10

Renormalization of operators using the background field method

In the following chapters, in order to probe the hadron properties, we will have always to deal with *local* hadronic currents or/and operators built from quark or/and gluon fields, but not only with Green's functions. Therefore, it is of prime importance to study the renormalization of such operators. Renormalization of composite operators has been studied [126,127] using background field technology and some further examples have been studied explicitly in perturbation theory [128].

10.1 Outline of the background field approach

The basic idea of the method is to write the gauge field appearing in the classical action as $A + Q$, where A is the background field and Q the quantum field which is the variable of integration in the functional integral.[1] The background field gauge is chosen, which maintains the gauge invariance in terms of the A field, but breaks the one of the Q field. This background field gauge invariance is further assured by coupling external sources only to the Q field, which allows one to perform quantum calculations without losing the gauge invariance of the background field. More explicitly, let us consider the generating functional in Yang–Mills theory:[2]

$$Z[J] = \int \mathcal{D}Q\mathcal{D}\phi\mathcal{D}\bar{\phi}\ \exp\left(i\int\ d^4x[\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm FP} + \mathcal{L}_{\rm source} + \mathcal{L}_{\rm gauge}]\right)\ . \tag{10.1}$$

where $\mathcal{L}_{\rm YM},\ \mathcal{L}_{\rm FP}$ are the QCD and Faddeev–Popov Lagrangians defined in Eq. (5.14) without the fermion fields, and $\mathcal{L}_{\rm source}$ is defined in Eq. (6.15). The gauge fixing term is:

$$\mathcal{L}_{\rm gauge} = -\frac{1}{2\alpha_G}(G^a)^2\ , \tag{10.2}$$

where (G^a) is, for example, $G^a = \partial_\mu Q^a_\mu$. Doing the shift:

$$Q^\mu_a(x) \to Q^\mu_a(x) + A^\mu_a(x)\ , \tag{10.3}$$

[1] We shall follow closely the discussion in [127].
[2] Fermions do not play a role in this approach as they can be treated in the usual way.

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where $A_a^\mu(x)$ is the background field, the functional integral becomes:

$$\tilde{Z}[J, A] = \int \mathcal{D}Q\mathcal{D}\phi\mathcal{D}\bar{\phi}\ \exp\left(i\int d^4x[\tilde{\mathcal{L}}_{\rm YM} + \tilde{\mathcal{L}}_{\rm source} + \tilde{\mathcal{L}}_{\rm gauge}]\right) , \tag{10.4}$$

where:

$$\begin{aligned}\tilde{\mathcal{L}}_{\rm YM} &= -\frac{1}{4}\left(\partial^\mu(A+Q)_a^\nu - \partial^\nu(A+Q)_a^\mu\right)\left(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu\right) ,\\ \mathcal{L}_{\rm FP} &= -\partial_\mu\bar{\varphi}_a\left(\partial^\mu\delta_{ab} - gf_{abc}(A+Q)_c^\mu\right)\varphi^a ,\\ \tilde{\mathcal{L}}_{\rm source} &= Q_\mu J^\mu + \bar{\chi}\varphi + \bar{\varphi}\chi ,\end{aligned} \tag{10.5}$$

where the term $A_\mu J^\mu$ in the source has been omitted as A_μ is an external field to which one does not need to attach a source. The gauge fixing term (*background gauge*) can be chosen as:

$$\tilde{\mathcal{L}}_{\rm gauge} = -\frac{1}{2\alpha_G}(G^a)^2 , \tag{10.6}$$

where:

$$G^a = \partial^\mu Q_\mu^a + gf^{abc}A_b^\mu Q_c^\mu . \tag{10.7}$$

Like in the conventional approach, one can define the connected Green functions:

$$\tilde{W}[J] = -i\ln\tilde{Z}[J] , \tag{10.8}$$

and the effective action:

$$\tilde{\Gamma}[\tilde{Q}] = \tilde{W}[J] - \int d^4x\ J_\mu^a\tilde{Q}_\mu^a , \tag{10.9}$$

where:

$$\tilde{Q}_\mu^a = \delta\tilde{W}/\delta J_\mu^a . \tag{10.10}$$

Using the change of variable:

$$Q_\mu^a \to Q_\mu^a - f^{abc}\theta^b Q_\mu^c , \tag{10.11}$$

it is easy to show that $\tilde{Z}[J]$ and hence $\tilde{W}[J]$ are invariant under the infinitesimal transformations:

$$\begin{aligned}\delta A_\mu^a &= gf_{abc}\theta_b A_\mu^c - \partial_\mu\theta_a ,\\ \delta J_\mu^a &= gf_{abc}\theta_b J_\mu^c .\end{aligned} \tag{10.12}$$

Then, it follows that $\tilde{\Gamma}[\tilde{Q}, A]$ is invariant under the infinitesimal transformations:

$$\delta A_\mu^a = gf_{abc}\theta_b A_\mu^c - \partial_\mu\theta_a , \tag{10.13}$$

and:

$$\delta\tilde{Q}_\mu^a = gf_{abc}\theta_b\tilde{Q}_\mu^c , \tag{10.14}$$

in the background field gauge. In particular, $\tilde{\Gamma}[0, A]$ should be an explicitly gauge-invariant functional of A, since Eq. (10.13) is an ordinary gauge transformation of the background field. The quantity $\tilde{\Gamma}[\tilde{0}, A]$ is the gauge-invariant effective action which one computes in the background field method. One can show that:

$$\tilde{\Gamma}[\tilde{0}, A] = \Gamma[\bar{Q}]|_{\bar{Q}=A} , \tag{10.15}$$

where the latter is the usual action calculated in an unconventional gauge depending on A. Therefore, $\tilde{\Gamma}[\tilde{0}, A]$ can be used to generate the S-matrix of a gauge theory in the same way as the usual effective action is used. Feynman rules in the background gauge formalism can be generated from $\tilde{\mathcal{L}}_{\text{gauge}}$ in Eq. (10.6). Since the effective action only involves 1PI diagrams, vertices with only one outgoing quantum line will never contribute. Furthermore, the propagator of the A field is not defined, which does not matter as it is a classical field which never appears in the loop. Compared with ordinary Feynman rules the only difference is the appearance of the A field in external legs, which one denotes by a blob. These Feynman rules are given in Appendix E.

10.2 On the UV divergences and β-function calculation

The UV divergences of $\tilde{\Gamma}[\tilde{0}, A]$ can be absorbed by the renormalizations Z_A, Z_g, Z_{α_G} of the A field, the coupling constant and the gauge parameter, as it is a sum of a 1PI diagrams with A-field external legs and Q fields inside the loops. The renormalization of the gauge parameter can be avoided by working in the Landau gauge $\alpha_G = 0$. Because explicit gauge invariance is retained in the background field method, the renormalization constants Z_A and Z_G are related, and the infinities must take the gauge-invariant form of a divergent constant times the product of field strength $G^a_{\mu\nu} G^{\mu\nu}_a$. Let's now consider the bare field strength:

$$G^{a,B}_{\mu\nu} = Z_A^{1/2}\left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} Z_g Z_A^{1/2} A^b_\mu A^c_\nu\right] , \tag{10.16}$$

where we have used the fact that A_μ is a classical field for renormalizing $A^b_\mu A^c_\nu$. It will only take the form constant times $G^a_{\mu\nu}$ if:

$$Z_g Z^{1/2} = 1 , \tag{10.17}$$

which is a relation analogous to the one in QED. Equation (10.17) simplifies the computation of the β-function as illustrated in the explicit calculation of [127]. In the following, we give another application of the method to the renormalization of some composite operators.

10.3 Renormalization of composite operators

The first thing to do is to classify these operators into three classes:

Class I: gauge-invariant and *do not vanish* after using the equation of motion.
Class II: gauge-invariant but *vanish* after using the equation of motion.
Class III: gauge-dependent operators.

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Therefore any composite *renormalized* operators can be written as:

$$\mathcal{O} = Z_I O_I^B + Z_{II} O_{II}^B + Z_{III} O_{III}^B . \tag{10.18}$$

The great advantage of the background field techniques is that for graphs with external quark and background fields, one only needs gauge-invariant counterterms, i.e:

$$Z_{III} = 0 , \tag{10.19}$$

which is a consequence of the background field gauge invariance under quantization and renormalization. We shall now study some useful examples.

10.3.1 The vector and axial-vector currents

A classics example of composite operator is the local electromagnetic or neutral vector current:

$$V^\mu(x) = \bar\psi \gamma^\mu \psi(x) , \tag{10.20}$$

which is conserved to all orders of perturbation theory:

$$\partial_\mu V^\mu(x) = 0 , \tag{10.21}$$

and does not require any renormalization. The axial-vector current:

$$A_{ij}^\mu(x) = \bar\psi_i \gamma^\mu \gamma_5 \psi_j(x) , \tag{10.22}$$

is partially conserved for $SU(n)_L \times SU(n)_R$:

$$\partial_\mu A_{ij}^\mu(x) = (m_i + m_j)\bar\psi_i \, (i\gamma_5)\, \psi_j(x) . \tag{10.23}$$

It can be seen that for the divergence of the axial current, the mass renormalization compensates that of the operator $\bar\psi\gamma_5\psi$, such that at the end it does not get renormalized. We shall see, in the following, that for the $U(1)_A$ current, it needs to be renormalized.

10.3.2 Renormalization of $G_{\mu\nu}G^{\mu\nu}$

Let us illustrate the approach by studying the renormalization of the $G^a_{\mu\nu}G_a^{\mu\nu}$ gluon operator in the presence of massive quarks. For that, we have to take all *bare* (B) operators of dimension-four:

$$\begin{aligned} O_1^B &= -\frac{i}{4} GG , \\ O_2^B &= -\sum_j \bar\psi_j(\hat D + i m_j)\psi_j , \\ O_3^B &= i \sum_j m_j \bar\psi_j \psi_j , \end{aligned} \tag{10.24}$$

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where $\hat{D}$ is the covariant derivative. The renormalized O_1^R operator is, in general, a combination of these three bare operators:

$$O_1^R = Z_{11} O_1^B + Z_{12} O_2^B + Z_{13} O_3^B \, . \tag{10.25}$$

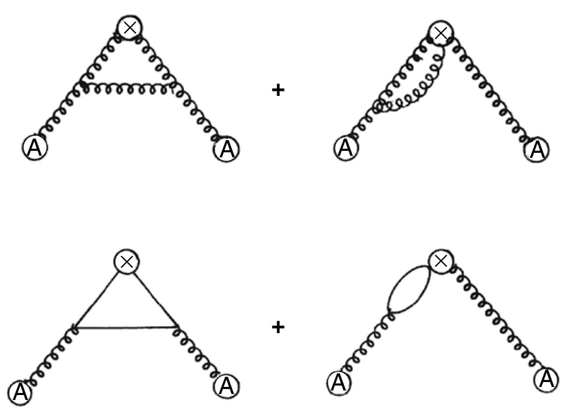
The renormalization constants Z_{ij} are mass-independent in the $\overline{MS}$ scheme, where one can notice that Z_{11} and Z_{12} can already be obtained in the massless limit. In order to evaluate the Z_{ij}, one inserts the zero momentum O_1 operator into the gluon and quark propagators:

$$\begin{aligned} \langle A_a^\mu \; O_1 \; A_b^\nu \rangle &= Z_\alpha^{-1} Z_{11} \langle A_a^\mu \; O_1 \; A_b^\nu \rangle + Z_{12} \langle A_a^\mu \; O_2 \; A_b^\nu \rangle \, , \\ \langle \bar{\psi} \; O_1 \; \psi \rangle &= Z_{11} \langle \bar{\psi} \; O_1 \; \psi \rangle + Z_{2F} Z_{12} \langle \bar{\psi} \; O_2 \; \psi \rangle + Z_{2F} Z_{13} \langle \bar{\psi} \; O_3 \; \psi \rangle \, . \end{aligned} \tag{10.26}$$

In practice, the insertions of O_1^B and O_2^B into the gluon propagator corresponds to the Feynman rules:

$$\begin{aligned} O_1 &\to -i \delta_{ab} (p^2 g_{\mu\nu} - p_\mu p_\nu) \, , \\ O_2 &\to i \hat{p} \, , \end{aligned} \tag{10.27}$$

and one has to calculate respectively:



(10.28)

The insertions of O_1^B, O_2^B and O_3^B into the quark propagator correspond respectively to:

$$\begin{aligned} O_1 &\to i g \frac{\lambda_a}{2} \gamma^\mu \, , \\ O_2 &\to i (\hat{p} - m_{j,B}) \, , \\ O_3 &\to i m_{j,B} \, , \end{aligned} \tag{10.29}$$

which can be represented by the following diagrams (Fig. 10.1):

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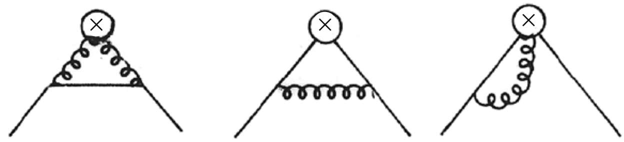

(10.30)

Evaluations of the previous diagrams give in the Landau gauge [128]:

$$Z_{11}^{(2)} = Z_{\alpha}^{(2)}, \qquad Z_{12}^{(2)} = 0, \qquad Z_{13}^{(2)} = -\frac{\gamma_1}{\epsilon}\left(\frac{\alpha_s}{\pi}\right), \tag{10.31}$$

as $Z_{2F}^{(2)} = 1$ in the Landau gauge (the index (2) means second order in α_s). Therefore, one can deduce:

$$(GG)_R = \left(1 + \left(\frac{\alpha_s}{\pi}\right)\frac{\beta_1}{\epsilon}\right)(GG)_B + 4\frac{\gamma_1}{\epsilon}\left(\frac{\alpha_s}{\pi}\right)\sum_j m_{j,B}(\bar{\psi}_j\psi_j)_B\ , \tag{10.32}$$

i.e., GG is not multiplicatively renormalizable. However, one can deduce from this expression the finite non-renormalized combination:

$$\theta_\mu^\mu = \frac{1}{4}\beta(\alpha_s)GG + \sum_j \gamma_m(\alpha_s)m_j\bar{\psi}_j\psi_j\ , \tag{10.33}$$

which is the trace of the energy-momentum tensor; $\beta(\alpha_s)$ and $\gamma_m(\alpha_s)$ are the β function and the mass-anomalous dimension defined in the previous section. The non-renormalization of θ_μ^μ is also preserved by higher-order terms [128].

10.3.3 Renormalization of the axial anomaly

The renormalization of the axial anomaly has been also discussed in [129]. Here, the different lowest dimension gauge-invariant pseudoscalar operators are:

$$\begin{aligned} O_1 &= -\frac{i}{4}G\tilde{G}\ , \\ O_2 &= \sum_j \bar{\psi}_j\gamma_5(i\hat{D} - m_j)\psi_j\ , \\ O_3 &= i\partial^\mu\sum_j(\bar{\psi}_j\gamma_\mu\gamma_5\psi_j)\ , \\ O_4 &= \sum_j m_j\bar{\psi}_j\gamma_5\psi_j\ . \end{aligned} \tag{10.34}$$

Previously background field techniques have been used for studying the renormalizations of these different operators, whereas the one of O_2 has not been studied because it does not appear in the triangle anomaly equation:

$$\partial^\mu\left(J_{\mu 5} \equiv \sum_j \bar{\psi}_j\gamma_\mu\gamma_5\psi_j\right) = 2i\sum_j m_j\bar{\psi}_j\gamma_5\psi_j - \left(2Q \equiv \frac{\alpha_s}{4\pi}n_f G\tilde{G}\right), \tag{10.35}$$

where n_f is the number of quark flavours. After renormalizations, the divergence of the flavour singlet current and the gluon topological charge density mix as follows:

$$J^R_{\mu 5} = Z J^B_{\mu 5} ,$$
$$Q^R = Q^B - \frac{1}{2n_f}(1-Z)\partial^\mu J^B_{\mu 5} , \tag{10.36}$$

where the renormalization constant is in n-dimension space–time ($n \equiv 4-\epsilon$), and reads:

$$Z = 1 + \left(\frac{\alpha_s}{\pi}\right)^2 \frac{3}{4}\frac{4}{3} n_f \frac{1}{\epsilon} . \tag{10.37}$$

10.3.4 Renormalizations of higher-dimension operators

The renormalization of dimension-five and -six operators have been studied in [130,131] and reviewed in detail in [3]. In the chiral limit, one can built the RGI mixed quark-gluon $d = 5$ operator for N colours and n_f flavours:

$$\langle \bar{O}_5 \rangle = \alpha_s^{-(\gamma_5/\beta_1)} \left\langle g\bar{\psi}\sigma^{\mu\nu}\frac{\lambda_a}{2}\psi G^a_{\mu\nu} \right\rangle , \tag{10.38}$$

with:

$$\gamma_5 = -\frac{(N^2-5)}{4N} , \qquad \beta_1 = -\frac{1}{6}(11N - 2n_f) . \tag{10.39}$$

The triple gluon condensate does not mix under renormalization, and one can form the renormalization group invariant (RGI) operator:

$$\langle \bar{O}_G \rangle = \alpha_s^{-(\gamma_G/\beta_1)} \langle g f_{abc} G^a G^b G^c \rangle \ : \qquad \gamma_G = \frac{2+7N}{6} . \tag{10.40}$$

The renormalization of the four-quark operators involves, in general, the mixing of different operators, such that the four-quark condensate:

$$\langle O_2 \rangle = \langle g^2 \bar{\psi}\psi\bar{\psi}\psi \rangle , \tag{10.41}$$

retained in the QSSR analysis within the vacuum saturation cannot be made RGI but possesses an intrinsic μ dependence. This μ dependence is only absent in the large N_c-limit, where only the diagonal renormalization constant $Z_{2,2}$ (notation in [130]) contributes. Therefore, only in this limit, one can form a RGI condensate:

$$\langle \bar{O}_2 \rangle = \alpha_s^{-(\gamma_2/\beta_1)} \langle O_\psi \rangle \ : \qquad \gamma_2 = \frac{143N}{33} . \tag{10.42}$$

We shall see later on, the importance of these operators in the context of QSSR.