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GENERALIZATIONS OF THE FUNDAMENTAL THEOREM OF PROJECTIVE GEOMETRY

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The fundamental theorem of projective geometry states that every line-preserving bijection of a Desarguesian projective plane is induced from a semilinear transformation. This theorem extends to higher-dimensional projective spaces and has an affine version, possibly due to Darboux, which states that every line-preserving bijection of an affine space is composed of a linear transformation and a translation and a mapping induced by an automorphism of the underlying field. In the 1800s, properties of maps such as conformal and Möbius transformations were worked out and, in particular, Liouville showed that sufficiently smooth conformal mappings of the real plane arise from the action of the conformal group. (Liouville's contribution is discussed in [4].) In the 1950s and 1960s more work was done on conformal mappings and the regularity assumptions of Liouville's theorem were relaxed by analysts such as Gehring [5] and Rešetnjak [8]: theorems were proved about conformal mappings defined on open subsets of Euclidean space, rather than on the whole space or the one-point compactification of the whole space. Earlier, in the 1930s, Carathéodory [2] had considered maps of spheres preserving circles in the spheres, and maps of subsets of the sphere preserving arcs as well. Maps of subsets of the real plane preserving line segments do not seem to have been considered until quite recently (except that mappings of the disc that preserve line segments can be interpreted as mappings of the Klein model of the hyperbolic plane that preserve geodesics; it was apparently well known that these come from the isometry group of the hyperbolic plane).

In the 1970s, Tits [9] extended the fundamental theorem of projective geometry to buildings, a combinatorial construction. He associated 'spherical buildings' to semisimple groups over arbitrary fields, and showed that certain bijections of the

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spherical building of rank at least two associated to a semisimple group G come from the action of G, possibly composed with a mapping induced by an automorphism of the underlying field; namely, the bijections that preserves the fibrations of the building associated to the fibrations of G by the cosets of parabolic subgroups P containing a fixed minimal parabolic subgroup P_0.

A little later, Mostow [7] considered certain mappings of G/P_0 in the proof of his 'rigidity theorem', and showed that these come from the action of the group G; this involved real Lie groups only. In the 1990s, others, including Gromov and Schoen [6] and Corlette [3], extended Mostow's ideas using geometric tools such as harmonic mappings; some of these extensions dealt with local fields. At around the same time, in the case of real Lie groups, Yamaguchi [10] proved that smooth, locally defined mappings of the spaces G/P that preserved the fibrations considered by Tits also arise from the action of the group G.

This thesis unifies the geometric themes of the classical authors with the group-theoretic approach of Tits. Associated to the classical real Lie groups G, there are spaces of the form G/P for some parabolic subgroup P of geometric significance; for example, some G/P are quasi-spheres. We consider bijections of these spaces that preserve their geometry, such as transformations of quasi-spheres that preserve quasi-circles, and show geometrically that these arise from the G-action. Thus, Tits' results follow from purely geometric considerations. However, we do more: we prove local versions of these theorems (along the lines of the result of Carathéodory mentioned above) and thereby generalize the work of Yamaguchi. Moreover, we prove local theorems over arbitrary nondiscrete fields, thereby extending theorems that seem to rely on real differential geometry into a much broader context. Indeed, our work seems to hint that there might be some form of 'topological geometry' that lies between algebraic geometry and differential geometry.

Some recent work was important for this thesis. In [1], Čap *et al.*, using an argument that involves order, show that maps of open sets in the plane that preserve collinearity come from projective transformations. We show that this implies a local version of Tits' 'fundamental theorem of projective geometry' for the group $\mathrm{SL}(3, \mathbb{R})$. In this thesis, we give an alternative proof of the collinearity-preserving result that works for an arbitrary nondiscrete topological field.

Part I of this thesis includes a review of the 'fundamental theorem of projective geometry' over the real and complex numbers $\mathbb{R}$ and $\mathbb{C}$, as well as the quaternions $\mathbb{H}$. Both global and local theorems are discussed; the former are well known but the latter are new. In some sense, the key result is that a mapping defined on open subsets of the real plane, that preserve line segments, and whose range has sufficiently many points in general position, is a projective mapping. We include two proofs of this: the proof of Čap *et al.* [1] and the present author's own proof. In Part II, we consider extensions to other classical groups. Here there are a number of key geometric results. One result, which extends Carathéodory's theorem, is that mappings that send quasi-circles into quasi-circles come from the group G. In this part, we also consider flag manifolds appropriate to the various classical groups and show how the extension of Carathéodory's theorem can be interpreted as a result about mappings of a flag manifold.

Part III of the thesis deals with nondiscrete topological fields, and more general rings, including adeles. We extend many of the results for classical Lie groups and geometries into this generality. The major result here is both topological and algebraic in nature: 'local homomorphisms' extend to global homomorphisms. This enables us to extend the local version of the fundamental theorem of projective geometry to topological fields and division rings and even some rings that are not division rings. Finally, in Part IV, we consider some results where the hypotheses involve measurable sets rather than open sets.

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