + \text{ann }\sigma \in \text{ann}_{\bar{R}}(I_R + \text{ann }\sigma).$$

Then $\langle 1, -x\rangle\varphi \in \text{ann }\sigma$ for all $x \in G$ and so

$$\varphi\sigma \in \text{ann}_R(I_R) = \{0, p_0\sigma\}.$$

Hence $\varphi \equiv 0$ or p_0 (mod ann σ).

(2) Let $T = WF(\sqrt{x})$ and $s_*:T \to R$ be the Scharlau transfer. Let $\text{ann}_R(IF) = \{0, p\}$ and suppose $p_1 \in \text{ann}_T(IF(\sqrt{x}))$. Then for all $a \in \dot{F}$,

$$0 = s_*(\langle 1, -a\rangle p_1) = \langle 1, -a\rangle s_*(p_1).$$

So $s_*(p_1) \in \{0, p\}$.

First suppose $s_*(p_1) = 0$. Then $p_1 = \varphi \otimes F(\sqrt{x})$ for some $\varphi \in R$. For all $z \in F(\sqrt{x})^{\cdot}$,

$$0 = s_*(\langle 1, -z\rangle \otimes \varphi \otimes F(\sqrt{x})) = \varphi \otimes s_*(\langle 1, -z\rangle).$$

We thus have $\varphi\ \text{ann}\langle 1, -x\rangle = 0$. Hence

$$\varphi \in \text{ann}(\text{ann}\langle 1, -x\rangle) = R \cdot \langle 1, -x\rangle,$$

since R is injective ((2.1) and [**5**, 19.10]). But then

$$p_1 = \varphi \otimes F(\sqrt{x}) = 0.$$

We have shown that if $p_1 \in \text{ann}_T(IF(\sqrt{x}))$ and $p_1 \neq 0$ then $s_*(p_1) = p$. In particular,

$$|\text{ann}_T(IF(\sqrt{x}))| = 2$$

and T is Gorenstein by (2.1).

Since group extensions of Witt rings of local type are Gorenstein, (2.6) (2) gives a slight extension of [**2**, Theorem 3.8].

If W is an R-submodule of R^n, let

$$W^{\perp} = \{(r_1, \ldots, r_n) \in R^n | \sum r_i w_i = 0 \quad \text{for all } (w_1, \ldots, w_n) \in W\}.$$

The following computation, involving Ext, is well known (cf. [**5**, 19.10]).

PROPOSITION 2.7. *Let R be Gorenstein and let W be an R-submodule of R^n. Then $W^{\perp\perp} = W$.*

We may easily deduce several striking properties of Gorenstein Witt rings. For the next six corollaries R denotes a local Gorenstein Witt ring with $\nu(R) = n + 1$ and $\mathrm{ann}(I_R) = \{0, p\}$.

COROLLARY 2.8. *If* $I \subset R$ *is an ideal then* $\mathrm{ann}(\mathrm{ann}\ I) = I$.

Proof. $I^{\perp} = \mathrm{ann}\ I$, so apply (2.7).

Recall that a form φ is *round* if $G(\varphi) = D(\varphi)$. If $a \in G$ the *radical of* a is

$$\mathrm{rad}(a) \equiv \{b \in G | D\langle 1, -a\rangle \subset D\langle 1, -b\rangle\}.$$

COROLLARY 2.9. *If* φ_1 *and* φ_2 *are round forms with* $D(\varphi_1) \subset D(\varphi_2)$ *then* $\varphi_1 | \varphi_2$. *In particular, for* $a \in G$ *we have* $\mathrm{rad}(a) = \{1, a\}$.

Proof. We have that $\mathrm{ann}\ \varphi_1$ is generated by $\langle 1, -a\rangle$ with $a \in D(\varphi_1)$ [**11**, 4.23]. So

$$\varphi_2 \in \mathrm{ann}(\mathrm{ann}(\varphi_1)) = (\varphi_1)$$

by (2.8).

COROLLARY 2.10. $\mathrm{ann}\ I_R^2 = I_R^{n-1}$.

Proof. It suffices to show $I_R^2 = \mathrm{ann}\ I_R^{n-1}$ by (2.8). Clearly $I_R^2 \subset \mathrm{ann}\ I_R^{n-1}$ so if equality does not hold there exists $0 \neq \langle 1, a\rangle \in \mathrm{ann}\ I_R^{n-1}$. Thus for any $b_1, \dots, b_{n-1} \in G$ we have

$$\langle\langle a, b_1, \dots, b_{n-1}\rangle\rangle = 0.$$

Thus

$$\langle\langle a, b_1, \dots, b_{n-2}\rangle\rangle \in \mathrm{ann}\ I_R = \{0, p\},$$

and so for any $b_1, \dots, b_{n-2} \in G$ we have

$$\langle\langle a, b_1, \dots, b_{n-2}\rangle\rangle = 0.$$

Continuing yields $\langle 1, a\rangle = 0$, a contradiction.

COROLLARY 2.11. *If* $\varphi \in I_R^{n-1}\backslash I_R^n$ *then* $G(\varphi)$ *has index* 2 *in* G. *If* $H \subset G$ *is a subgroup of index* 2 *then there exists* $\varphi \in I_R^{n-1}\backslash I_R^n$ *with* $G(\varphi) = H$. *In particular*, $|I_R^{n-1}| = 2|G|$.

Proof. If $\varphi \in I_R^{n-1}\backslash I_R^n$ then for all $a \in G$, $\langle 1, a\rangle\varphi \in \{0, p\}$. Since $\varphi \notin \mathrm{ann}\ I_R$, $G(\varphi)$ has index 2 in G.

Now suppose $H \subset G$ is a subgroup of index 2. Let I be the ideal of R generated by all $\langle 1, -h\rangle$, $h \in H$. Assume there is no $\varphi \in I_R$ with $G(\varphi) = H$. Then for any $\varphi \in I_R$, $H \subset G(\varphi)$ if and only if $G = G(\varphi)$ if and only if $\varphi \in \{0, p\}$. Hence $\mathrm{ann}\ I = \{0, p\}$. But then (2.8) implies $I = \mathrm{ann}\{0, p\} = I_R$, a contradiction. So $G(\varphi) = H$ for some $\varphi \in I_R$.

We finish by showing $\varphi \in I_R^{n-1}$. Let $b \notin G(\varphi)$. If $-1 \in G(\varphi)$ then

$$G = \{1, b\}G(\varphi) \subset G(\langle 1, -b\rangle\varphi).$$

If $-1 \notin G(\varphi)$ then $\langle 1, -b\rangle\varphi = 2\varphi$ so again $G(\langle 1, -b\rangle\varphi) = G$ by (2.2). Hence for all $b \in G$, $\langle 1, -b\rangle\varphi \in \text{ann } I_R$. That is,

$$\varphi \in \text{ann } I_R^2 = I_R^{n-1}$$

by (2.10). Clearly $\varphi \neq 0$ or p, so $\varphi \in I_R^{n-1}\backslash I_R^n$.

For the last statement, let $H \subset G$ be a subgroup of index 2 and suppose $\varphi_1, \varphi_2 \in I_R^{n-1}\backslash I_R^n$ satisfy $G(\varphi_1) = H = G(\varphi_2)$. Then $\text{ann}(\varphi_1) = \text{ann}(\varphi_2)$ and $\varphi_2 = u\varphi_1$ for some unit $u \in R$ (2.8). Write $u = \langle d\rangle + u_0$, with $u_0 \in I_R^2$. Then $\varphi_2 = \langle d\rangle\varphi_1$ as $u_0\varphi_1 = 0$. Since $i_G G(\varphi_1) = 2$, we see there are exactly two elements in $I_R^{n-1}\backslash I_R^n$ having $G(\varphi) = H$. There are $|G| - 1$ many subgroups of index 2 in G, so

$$|I_R^{n-1}\backslash I_R^n| = 2(|G| - 1).$$

Thus $|I_R^{n-1}| = 2|G|$, as $|I_R^n| = 2$.

COROLLARY 2.12. *For any ideal $I \subset R$ and element $b \in R$ we have*

$$\text{ann}[I:(b)] = b \text{ ann } I.$$

Proof. Suppose I is generated by $a_1, \ldots, a_n \in R$. Set

$$V = R \cdot (a_1, \ldots, a_n) \subset R^n \quad \text{and}$$

$$W = R \cdot (b, a_1, \ldots, a_n) \subset R^{n+1}.$$

Then

$$W^\perp = \{(r, \overline{s}) \in R^{n+1} | rb + \overline{sa} = 0\},$$

where $\overline{a} = (a_1, \ldots, a_n) \in R^n$ and $\overline{sa} = \sum s_i a_i$. Note that $(0, \overline{s}) \in W^\perp$ if and only if $\overline{s} \in V^\perp$ and if we set

$$W_1^\perp = \{r \in R | (r, \overline{s}) \in W^\perp\}$$

then $W_1^\perp = [I:(b)]$.

Now

$$W^{\perp\perp} = \{(t, \overline{u}) \in R^{n+1} | tr + \overline{us} = 0 \quad \text{for all } (r, \overline{s}) \in W^\perp\}.$$

In particular, if $(t, \overline{u}) \in W^{\perp\perp}$ then $\overline{us} = 0$ for all $\overline{s} \in V^\perp$. So $\overline{u} \in V^{\perp\perp} = V$ by (2.7), that is, $\overline{u} = r\overline{a}$ for some $r \in R$. Set

$$W_1 = \{(t, 0) \in W^{\perp\perp}\}.$$

Then $0 = W^{\perp\perp}/W \approx W_1/W_1 \cap W$. Since

$$W_1 = \{(t, 0) | \quad tr = u \text{ for all } r \in W_1^\perp\} = \text{ann}[I:(b)],$$

and

$$W_1 \cap W = b \text{ ann } I$$

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we have $\operatorname{ann}[I:(b)] = b \operatorname{ann} I$.

COROLLARY 2.13. *For ideals* $I, J \subset R$:
(1) $I \cap J = \operatorname{ann}(\operatorname{ann} I + \operatorname{ann} J)$,
(2) $I + J = \operatorname{ann}(\operatorname{ann} I \cap \operatorname{ann} J)$,
(3) $I \cdot J = \operatorname{ann}[\operatorname{ann} J:I]$.

Proof. $\operatorname{ann} I \cap \operatorname{ann} J = \operatorname{ann}(I + J)$ which yields (1) and (2) by (2.8). For (3), let I be generated by $b_1, \dots, b_n$. Then

$$\operatorname{ann}[\operatorname{ann} J:I] = \operatorname{ann}(\cap[\operatorname{ann} J:(b_i)]) = \sum \operatorname{ann}[\operatorname{ann} J:(b_i)]$$

by (1) and (2.8). Then (2.12) and (2.8) yield

$$\operatorname{ann}[\operatorname{ann} J:I] = \sum b_i J = IJ.$$

Each of the preceding corollaries, except (2.10), characterizes the group extensions of local type among those Witt rings of elementary type. We turn now to the question of when an arbitrary Gorenstein Witt ring is a group extension of a Witt ring of local type. We begin with a reduction theorem.

THEOREM 2.14. *Let R be a local noetherian Witt ring with $\nu(R) = n + 4$ ($n \geqq 1$). Suppose that for all $x \in G\backslash\{1\}$, $R/\operatorname{ann}\langle 1, -x\rangle$ is a Witt ring of local type extended by a group of order 2^n. Then R is also a group ring extension of a Witt ring of local type.*

Proof. We are assuming that for all $x \in G\backslash\{1\}$ there exist groups $H(x)$ and $\Delta(x)$ such that:
(i) $G = H(x) \times \Delta(x)$
(ii) $D\langle 1, -x\rangle \subset H(x)$, $-1 \in H(x)$
(iii) $|\Delta(x)| = 2^n$
(iv) $D\langle\langle -x, -y\rangle\rangle = \{1, -y\}D\langle 1, -x\rangle$, for all $y \in G\backslash H(x)$
(v) $|D\langle\langle -x, -y\rangle\rangle| = |G|/2^{n+1}$, for all $y \in H(x)\backslash D\langle 1, -x\rangle$.
Step 1. There exists an $x \in G$ with $|D\langle 1, -x\rangle| < |G|/2^{n+2}$.
If for any $x \in G\backslash\{1\}$ we have $|D\langle 1, -x\rangle| > |G|/2^{n+2}$ then for $y \in H(x)\backslash D\langle 1, -x\rangle$, $D\langle\langle -x, -y\rangle\rangle = D\langle 1, -x\rangle$. In particular, $-y \in D\langle 1, -x\rangle$ and $\langle\langle -x, -y\rangle\rangle = \langle\langle 1, -x\rangle\rangle$. Then $D\langle 1, -x\rangle = G$ by (2.2) which contradicts (ii), (iii). Hence for all $x \in G\backslash\{1\}$,

$$|D\langle 1, -x\rangle| \leqq |G|/2^{n+2}.$$

Suppose then that $|D\langle 1, -x\rangle| = |G|/2^{n+2}$ for all $x \in G\backslash\{1\}$. Fix $x \in D\langle 1, 1\rangle$ so that $x \in D\langle 1, -x\rangle$. For all $z \notin D\langle 1, -x\rangle$ (iv) and (v) imply

$$D\langle\langle -x, -z\rangle\rangle = \{1, u\}D\langle 1, -x\rangle \quad \text{for some } u \in G\backslash D\langle 1, -x\rangle.$$

So

$$D\langle 1, -z\rangle \subset \{1, u\}D\langle 1, -x\rangle \quad \text{and}$$

$$|D\langle 1, -x\rangle \cap D\langle 1, -z\rangle| = \frac{1}{2}|D\langle 1, -z\rangle| = |G|/2^{n+3}.$$

We use the basic counting formula of [**8**, Proposition 8]:

$$(*) \quad \sum_{y\neq 1,x} |D\langle 1, -x\rangle \cap D\langle 1, -y\rangle|$$

$$= |G| - 2|D\langle 1, -x\rangle| + \sum_{y\in D\langle 1,-x\rangle\backslash\{1\}} |D\langle 1, -y\rangle|.$$

Let $g = |G|$ and let T be the sum of $|D\langle 1, -x\rangle \cap D\langle 1, -y\rangle|$ over those y in $D\langle 1, -x\rangle\backslash\{1, x\}$. Then (*) is:

$$2^{-(2n+5)}[(2^{n+2}g - g)g + 2^{2n+5}T]$$

$$= 2^{-(2n+4)}[2^{2n+4}g - 2^{n+3}g + (g - 2^{n+2})g]$$

$$(2^{n+2} - 3)g^2 + T = 2^{n+3}(2^{n+2} - 3)g.$$

As $T > 0$, we have $g < 2^{n+3}$ and hence $g = 2^{n+2}$, contradicting our supposition.

Step 2. If $|D\langle 1, -x\rangle| < |G|/2^{n+2}$ and $z \in G\backslash H(x)$ then

(a) $|D\langle 1, -x\rangle| = |D\langle 1, -z\rangle|$

(b) $|D\langle 1, -x\rangle \cap D\langle 1, -z\rangle| = \frac{1}{2}|D\langle 1, -x\rangle|$

(c) If $y \in H(x)\backslash D\langle 1, -x\rangle$ and $|D\langle 1, -y\rangle < |G|/2^{n+2}$ then $|D\langle 1, -y\rangle| = |D\langle 1, -x\rangle|$.

We have, by (iv), that

$$D\langle\langle -x, -z\rangle\rangle = \{1, -z\}D\langle 1, -x\rangle$$

(note that $\pm z \notin D\langle 1, -x\rangle$ by (2.2)). Suppose $x \in H(z)$. Then, as $z \notin D\langle 1, -x\rangle$,

$$|D\langle\langle -x, -z\rangle\rangle| = |G|/2^{n+1}$$

and so

$$|D\langle 1, -x\rangle| = |G|/2^{n+2}$$

contrary to our assumption. Hence $x \in G\backslash H(z)$ and

$$\{1, -x\}D\langle 1, -z\rangle = D\langle\langle -x, -z\rangle\rangle = \{1, -z\}D\langle 1, -x\rangle.$$

This implies (a) and (b).

For (c), we may choose $w \notin H(x) \cup H(y)$ since G cannot be the union of two proper subgroups. By (a),

$$|D\langle 1, -w\rangle| = |D\langle 1, -x\rangle|.$$

Applying (a) with x replaced by y gives that

$$|D\langle 1, -w\rangle| = |D\langle 1, -y\rangle|$$

also.

Step 3. There exists an $m < |G|/2^{n+2}$ so that for all $x \in G\backslash\{1\}$ either $|D\langle 1, -x\rangle| = m$ or $|D\langle 1, -x\rangle| = |G|/2^{n+2}$.

Fix an $x \in G\backslash\{1\}$ with $|D\langle 1, -x\rangle| < |G|/2^{n+2}$ using Step 1 and set $m = |D\langle 1, -x\rangle|$. If $y \notin D\langle 1, -x\rangle$ then Step 2 shows $|D\langle 1, -y\rangle| = m$ or $|G|/2^{n+2}$. Let $y \in D\langle 1, -x\rangle$. There must exist a $z \in G\backslash H(x)$ with $y \notin D\langle 1, -z\rangle$ since otherwise $G\backslash H(x) \subset D\langle 1, -y\rangle$ and so $G \subset D\langle 1, -y\rangle$. Applying Step 2 with x replaced by z gives $|D\langle 1, -y\rangle| = |G|/2^{n+2}$ or $|D\langle 1, -z\rangle|$. Since $|D\langle 1, -z\rangle| = m$ we are done.

Step 4. Finish.

We finish by showing $m = 2$ and hence that R is a group ring. If t is two-sided rigid then so are all elements of $\Delta(t)$. Hence $R = S[\Delta]$ with $|\Delta| \geqq 2^{n+1}$ and S not a group ring. If $1 \neq y \in G_s$, the group associated to S, then $|D\langle 1, -y\rangle| \neq 2$ and so

$$|D\langle 1, -y\rangle| = |G|/2^{n+2} \geqq \frac{1}{2}|G_s|.$$

Thus S is of local type.

We again use the counting formula (*). Fix x with $|D\langle 1, -x\rangle| = m$. Set

$$g = |G|, \quad k = |G|/2^{n+2},$$
$$a = |\{z \in D\langle 1, -x\rangle\backslash\{1\} \mid |D\langle 1, -z\rangle| = m\}|$$

and let T be the sum of $|D\langle 1, -y\rangle \cap D\langle 1, -x\rangle|$ for $y \in D\langle 1, -x\rangle\backslash \{1, x\}$. Then (*) yields:

$$\frac{1}{2}m(g - k) + T = g - 2m + am + (m - a - 1)k$$
$$T = g - 2m - a(k - m) - k - \frac{1}{2}m(g - k)$$
$$g > 2m + k + a(k - m) + \frac{m}{2}(g - k)$$
$$g > \frac{1}{2}m(g - k) = 2^{-1}mg(1 - 2^{-(n+2)}).$$

We thus have $m < 2^{n+3}/(2^{n+2} - 1) < 3$ (as $n \geqq 1$). But then $m = 2$ as desired.

To prove group ring extensions of Witt rings of local type are the only Gorenstein Witt rings the reduction theorem (2.14) (along with (2.5)) shows it suffices to do the case $\nu(R) = 4$. We are unable to do this without added assumptions. However we will now concentrate on the case $\nu(R) = 4$.

Recall that for $x \in G$,

$$Q(x) = \{ \ll -x, -y \gg + I_R^3 | \, y \in G \}$$

is a subgroup of I_R^2/I_R^3.

PROPOSITION 2.15. *Let R be Gorenstein with $\nu(R) = 4$. For any $x, y \in G$ we have*:

$$|Q(x) \cap Q(y)| \, |Q(x) \cap Q(xy)| = |Q(x)|.$$

Proof. Let

$$k = |Q(x)/Q(x) \cap Q(y)|.$$

The map $G \to Q(x)/Q(x) \cap Q(y)$ sending g to the coset represented by $\ll -x, -g \gg$ has kernel $D\langle 1, -x\rangle D\langle 1, -xy\rangle$. Thus

$$k = [G:D\langle 1, -x\rangle D\langle 1, -xy\rangle].$$

Any $\sigma \in Q(x) \cap Q(xy)$ represents $D\langle 1, -x\rangle D\langle 1, -xy\rangle$ and $[G:D(\sigma)] \leqq 2$ by (2.4). Since there are k subgroups of index $\leqq 2$ containing $D\langle 1, -x\rangle D\langle 1, -xy\rangle$ and $\sigma = \sigma'$ if and only if $D(\sigma) = D(\sigma')$ by (2.9), we have

$$|Q(x) \cap Q(xy)| \leqq k.$$

Further, for each subgroup H of index 2 containing $D\langle 1, -x\rangle D\langle 1, -xy\rangle$ there exists a $\varphi \in I^2F$ with $G(\varphi) = H$ (2.11). Now $D\langle 1, -x\rangle \subset G(\varphi)$ implies

$$\varphi \in \operatorname{ann}(\operatorname{ann}\langle 1, -x\rangle) = (\langle 1, -x\rangle),$$

by (2.8), and so we may assume $\varphi = \ll -x, -w \gg$ for some $w \in G$. Also $D\langle 1, -xy\rangle \subset G(\varphi)$ implies $\varphi \in Q(x) \cap Q(xy)$. Thus

$$k \leqq |Q(x) \cap Q(xy)|$$

and we are done.

COROLLARY 2.16. *Let R be Gorenstein with $\nu(R) = 4$. If for some $x, y \in G$, $Q(x) \subset Q(y)$ then $x = 1$ or y.*

Proof. $Q(x) \subset Q(y)$ implies $|Q(x) \cap Q(xy)| = 1$ by (2.15). Then

$$|D\langle 1, -y\rangle | = |D\langle 1, -x\rangle \cap D\langle 1, -xy\rangle |$$

by [**11**, 5.2] and so $D\langle 1, -y\rangle \subset D\langle 1, -x\rangle$. Thus $x \in \{1, y\}$ by (2.9).

COROLLARY 2.17. *Let R be Gorenstein with $\nu(R) = 4$. If $x, y \in G$ then*:

$$|D\langle 1, -x\rangle|\,|D\langle 1, -y\rangle|\,|D\langle 1, -xy\rangle|$$
$$= |G|\,|D\langle 1, -x\rangle \cap D\langle 1, -y\rangle|^2.$$

Proof. Combine (2.15) and the equation [**11**, 5.2]:

$$|D\langle 1, -xy\rangle|/|D\langle 1, -x\rangle \cap D\langle 1, -y\rangle| = |Q(x) \cap Q(y)|.$$

THEOREM 2.18. *Let R be Gorenstein with $\nu(R) = 4$. Assume that for all $x \neq 1$ and subgroups $H \subset D\langle 1, -x\rangle$ of index 2 that there exists a $y \in G$ with*

$$H = D\langle 1, -x\rangle \cap D\langle 1, -y\rangle.$$

Then $R \approx L[\Delta_1]$ where L is of local type.

Proof. Step 1. There exists $x \in G\backslash\{1\}$ with

$$[G:D\langle 1, -x\rangle] = 4.$$

Choose any $x \neq 1$ with $|D\langle 1, -x\rangle|$ maximal. Let $H \subset D\langle 1, -x\rangle$ be a subgroup of index 2 and let $y \in G$ satisfy

$$H = D\langle 1, -x\rangle \cap D\langle 1, -y\rangle.$$

Then

$$|D\langle 1, -x\rangle \cap D\langle 1, -y\rangle| = \frac{1}{2}|D\langle 1, -x\rangle|.$$

If $|D\langle 1, -y\rangle| < |D\langle 1, -x\rangle|$ then $D\langle 1, -y\rangle \subset D\langle 1, -x\rangle$, contradicting (2.9). So $|D\langle 1, -y\rangle| = |D\langle 1, -x\rangle|$ by maximality. Similarly, $|D\langle 1, -x\rangle| = |D\langle 1, -xy\rangle|$. Apply (2.17) to get

$$[G:D\langle 1, -x\rangle] = 4.$$

Step 2. $K \equiv \{x \in G|\ [G:D\langle 1, -x\rangle] \leqq 4\}$ is a subgroup of index at most 2 in G.

We first note that for no x is $D\langle 1, -x\rangle$ of index 2, as $\nu(R) = 4$. Let $x, y \in K$. Then $|Q(x) \cap Q(y)| = 2$ by (2.16) and so $|Q(x) \cap Q(xy)| = 2$ by (2.15). Similarly, $|Q(y) \cap Q(xy)| = 2$. Applying (2.15) again yields $|Q(xy)| = 4$ and $xy \in K$.

If $x \in K\backslash\{1\}$ then for each $H \subset D\langle 1, -x\rangle$ of index 2 there exists y_H with

$$H = D\langle 1, -x\rangle \cap D\langle 1, -y_H\rangle.$$

Thus $|D\langle 1, -y_H\rangle| \geqq |D\langle 1, -x\rangle|$ by (2.9) and so y_H and xy_H both lie in K. Hence

$$|K| \geqq 2|D\langle 1, -x\rangle| = \frac{1}{2}|G|.$$

Step 3. Finish.

We first note that if $y \equiv z \pmod K$ then $|Q(y)| = |Q(z)|$. Namely, suppose $y = xz$ with $x \in K$. As in Step 2,

$$|Q(x) \cap Q(y)| = 2 = |Q(x) \cap Q(z)|.$$

Hence

$$|Q(y)| = 2|Q(y) \cap Q(z)| = |Q(z)|$$

by (2.15). We let k denote the common size $|Q(y)|$, $y \notin K$.

We count using [**8**, Proposition 8]. Let $g = |G|$ and fix $x \in K\backslash\{1\}$. Then:

$$\sum_{y \neq 1,x} |D\langle 1, -x\rangle \cap D\langle 1, -y\rangle| = \left[\frac{g}{2} - 2\right]\frac{g}{8} + \frac{g}{2} \cdot \frac{g}{2k},$$

since for $y \notin K$,

$$|D\langle 1, -x\rangle \cap D\langle 1, -y\rangle| = |D\langle 1, -xy\rangle|/|Q(x) \cap Q(y)| = g/2k$$

by Step 2. Also:

$$-2|Q(x)| + \sum_{y \in D\langle 1,-x\rangle} |D\langle 1, -y\rangle| = \frac{g}{2} + (p-1)\frac{g}{4} + \left[\frac{g}{4} - p\right]\frac{g}{k},$$

where $p = |D\langle 1, -x\rangle \cap K|$. We thus obtain:

$$g^2 = 8g + 16pg(1/4 - 1/k).$$

Now $p \leqq |D\langle 1, -x\rangle| = g/4$ so that

$$g^2 \leqq 8g + 4g^2(1/4 - 1/k).$$

Thus $4g^2/k \leqq 8g$ and $k \geqq g/2$. This implies that for each $y \notin K$ that $|D\langle 1, -y\rangle| = 2$. Hence $R \approx L[\Delta_1]$ with L of local type.

COROLLARY 2.19. *Let R be Gorenstein with $\nu(R) = n + 4$ $(n \geqq 0)$. Assume: for any $\sigma \in P_nR$, $x \notin D(\sigma)$ and subgroup H with*

$$D(\sigma) \subset H \subset D(\langle 1, -x\rangle\sigma) \quad \text{and} \quad [D(\langle 1, -x\rangle\sigma):H] = 2$$

that there exists a $y \in G$ with

$$H = D(\langle 1, -x\rangle\sigma) \cap D(\langle 1, -y\rangle\sigma).$$

Then $R \approx L[\Delta_{n+1}]$ where L is of local type.

Proof. Combine (2.14) and (2.18).

We conclude with an ideal theoretic version of the extra assumption in (2.18). Let $H \subset G$ be a subgroup. Let I_H be the 1-Pfister ideal $(\{\langle 1, -h\rangle | h \in H\})$ (cf. [**4**]). Let

$$J_H = \{\varphi \in I_R^2 | H \subset G(\varphi)\} \quad \text{and}$$

$$C(H) = \{z \in G |\ H \subset D\langle 1, -z\rangle\}.$$

If R is a local Gorenstein Witt ring and $\nu(R) = 4$ then $\operatorname{ann} I_H = I_{C(H)} + J_H$.

PROPOSITION 2.20. *Let R be Gorenstein with $\nu(R) = 4$. Let $I = I_H$ be a 1-Pfister ideal. The following are equivalent*:

(1) $\operatorname{ann} I$ *is a* 1-*Pfister ideal.*

(2) $I_R \cdot \operatorname{ann} I = I_R^2 \cap \operatorname{ann} I$.

(3) $C^2(H) = H$.

Proof. (1) → (2) is [**4**, 2.15]. (2) → (3):

$$[I_R^3 : \operatorname{ann} I] = \operatorname{ann}(\operatorname{ann} I \cdot \operatorname{ann} I_R^3)$$

by (2.13). Thus

$$\begin{aligned}[I_R^3 : \operatorname{ann} I] &= \operatorname{ann}(I_R \operatorname{ann} I) = \operatorname{ann}(I_R^2 \cap \operatorname{ann} I) \\ &= \operatorname{ann} I_R^2 + \operatorname{ann}(\operatorname{ann} I) = I + I_R^2,\end{aligned}$$

using (2.13), (2.8) and (2.10). Computing differently yields

$$\begin{aligned}[I_R^3 : \operatorname{ann} I] &= [I_R^3 : I_{C(H)} + J_H] = [I_R^3 : I_{C(H)}] \\ &= \operatorname{ann} I_{C(H)} + I_R^2 = I_{C^2(H)} + I_R^2.\end{aligned}$$

Thus $I_{C^2(H)} + I_R^2 = I_H + I_R^2$ and so $C^2(H) = H$.

(3) → (1): Here

$$\operatorname{ann} I_{C(H)} = I_{C^2(H)} + J_{C(H)} = I_H + J_{C(H)}.$$

Thus, by (2.8),

$$I_{C(H)} = \operatorname{ann} I_H \cap \operatorname{ann} J_{C(H)} = (I_{C(H)} + J_H) \cap (I_{C(H)} + I_R^2).$$

In particular, $J_H \subset I_{C(H)}$. So

$$\operatorname{ann} I_H = I_{C(H)} + J_H = I_{C(H)}$$

is 1-Pfister.

COROLLARY 2.21. *Let R be Gorenstein with $\nu(R) = 4$. Then $R \approx L[\Delta_1]$ with L of local type if and only if for all* 1-*Pfister ideals I not containing I_R^2 we have*

$$I_R \operatorname{ann} I = I_R^2 \cap \operatorname{ann} I.$$

Proof. The implication (→) is simple to check using (2.20) (3). For the reverse implication we check the condition of (2.18). Let $x \in G\backslash\{1\}$ and let H be a subgroup of $D\langle 1, -x\rangle$ of index 2. If $\varphi \in I_H \cap I_R^2$ then $x \in G(\varphi)$, so that $I_R^2 \not\subset I_H$ (2.11). Hence our assumption and (2.20) give $C^2(H) = H$. Then $\{1, x\} \subsetneq C(H)$. Let $y \in C(H)\backslash\{1, x\}$. Then $H \subset D\langle 1, -y\rangle$ and $D\langle 1, -x\rangle \cap D\langle 1, -y\rangle \neq D\langle 1, -x\rangle$ by (2.9). So

$$H = D\langle 1, -x\rangle \cap D\langle 1, -y\rangle$$

as desired.

THEOREM 2.22. *Let R be Gorenstein with $\nu(R) = n + 4$ ($n \geqq 0$). The following are equivalent*:

(1) $R \approx L[\Delta_{n+1}]$, *where L is of local type.*

(2) *For all $\sigma \in P_nR$ and all* 1-*Pfister ideals I not containing I_R^2 we have*

$$I_R \operatorname{ann}(\sigma I) = I_R^2 \cap \operatorname{ann}(\sigma I).$$

Proof. Combine (2.21) and (2.14).

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