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Boundary Quotient C*-algebras of Products of Odometers

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Abstract. In this paper, we study the boundary quotient C*-algebras associated with products of odometers. One of our main results shows that the boundary quotient C*-algebra of the standard product of k odometers over n_i-letter alphabets ($1 \le i \le k$) is always nuclear, and that it is a UCT Kirchberg algebra if and only if $\{\ln n_i : 1 \le i \le k\}$ is rationally independent, if and only if the associated single-vertex k-graph C*-algebra is simple. To achieve this, one of our main steps is to construct a topological k-graph such that its associated Cuntz–Pimsner C*-algebra is isomorphic to the boundary quotient C*-algebra. Some relations between the boundary quotient C*-algebra and the C*-algebra $\mathrm{Q}_{\mathbb{N}}$ introduced by Cuntz are also investigated.

1 Introduction

In [Li12], Xin Li associated several C*-algebras with a discrete left cancellative semigroup P. One of them is called the *full C*-algebra* $\mathrm{C}^*(P)$ *of* P, and is generated by an isometric representation of P and a family of projections parametrized by a family of right ideals of P satisfying certain relations. Since then the study of semigroup C*-algebras has been regaining a lot of attention; see, for example, [ABLS16, BOS15, BLS16, BRRW14, Stam16, Star15] and the references therein. In [BRRW14], Brownlowe, Ramagge, Robertson, and Whittaker defined a quotient C*-algebra $\mathcal{Q}(P)$ of $\mathrm{C}^*(P)$. They called it the *boundary quotient of* $\mathrm{C}^*(P)$. In fact, [BRRW14, Definition 5.1] applies to right least common multiple (LCM) semigroups only, but [BRRW14, Remark 5.5] proposes a definition for arbitrary left cancellative semigroups. Roughly speaking, if we think of $\mathrm{C}^*(P)$ as a "Toeplitz type" C*-algebra, then $\mathcal{Q}(P)$ is of "Cuntz–Pimsner type".

In [BRRW14, Section 6], the authors investigated many examples of the boundary quotients of the full C*-algebras of semigroups coming from Zappa and Szép products, which are also right LCM semigroups. The last example there, *i.e.,* [BRRW14, Subsection 6.6], is concerned with the standard product of two odometers

$$\big(\mathbb{Z}, \{0, 1, \dots, n-1\}\big) \quad \text{and} \quad \big(\mathbb{Z}, \{0, 1, \dots, m-1\}\big),$$

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where m and n are two coprime positive integers greater than 1. If we "divide" the elements in $\{0,1,\dots,mn-1\}$ by n and m, respectively, then we get a bijection θ from $\{0,1,\dots,n-1\}\times\{0,1,\dots,m-1\}$ to $\{0,1,\dots,m-1\}\times\{0,1,\dots,n-1\}$. Then we obtain a special semigroup $\mathbb{F}^+_\theta$, which is actually a single-vertex 2-graph (see [DY09$_1$]). Since n and m are coprime, $\mathbb{F}^+_\theta$ is right LCM. Moreover, one can form the Zappa–Szép product $\mathbb{F}^+_\theta \bowtie \mathbb{Z}$, which also turns out to be right LCM and so falls into the class studied in [BRRW14]. As observed in [BRRW14], it is easy to see that the 2-graph C*-algebra $\mathcal{O}_\theta$ of $\mathbb{F}^+_\theta$ is simple, as the coprimeness of n and m implies the aperiodicity of $\mathbb{F}^+_\theta$ (see [DY09$_1$]). However, unlike the other examples, $\mathcal{Q}(\mathbb{F}^+_\theta \bowtie \mathbb{Z})$ was not well understood there.

Let $n_1,\dots,n_k$ be k positive integers (where $k=\infty$ is allowed). For each $1\le i\le k$, let $X_i := \{x_{\mathfrak{s}}^i : 0\le \mathfrak{s}\le n_i-1\}$, and let $\mathbb{Z}$ act on each X_i as an odometer. For $1\le i<j\le k$, completely similar to the above, one has a bijection $\theta_{ij}\colon X_i\times X_j\to X_j\times X_i$. This induces a single-vertex k-graph $\mathbb{F}^+_\theta$ where $\theta=\{\theta_{i,j} : 1\le i<j\le k\}$ ([DY09$_2$]). So one can form a Zappa–Szép product $\mathbb{F}^+_\theta\bowtie\mathbb{Z}$, which is called the *standard product of odometers* $\{(\mathbb{Z},\{0,1,\dots,n_i-1\})\}_{i=1}^k$. We first construct a family of topological k-graphs $\{\Lambda_{\mathbf{n}} : \mathbf{n}\in\mathbb{N}^k\}$ that are k-dimensional analogues of Katsura's topological graphs $\{E_{n,1} : n\in\mathbb{N}\}$ in [Kat08], and then show that their associated Cuntz–Pimsner C*-algebras $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ are isomorphic to $\mathcal{Q}(\mathbb{F}^+_\theta\bowtie\mathbb{Z})$. On the way to our main results, we carefully study the generators and relations of the boundary quotient C*-algebras of a class of Zappa–Szép products of the form $\mathbb{F}^+_\theta\bowtie G$, where $\mathbb{F}^+_\theta$ is a single-vertex k-graph and G is a group. We should mention that $\mathbb{F}^+_\theta$ here is not necessarily right LCM, and so $\mathbb{F}^+_\theta\bowtie G$ is not right LCM in general. Therefore, one cannot apply the results in the recent works on right LCM semigroups, such as [ABLS16, BOS15, BLS16, BRRW14, Stam16, Star15], to our cases. Our main result on the boundary quotient C*-algebras associated with standard product of odometers can be summarized as follows.

Theorem (Theorems 5.4 and 5.13) *Let $\mathbb{F}^+_\theta\bowtie\mathbb{Z}$ be the Zappa–Szép product induced by the standard product of k odometers $\{(\mathbb{Z},\{0,1,\dots,n_i-1\})\}_{i=1}^k$.*

(i) *$\mathcal{Q}(\mathbb{F}^+_\theta\bowtie\mathbb{Z})$ is isomorphic to $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$.*
(ii) *$\mathcal{Q}(\mathbb{F}^+_\theta\bowtie\mathbb{Z})$ is nuclear.*
(iii) *$\mathcal{Q}(\mathbb{F}^+_\theta\bowtie\mathbb{Z})$ is a unital UCT Kirchberg algebra $\Leftrightarrow$ $\{\ln n_i\}_{1\le i\le k}$ is rationally independent $\Leftrightarrow$ $\mathbb{F}^+_\theta$ is aperiodic $\Leftrightarrow$ the k-graph C*-algebra $\mathcal{O}_\theta$ of $\mathbb{F}^+_\theta$ is simple.*

Therefore, the boundary quotient C*-algebras $\mathcal{Q}(\mathbb{F}^+_\theta\bowtie\mathbb{Z})$ are classifiable by K-theory when $\{\ln n_i\}_{1\le i\le k}$ is rationally independent, due to the celebrated Kirchberg–Phillips classification ([Phi00]). Consequently, the above theorem with [BOS15, Theorem 6.1] provides a very clear picture for the boundary quotient C*-algebra given in [BRRW14, Subsection 6.6], as mentioned above.

As a byproduct, we also prove that there is a natural homomorphism from $\mathcal{Q}(\mathbb{F}^+_\theta\bowtie\mathbb{Z})$ into the C*-algebra $\mathcal{Q}_{\mathbb{N}}$ introduced by Cuntz [Cun08]. It turns out that this homomorphism is injective if and only if $\{\ln n_i\}_{1\le i\le k}$ is rationally independent. In particular, one has $\mathcal{Q}(\mathbb{F}^+_\theta\bowtie\mathbb{Z})\cong\mathcal{Q}_{\mathbb{N}}$ if $\{n_i\}_{i=1}^\infty$ is the set of all prime numbers.

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This paper is organized as follows. In Section 2, some necessary background, which will be used later, is given. With very careful analysis, in Section 3, we exhibit the generators and relations of the boundary quotient C*-algebras of Zappa–Szép products of the form $\mathbb{F}_\theta^+ \bowtie G$, where $\mathbb{F}_\theta^+$ is a single-vertex k-graph and G is a group (see Theorem 3.3). As an important application of the results in Section 3, we obtain a very simple presentation of the boundary quotient C*-algebra of the standard product of k odometers (see Definition 4.6) in Section 4. Roughly speaking, it is the universal C*-algebra generated by a unitary representation of G and a $*$-representation of $\mathbb{F}_\theta^+$ which are compatible with the odometer actions (see Theorem 4.9). In our main section, Section 5, we first construct the class of topological k-graphs $\{\Lambda_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^k\}$, which is a higher-dimensional analogue of a class of topological graphs $\{E_{n,1} : n \in \mathbb{N}\}$ given by Katsura [Kat08]. By Yamashita's construction in [Yam09], there is a product system $X(\Lambda_{\mathbf{n}})$ over $\mathbb{N}^k$. The first main result in this section shows that the associated Cuntz–Pimsner C*-algebra $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ of $\Lambda_{\mathbf{n}}$ is isomorphic to the boundary quotient C*-algebra of the standard product of k odometers (see Theorem 5.4). Then, motivated by and with the aid of some results in [Cun08, Kat08, Yam09], we prove Theorem 5.12, which says that $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is simple if and only if $\{\ln n_i : 1 \le i \le k\}$ is rationally independent, and $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is also purely infinite in these cases. The nuclearity of $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is obtained by applying some results from [CLSV11, Yee07] to our case. Also, $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ satisfies the Universal Coefficient Theorem (UCT) from [RS87] due to [Tu99]. Therefore, $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is a unital UCT Kirchberg algebra if and only if $\{\ln n_i\}_{1\le i\le k}$ is rationally independent.

2 Preliminaries

In this section, we provide some necessary background, which will be useful later. We also take this chance to fix our terminologies and notation.

Notation and Conventions

Let $\mathbb{N}$ be the additive semigroup of non-negative integers. Denote by $\mathbb{N}^\times$ the multiplicative semigroup of positive integers. Let $1 \le k \le \infty$. For any semigroup P, denote by P^k (resp. $\prod_{i=1}^k P$) the direct sum (resp. product) of k copies of P (they coincide if $k < \infty$). Let $\{e_i\}_{i=1}^k$ be the standard basis of $\mathbb{N}^k$. For $n \in \mathbb{N}^k$, we write $n = (n_1, \dots, n_k)$. For $n, m \in \mathbb{N}^k$, $z \in \prod_{i=1}^k \mathbb{T}$, denote by $n \vee m$ (resp. $n \wedge m$) the coordinatewise maximum (resp. minimum) of n and m, and let $z^n := \prod_{i=1}^k z_i^{n_i}$.

For $1 \le n \in \mathbb{N}$, let $[n] := \{0, 1, \dots, n-1\}$. By $\mathbb{F}_n^+$, we mean the unital free semigroup with n generators.

In this paper, k is an arbitrarily fixed positive integer that could also be ∞, unless otherwise specified.

All semigroups in this paper are assumed to be unital (and so are monoids). For a semigroup U, its identity is denoted by 1_U (or just 1 if the context is clear).

2.1 Cuntz–Pimsner Algebras of Product Systems Over $\mathbb{N}^k$

In this subsection we recap the notion of product systems over $\mathbb{N}^k$ from [Fow02].

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Let A be a C*-algebra. A *C*-correspondence* over A (see [Fow02, FMR03]) is a right Hilbert A-module X together with a $*$-homomorphism $\phi: A \to \mathcal{L}(X)$, which gives a left action of A on X by $a \cdot x := \phi(a)x$ for all $a \in A$ and $x \in X$. A (Toeplitz) *representation* of X in a C*-algebra B is a pair (ψ, π), where $\psi: X \to B$ is a linear map, and $\pi: A \to B$ is a homomorphism such that

$$\psi(a \cdot x) = \pi(a)\psi(x); \psi(x)^*\psi(y) = \pi(\langle x, y\rangle) \quad \text{for all} \quad a \in A \text{ and } x, y \in X.$$

Notice that the relation $\psi(x \cdot a) = \psi(x)\pi(a)$ holds automatically, due to the above second relation. It turns out that there is a homomorphism $\psi^{(1)}: \mathcal{K}(X) \to B$ satisfying

$$\psi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^* \quad \text{for all} \quad x, y \in X, \tag{2.1}$$

where $\Theta_{x,y}(z) := x \cdot \langle y, z\rangle_A$ for $z \in X$ is a generalized rank-one operator. A representation (ψ, π) is said to be *Cuntz–Pimsner covariant* if

$$\psi^{(1)}(\phi(a)) = \pi(a) \quad \text{for all} \quad a \in \phi^{-1}(\mathcal{K}(X)).$$

Recall that X is said to be *essential* if $\overline{\operatorname{span}}\{\phi(a)x : a \in A, x \in X\} = X$, and *regular* if the left action ϕ is injective and $\phi(A) \subseteq \mathcal{K}(X)$.

Definition 2.1 Let A be a C*-algebra, and let $X = \bigsqcup_{n\in\mathbb{N}^k} X_n$ be a semigroup such that X_n is a C*-correspondence over A for all $n \in \mathbb{N}^k$. Then X is called a *product system over* $\mathbb{N}^k$ *with coefficient* A if the following hold:

(i) $X_0 = A$;
(ii) $X_n \cdot X_m \subset X_{n+m}$ for all $n, m \in \mathbb{N}^k$;
(iii) for $n, m \in \mathbb{N}^k \setminus \{0\}$, there exists an isomorphism from $X_n \otimes_A X_m$ onto X_{n+m}, where $X_n \otimes_A X_m$ denotes the balanced tensor product, by sending $x \otimes y$ to xy for all $x \in X_n$ and $y \in X_m$;
(iv) for $n \in \mathbb{N}^k$, the multiplication $X_0 \cdot X_n$ is implemented by the left action of A on X_n, and the multiplication $X_n \cdot X_0$ is implemented by the right action of A on X_n.

Definition 2.2 Let A, B be C*-algebras, let X be a product system over $\mathbb{N}^k$ with coefficient A, and let $\psi: X \to B$ be a map. For $n \in \mathbb{N}^k$, denote by $\psi_n := \psi|_{X_n}$. Then ψ is called a (Toeplitz) *representation* of X if

(T1) (ψ_n, ψ_0) is a representation of X_n for all $n \in \mathbb{N}^k$;
(T2) $\psi_n(x)\psi_m(y) = \psi_{n+m}(xy)$ for all $n, m \in \mathbb{N}^k$, $x \in X_n$, $y \in X_m$.

We write $\psi_n^{(1)}$ for the homomorphism from $\mathcal{K}(X_n)$ to B as in (2.1). The representation ψ is said to be *Cuntz–Pimsner covariant* if (ψ_n, ψ_0) is Cuntz–Pimsner covariant for all $n \in \mathbb{N}^k$.

The product system X is said to be *essential* (resp. *regular*) if X_n is essential (resp. regular) for all $n \in \mathbb{N}^k$.

Standing Assumptions

All product systems are always assumed to be essential and regular throughout the rest of the paper. Under these assumptions, every Cuntz–Pimsner covariant representation is automatically Nica covariant (see [Fow02, Proposition 5.4]).

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Proposition 2.3 *Let X be a product system over $\mathbb{N}^k$ with coefficient A. Then there exists a universal Cuntz–Pimsner covariant representation $j_X\colon X \to \mathcal{O}_X$ such that j_X generates $\mathcal{O}_X$, and for any Cuntz–Pimsner covariant representation J of X into a C*-algebra B, there is a unique homomorphism $\widetilde{J}\colon \mathcal{O}_X \to B$ such that $\widetilde{J} \circ j_X = J$. The C*-algebra $\mathcal{O}_X$ is called the* Cuntz–Pimsner algebra *of X.*

For a representation ψ of X, a *gauge action* is a strongly continuous homomorphism $\alpha\colon \prod_{i=1}^k \mathbb{T} \to \mathrm{Aut}(\mathrm{C}^*(\psi(X)))$ such that $\alpha_z(\psi_n(x)) = z^n\psi_n(x)$ for all $z \in \prod_{i=1}^k \mathbb{T}$, $n \in \mathbb{N}^k$, $x \in X_n$. The universal Cuntz–Pimsner covariant representation j_X admits a gauge action $\gamma\colon \prod_{i=1}^k \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_X)$. The gauge-invariant uniqueness theorem for a product system over $\mathbb{N}^k$ is highly nontrivial to achieve. However, this problem was completely resolved by Carlsen, Larsen, Sims, and Vittadello in [CLSV11] (their nice work covers much more general product systems). Combining [CLSV11, Corollary 4.12] and [SY10, Corollary 5.2], we obtain the following version of the gauge-invariant uniqueness theorem, which is analogous to the one in [FMR03].

Theorem 2.4 *Let X be a product system over $\mathbb{N}^k$ with coefficient A and let ψ be a Cuntz–Pimsner covariant representation of X that admits a gauge action. Denote by $h\colon \mathcal{O}_X \to \mathrm{C}^*(\psi(X))$ the homomorphism induced from the universal property of $\mathcal{O}_X$. If $h|_{j_{X,0}(A)}$ is injective, then h is an isomorphism.*

For later use, let us record the following two simple lemmas.

Lemma 2.5 *Let A, B be C*-algebras where A is generated by $\mathcal{G}$. Let X be a C*-correspondence over A, which has a subset $\mathcal{F}$ whose linear span is dense in X. Let $\psi_0\colon \mathrm{span}\,\mathcal{F} \to B$ be a linear map, and let $\pi\colon A \to B$ be a homomorphism. Suppose that*

(i) $\mathcal{G} \cdot \mathcal{F} \subset \mathcal{F}$;
(ii) $\psi_0(a \cdot x) = \pi(a)\psi_0(x)$ *for all* $x \in \mathcal{F}$ *and* $a \in \mathcal{G}$;
(iii) $\psi_0(x)^*\psi_0(y) = \pi(\langle x, y\rangle_A)$ *for all* $x, y \in \mathcal{F}$.

Then ψ_0 is a bounded linear map with the unique extension ψ to X, and (ψ, π) is a representation of X. Moreover, if $\psi^{(1)}(\phi(a)) = \pi(a)$ for all $a \in \mathcal{G}$, then (ψ, π) is also Cuntz–Pimsner covariant.

Proof This is straightforward to prove and left to the reader. ∎

Lemma 2.6 *Let X be a product system over $\mathbb{N}^k$ with coefficient A and let $\psi\colon X \to B$ be a representation. Suppose that (ψ_{e_i}, ψ_0) is Cuntz–Pimsner covariant for all $1 \le i \le k$. Then ψ is Cuntz–Pimsner covariant.*

Proof For $1 \le i, j \le k$, there exists an isomorphism from $X_{e_i} \otimes X_{e_j}$ onto $X_{e_i+e_j}$ sending $x \otimes y$ to $x \cdot y$ for all $x \in X_{e_i}$, $y \in X_{e_j}$, and there exists a linear map $\psi_{e_i} \otimes \psi_{e_j}\colon X_{e_i} \otimes X_{e_j} \to B$ such that $\psi_{e_i} \otimes \psi_{e_j}(x \otimes y) = \psi_{e_i}(x)\psi_{e_j}(y)$ for all $x \in X_{e_i}$ and $y \in X_{e_j}$. Similar to the proof of [Pim97, Lemma 3.10], one can see that $(\psi_{e_i} \otimes \psi_{e_j}, \psi_0)$ is Cuntz–Pimsner covariant. Hence ψ is Cuntz–Pimsner covariant. ∎

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2.2 Topological k-graphs

In this subsection we recall the definition from [Yee07] of topological k-graphs, which are generalizations of k-graphs studied by Kumjian–Pask in [KP00]. Then we briefly recall the product system associated wiht each topological k-graph from [Yam09].

Definition 2.7 A *topological k-graph* is a k-graph Λ equipped with a locally compact Hausdorff topology such that

- the composition of paths is continuous and open;
- the range map r, the source map s, and the degree map d are all continuous;
- s is a local homeomorphism.

For $n \in \mathbb{N}^k$, let Λ^n be the set of all paths of degree n. The topological k-graph Λ is said to be *regular* if $r|_{\Lambda^{e_i}}$ is proper and surjective for all $1 \le i \le k$.

One can show that if Λ is regular, then $r|_{\Lambda^n}$ is proper and surjective for all $n \in \mathbb{N}^k$.

Let Λ be a regular topological k-graph. One can construct a product system $X(\Lambda)$ over $\mathbb{N}^k$ as follows. Given $n \in \mathbb{N}^k$, define a topological graph $E_n := (\Lambda^0, \Lambda^n, r, s)$. Let $X_n(\Lambda) := X(E_n)$ be the graph correspondence of E_n in the sense of Katsura (*cf.* [Kat04$_1$]). By [Kat04$_1$, Proposition 1.10],

$$X_n(\Lambda) = \left\{ x \in \mathrm{C}(\Lambda^n) : \langle x, x\rangle_{C_0(\Lambda^0)} \in C_0(\Lambda^0) \right\}.$$

For $n, m \in \mathbb{N}^k$, $x \in X_n$ and $y \in X_m$, define a diamond operation $x \diamond y \colon \Lambda^{n+m} \to \mathbb{C}$ by

$$x \diamond y(\mu) := x(\alpha)y(\beta) \text{ for } \mu \in \Lambda^{n+m} \text{ with } \mu = \alpha\beta, d(\alpha) = n, d(\beta) = m.$$

Notice that $x \diamond y$ is well defined due to the unique factorization of μ. Let

$$X(\Lambda) := \bigsqcup_{n\in\mathbb{N}^k} X_n(\Lambda).$$

Then $X(\Lambda)$ is a product system over $\mathbb{N}^k$ with coefficient $C_0(\Lambda^0)$ under $\diamond$. We call $X(\Lambda)$ the *product system associated with* Λ. Notice that $X(\Lambda)$ is essential and regular.

2.3 Single-vertex k-graphs

In this subsection we recap the theory of single-vertex k-graphs and their C*-algebras from [DY09$_1$, DY09$_2$]. Let Λ be a single-vertex k-graph. For $1 \le i \le k$, let $\{x_{\mathfrak{s}}^i : \mathfrak{s} \in [m_i]\}$ be the set of all edges in Λ of degree e_i. It follows from the factorization property of Λ that, for $1 \le i < j \le k$, there is a bijection $\theta_{ij} \colon [m_i] \times [m_j] \to [m_j] \times [m_i]$ satisfying the following *θ-commutation relations*:

$$x_{\mathfrak{s}}^i x_{\mathfrak{t}}^j = x_{\mathfrak{t}'}^j x_{\mathfrak{s}'}^i \quad \text{if} \quad \theta_{ij}(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{t}', \mathfrak{s}').$$

Then Λ coincides with the semigroup $\mathbb{F}_\theta^+$ defined by (*cf.* [DY09$_2$])

$$\mathbb{F}_\theta^+ = \left\langle x_{\mathfrak{s}}^i : \mathfrak{s} \in [m_i],\ 1 \le i \le k;\ x_{\mathfrak{s}}^i x_{\mathfrak{t}}^j = x_{\mathfrak{t}'}^j x_{\mathfrak{s}'}^i \text{ whenever } \theta_{ij}(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{t}', \mathfrak{s}') \right\rangle,$$

which is also occasionally written as

$$\mathbb{F}_\theta^+ = \left\langle x_{\mathfrak{s}}^i : \mathfrak{s} \in [m_i],\ 1 \le i \le k;\ \theta_{ij},\ 1 \le i < j \le k \right\rangle.$$

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It is worthwhile mentioning that $\mathbb{F}_\theta^+$ has the cancellation property due to the factorization property of Λ. It follows from the θ-commutation relations that every element $w \in \mathbb{F}_\theta^+$ has the normal form $w = x_{u_1}^1 \cdots x_{u_k}^k$ for some $u_i \in \mathbb{F}_{m_i}^+$ $(1 \le i \le k)$. Here we use the multi-index notation: $x_{u_i}^i = x_{\mathfrak{s}_1}^i \cdots x_{\mathfrak{s}_n}^i$ if $u_i = \mathfrak{s}_1 \cdots \mathfrak{s}_n \in \mathbb{F}_{m_i}^+$.

For $k = 2$, every permutation θ determines a 2-graph. But for $k \ge 3$, $\theta = \{\theta_{ij} : 1 \le i < j \le k\}$ determines a k-graph if and only if it satisfies a *cubic condition* (see, *e.g.*, [DY09$_2$, FS02] for its definition). Here, it is probably worth mentioning that this is also related to the Yang–Baxter equation (see [Yan16$_1$, Yan16$_2$]).

By a $*$-*representation* S of $\mathbb{F}_\theta^+$ in a C*-algebra $\mathcal{A}$, we mean that S is a semigroup homomorphism of $\mathbb{F}_\theta^+$ that is subject to the relations: $S_{x_\mathfrak{s}^i}^* S_{x_\mathfrak{s}^i} = 1$ $(\mathfrak{s} \in [m_i])$, for $1 \le i \le k$, and the defect free condition $\sum_{\mathfrak{s}\in[m_i]} S_{x_\mathfrak{s}^i} S_{x_\mathfrak{s}^i}^* = 1$. The *k-graph C*-algebra* $\mathcal{O}_\theta$ of $\mathbb{F}_\theta^+$ is defined to be the universal C*-algebra for $*$-representations of $\mathbb{F}_\theta^+$.

2.4 Zappa–Szép Products of Semigroups

In this subsection we review the definitions of the full C*-algebra of a left cancellative semigroup from [Li12], its boundary quotient C*-algebra from [BRRW14], and the Zappa–Szép product of two semigroups from [BRRW14] (see also [Bri05]). The odometer action is also given to induce a class of Zappa–Szép products.

Let P be a left cancellative semigroup. For $p \in P$, we also denote by p the left multiplication map $q \mapsto pq$. The set of *constructible right ideals* is defined as

$$\mathcal{J}(P) := \{p_1^{-1} q_1 \cdots p_n^{-1} q_n P : n \ge 1, p_1, q_1, \ldots, p_n, q_n \in P\} \cup \{\varnothing\}.$$

A finite subset F of $\mathcal{J}(P)$ is called a *foundation set* if for each $Y \in \mathcal{J}(P)$ there exists $X \in F$ such that $X \cap Y \neq \varnothing$.

For $p, q \in P$, we say that p is a *right multiple* of q if there exists $r \in P$ such that $p = qr$. P is said to be *right LCM* if any two elements of P having a right common multiple have a right least common multiple.

Definition 2.8 ([Li12, Definition 2.2], [BRRW14, Remark 5.5]) Given a left cancellative semigroup P, the *full semigroup C*-algebra* $\mathrm{C}^*(P)$ of P is the universal C*-algebra generated by a family of isometries $\{v_p\}_{p\in P}$ and a family of projections $\{e_X\}_{X\in\mathcal{J}(P)}$ satisfying the following relations:

(L1) $v_p v_q = v_{pq}$ for all $p, q \in P$;
(L2) $v_p e_X v_p^* = e_{pX}$ for all $p \in P, X \in \mathcal{J}(P)$;
(L3) $e_\varnothing = 0$ and $e_P = 1$;
(L4) $e_X e_Y = e_{X\cap Y}$ for all $X, Y \in \mathcal{J}(P)$.

The *boundary quotient* $\mathcal{Q}(P)$ of $\mathrm{C}^*(P)$ is the universal C*-algebra generated by a family of isometries $\{v_p\}_{p\in P}$ and a family of projections $\{e_X\}_{X\in\mathcal{J}(P)}$ satisfying Conditions (L1)–(L4), and furthermore

(Q5) $\prod_{X\in F}(1 - e_X) = 0$ for all foundation sets $F \subset \mathcal{J}(P)$.

In this paper, $\mathcal{Q}(P)$ is simply called the *boundary quotient C*-algebra of* P.

Definition 2.9 ([BRRW14, Definition 3.1]) Let U and A be semigroups. Suppose there are two maps $A \times U \to U, (a, u) \mapsto a \cdot u$ and $A \times U \to A, (a, u) \mapsto a|_u$ such that

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for all $a, b \in A$ and $u, v \in U$, we have

(B1) $1_A \cdot u = u$;
(B2) $(ab) \cdot u = a \cdot (b \cdot u)$;
(B3) $a \cdot 1_U = 1_U$;
(B4) $a \cdot (uv) = (a \cdot u)(a|_u \cdot v)$;
(B5) $a|_{1_U} = a$;
(B6) $a|_{uv} = a|_u|_v$;
(B7) $1_A|_u = 1_A$;
(B8) $(ab)|_u = a|_{b\cdot u}b|_u$.

Let, $U \bowtie A := U \times A$, equipped with the multiplication

$$(u, a)(v, b) := (u(a \cdot v), a|_v \cdot b) \quad \text{for all} \quad (u, a), (v, b) \in U \times A.$$

Then $U \bowtie A$ is a semigroup under this multiplication, called the *(external) Zappa–Szép product*.

We call $a \cdot u$ the *action* of a on u, and $a|_u$ the *restriction* of a to u.

Let us record the following remark for later use.

Remark 2.10 If U and A in Definition 2.9 are both left cancellative semigroups, and if for any $a \in A$, the map $u \mapsto a \cdot u$ is an injection on U, then $U \bowtie A$ is also left cancellative.

One very useful way to produce Zappa–Szép products is from self-similar actions.

Definition 2.11 ([Nek05, Definition 1.5.1]) Let X be a non-empty finite set. Consider the free semigroup X^* generated by X. Suppose that a group G acts faithfully on X^*. Then this action is called *self-similar* if

(i) $g \cdot \varnothing = \varnothing$ for all $g \in G$;
(ii) for $g \in \mathbb{Z}, x \in X$, there exist unique $y \in X, h \in G$ such that $g \cdot (xw) = y(h \cdot w)$ for all $w \in X^*$.

We also call (G, X) a *self-similar action*.

If we let $g \cdot x := y$ and $g|_x := h$, then these two maps induce two maps $G \times X^* \to X^*, (g, u) \mapsto g \cdot u$ and $G \times X^* \to G, (g, u) \mapsto g|_u$ satisfying Conditions (B1)–(B8) of Definition 2.9. Identifying X^* with $\mathbb{F}^+_{|X|}$, we obtain a Zappa–Szép product $\mathbb{F}^+_{|X|} \bowtie G$.

A very important example of self-similar actions (see [LRRW14, Nek05]), which will be frequently used later, is given below.

Example 2.12 (Odometers) Let $n \geq 1$ and $X = \{x_{\mathfrak{s}} : \mathfrak{s} \in [n]\}$. Define

$$1 \cdot x_{\mathfrak{s}} = x_{(\mathfrak{s}+1) \bmod n} \quad \text{for } \mathfrak{s} \in [n],$$

$$1|_{x_{\mathfrak{s}}} = \begin{cases} 0 & \text{if } \mathfrak{s} < n-1, \\ 1 & \text{if } \mathfrak{s} = n-1. \end{cases}$$

This determines a self-similar action $(\mathbb{Z}, X)$, which is known as an *odometer* or an *adding machine*.

The following lemma will be used later and is of independent interest as well.

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Lemma 2.13 *Let U be a left cancellative semigroup and let G be a group. Let $G \times U \to U, (a,u) \mapsto a \cdot u$, and $G \times U \to G, (a,u) \mapsto a|_u$ be two maps satisfying Conditions* (B1)–(B8) *of Definition 2.9. Suppose that for $u \in U, b \in G$, there exists $a \in G$ such that $a|_u = b$. Then $UG = GU$ and $\mathcal{J}(U \bowtie G) = \mathcal{J}(U) \times G$ (with $\varnothing \times G := \varnothing$). Here, U is identified with $(U, 1_G) \leq U \bowtie G$ and similarly for G.*

Proof Let $(u,g) \in U \bowtie G$. By assumption, there is $h \in G$ such that $h|_u = g^{-1}$. From Definition 2.9 (B2) and (B8), one has that $h^{-1}(h(u)) = u$ and $h^{-1}|_{h(u)} = g$. Then

$$(u,g) = (u,1_G)(1_U,g) = (1_U,h^{-1})(h(u),1_G).$$

Thus, $UG = GU$. The rest of this lemma follows immediately. ∎

3 Generators and Relations of $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie G)$

When applying the construction given in Definition 2.8 to the Zappa–Szép product $U \bowtie A$ of two semigroups A and U, usually we find it hard to understand its boundary quotient C*-algebra $\mathcal{Q}(U \bowtie A)$. This is not surprising due to several factors: for instance, the constructible right ideals of $U \bowtie A$ could be very complex; its foundation sets are not easy to describe.

In this section, we study a class of Zappa–Szép products $\mathbb{F}_\theta^+ \bowtie G$, where G is a group and $\mathbb{F}_\theta^+$ is a single-vertex k-graph such that the restriction map satisfies a certain condition. In this case, $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie G)$ can be nicely presented by a unitary representation of G and a $*$-representation of $\mathbb{F}_\theta^+$ such that they are compatible with the action and restriction maps.

Lemma 3.1 *Let U be a left cancellative semigroup and G be a group. Let $G \times U \to U, (a,u) \mapsto a \cdot u$ and $G \times U \to G, (a,u) \mapsto a|_u$ be two maps satisfying conditions* (B1)–(B8) *of Definition 2.9. Suppose that for $u \in U, b \in G$, there exists $a \in G$ such that $a|_u = b$. Then $\mathcal{Q}(U \bowtie G)$ is isomorphic to the universal C*-algebra $\mathfrak{A}$ generated by a family of isometries $\{t_u\}_{u \in U}$, a family of projections $\{q_X\}_{X \in \mathcal{J}(U)}$, and a family of unitaries $\{s_a\}_{a \in G}$ satisfying the following properties. For $u, v \in U, X, Y \in \mathcal{J}(U), a, b \in G$,*

(i) $t_u t_v = t_{uv}$;
(ii) $t_u q_X t_u^* = q_{uX}$;
(iii) $s_a q_X s_a^* = q_{a \cdot X}$;[1]
(iv) $q_\varnothing = 0$ *and* $q_U = 1$;
(v) $q_X q_Y = q_{X \cap Y}$;
(vi) $\prod_{X \in F}(1 - q_X) = 0$ *for every foundation set* $F \subset \mathcal{J}(U)$;
(vii) $s_a s_b = s_{ab}$;
(viii) $s_a t_u = t_{a \cdot u} s_{a|_u}$.

Proof By Lemma 2.13, one can assume that $\{\delta_{(u,a)}\}_{(u,a) \in U \bowtie G}$ (resp. $\{e_{X \times G}\}_{X \in \mathcal{J}(U)}$) are the families of isometries (resp. projections) that generate $\mathcal{Q}(U \bowtie G)$.

For $(u,a) \in U \bowtie G$ and $X \in \mathcal{J}(U)$, define

$$\Delta_{(u,a)} := t_u s_a \text{ and } E_{X \times G} := q_X.$$

[1] Notice that $a \cdot X \in \mathcal{J}(U)$ by Lemma 2.13.

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Given $(u,a),(v,b)\in U\bowtie G$ and $X,Y\in\mathcal{J}(U)$, we have the following properties:

- $\Delta_{(u,a)}\Delta_{(v,b)}=t_us_at_vs_b=t_{u(a\cdot v)}s_{a|_vb}=\Delta_{(u,a)(v,b)}$.
- $\Delta_{(u,a)}E_{X\times G}\Delta^*_{(u,a)}=t_us_aq_Xs_a^*t_u^*=q_{u(a\cdot X)}=E_{ua\cdot X\times G}=E_{(u,a)X\times G}$.
- $E_\varnothing=q_\varnothing=0$ and $E_{U\bowtie G}=q_U=1$.
- $E_{X\times G}E_{Y\times G}=q_Xq_Y=q_{X\cap Y}=E_{(X\times G)\cap(Y\times G)}$.
- For a foundation set $\{X_i\times G\}_{1\le i\le n}$ of $\mathcal{J}(U\bowtie G)$, since $\{X_i\}_{i=1}^n$ is a foundation set of $\mathcal{J}(U)$, we have

$$\prod_{i=1}^n(1-E_{X_i\times G})=\prod_{i=1}^n(1-q_{X_i})=0.$$

Hence, relations (L1)–(L4) and (Q5) of Definition 2.9 hold. By the universal property of $\mathcal{Q}(U\bowtie G)$, there exists a homomorphism $\rho\colon\mathcal{Q}(U\bowtie G)\to\mathfrak{A}$ such that $\rho(\delta_{u,a})=\Delta_{(u,a)}$ and $\rho(e_{X\times G})=E_{X\times G}$ for all $(u,a)\in U\bowtie G$ and $X\in\mathcal{J}(U)$.

Conversely, for $u\in U$, $X\in\mathcal{J}(U)$ and $a\in G$, define

$$T_u:=\delta_{(u,1_G)},\quad Q_X:=e_{X\times G},\quad S_a:=\delta_{(1_U,a)}.$$

For $a\in G$, we compute that

$$S_aS_a^*:=\delta_{(1_U,a)}e_{U\bowtie G}\delta^*_{(1_U,a)}=e_{(1_U,a)(U\bowtie G)}=e_{U\bowtie G}=1.$$

So S_a is unitary. Then one can easily check that conditions (i)–(viii) hold. By the universal property of $\mathfrak{A}$, there exists a homomorphism $\pi\colon\mathfrak{A}\to\mathcal{Q}(U\bowtie G)$ such that $\pi(t_u)=T_u,\pi(q_X)=Q_X,\pi(s_a)=S_a$ for all $u\in U,X\in\mathcal{J}(U),a\in G$.

Finally, it is straightforward to see that $\pi\circ\rho=\mathrm{id}$ and $\rho\circ\pi=\mathrm{id}$. Therefore, $\mathcal{Q}(U\bowtie G)$ is isomorphic to $\mathfrak{A}$. ∎

Lemma 3.2 *Let $\mathbb{F}_\theta^+$ be a single-vertex k-graph, and let T be a $*$-representation of $\mathbb{F}_\theta^+$ in a C*-algebra $\mathcal{A}$. Given $\mu_1,\nu_1,\dots,\mu_n,\nu_n$ in $\mathbb{F}_\theta^+$, denote by*

$$F:=\left\{(\alpha_i,\beta_i)_{i=1}^n\in\prod_{j=1}^{2n}\mathbb{F}_\theta^+\ :\ (\alpha_i,\beta_i)\in\begin{cases}(\mathbb{F}_\theta^+)^{\min}(\mu_i,\nu_i\alpha_{i-1}) & \text{if } 1<i\le n,\\ (\mathbb{F}_\theta^+)^{\min}(\mu_i,\nu_i) & \text{if } i=1,\end{cases}\right\}$$

where $(\mathbb{F}_\theta^+)^{\min}(\mu,\nu)$ denotes the set of minimal common extensions of μ and ν. Then the following statement hold true.

(i) *For distinct tuples $(\alpha_i,\beta_i)_{i=1}^n$ and $(\gamma_i,\omega_i)_{i=1}^n\in F$, we have $d(\alpha_n)=d(\gamma_n)$, $d(\beta_n)=d(\omega_n)$, $\alpha_n\ne\gamma_n$, and $\beta_n\ne\omega_n$.*

(ii) $\mu_n^{-1}\nu_n\cdots\mu_1^{-1}\nu_1\mathbb{F}_\theta^+=\bigcup_{(\alpha_i,\beta_i)_{i=1}^n\in F}\alpha_n\mathbb{F}_\theta^+$.

(iii) *Each constructible right ideal of $\mathbb{F}_\theta^+$ has a unique representation as the union of disjoint principal right ideals of $\mathbb{F}_\theta^+$.*

(iv) $\mathcal{J}(\mathbb{F}_\theta^+)=\left\{\bigcup_{i=1}^n\alpha_i\mathbb{F}_\theta^+:d(\alpha_1)=\cdots=d(\alpha_n)\right\}$.

(v) *For any finite subset $F\subset\mathbb{F}_\theta^+$, we have $\{\alpha\mathbb{F}_\theta^+\}_{\alpha\in F}$ is a foundation set of $\mathcal{J}(\mathbb{F}_\theta^+)$ if and only if F is exhaustive (see [RSY04, Definition 2.4]).*

Proof (i) follows from the unique factorization property of $\mathbb{F}_\theta^+$.

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We verify (ii) by induction. It is straightforward to see that (ii) holds for $n = 1$. Suppose that (ii) holds for $n \geq 1$. Let

$$F' := \left\{ (\alpha_i, \beta_i)_{i=1}^{n+1} \in \prod_{j=1}^{2(n+1)} \mathbb{F}_\theta^+ : (\alpha_i, \beta_i) \in \begin{cases} (\mathbb{F}_\theta^+)^{\min}(\mu_i, \nu_i \alpha_{i-1}) & \text{if } i > 1, \\ (\mathbb{F}_\theta^+)^{\min}(\mu_i, \nu_i) & \text{if } i = 1. \end{cases} \right\}$$

Then

$$\begin{aligned} &\mu_{n+1}^{-1}\nu_{n+1}\mu_n^{-1}\nu_n \cdots \mu_1^{-1}\nu_1 \mathbb{F}_\theta^+ \\ &\quad = \mu_{n+1}^{-1}\nu_{n+1}\Big(\bigcup_{(\alpha_i,\beta_i)_{i=1}^n \in F} \alpha_n \mathbb{F}_\theta^+ \Big) \\ &\quad = \bigcup_{(\alpha_i,\beta_i)_{i=1}^n \in F} \mu_{n+1}^{-1}\nu_{n+1}(\alpha_n \mathbb{F}_\theta^+) \\ &\quad = \bigcup_{(\alpha_i,\beta_i)_{i=1}^n \in F} \mu_{n+1}^{-1}\nu_{n+1}\alpha_n \mathbb{F}_\theta^+ \\ &\quad = \bigcup_{(\alpha_i,\beta_i)_{i=1}^n \in F} \bigcup_{(\alpha_{n+1},\beta_{n+1}) \in (\mathbb{F}_\theta^+)^{\min}(\mu_{n+1}, \nu_{n+1}\alpha_n)} \alpha_{n+1}\mathbb{F}_\theta^+ \\ &\quad = \bigcup_{(\alpha_i,\beta_i)_{i=1}^{n+1} \in F'} \alpha_{n+1}\mathbb{F}_\theta^+. \end{aligned}$$

So this proves (ii), and (iii)–(v) easily follow from (ii). ∎

Theorem 3.3 *Let $\mathbb{F}_\theta^+$ be a single-vertex k-graph, G be a group, and let $G \times \mathbb{F}_\theta^+ \to \mathbb{F}_\theta^+, (g, \mu) \mapsto g \cdot \mu$ and $G \times \mathbb{F}_\theta^+ \to G, (g, \mu) \mapsto g|_\mu$ be two maps satisfying conditions* (B1)–(B8) *of Definition 2.9. Suppose that for $\mu \in \mathbb{F}_\theta^+, h \in G$, there exists $g \in G$ such that $g|_\mu = h$. Then $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie G)$ is isomorphic to the universal C*-algebra $\mathcal{A}$ generated by a unitary representation u of G and a $*$-representation v of $\mathbb{F}_\theta^+$ satisfying*

$$u_g v_\mu = v_{g\cdot\mu} u_{g|_\mu} \quad \text{for all} \quad \mu \in \mathbb{F}_\theta^+ \text{ and } g \in G. \tag{3.1}$$

Proof We apply the characterization of $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie G)$ from Lemma 3.1. That is, $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie G)$ is the universal C*-algebra generated by a family of isometries $\{t_\mu\}_{\mu \in \mathbb{F}_\theta^+}$, a family of projections $\{q_X\}_{X \in \mathcal{J}(\mathbb{F}_\theta^+)}$, and a family of unitaries $\{s_a\}_{a \in G}$ satisfying conditions (i)–(viii) of Lemma 3.1.

First of all, for $\mu \in \mathbb{F}_\theta^+$, $g \in G$ and $X = \bigcup_{i=1}^n \alpha_i \mathbb{F}_\theta^+ \in \mathcal{J}(\mathbb{F}_\theta^+)$, define $T_\mu := v_\mu, S_g := u_g, Q_X := \sum_{i=1}^n v_{\alpha_i} v_{\alpha_i}^*$. It is clear that T_μ and S_g are isometric and unitary, respectively. Also notice that Q_X is a well-defined projection due to Lemma 3.2. In what follows, we only verify that $\{T_\mu, S_g, Q_X : \mu \in \mathbb{F}_\theta^+, g \in G, X \in \mathcal{J}(\mathbb{F}_\theta^+)\}$ satisfies conditions (v) and (vi) of Lemma 3.1, as the other conditions hold easily.

To prove condition (v) of Lemma 3.1, let us fix $X = \bigcup_{i=1}^n \alpha_i \mathbb{F}_\theta^+$ and $Y = \bigcup_{j=1}^m \beta_j \mathbb{F}_\theta^+$ in $\mathcal{J}(\mathbb{F}_\theta^+)$. Then $X \cap Y = \bigcup_{i,j} \bigcup_{(\mu,\nu) \in \Lambda^{\min}(\alpha_i, \beta_j)} \alpha_i \mu \mathbb{F}_\theta^+$. So

$$\begin{aligned} Q_X Q_Y &= \sum_{i,j} v_{\alpha_i} v_{\alpha_i}^* v_{\beta_j} v_{\beta_j}^* \\ &= \sum_{i,j} \sum_{(\mu,\nu) \in (\mathbb{F}_\theta^+)^{\min}(\alpha_i, \beta_j)} v_{\alpha_i \mu} v_{\beta_j \nu}^* \quad \text{(see [KP00, Lemma 3.1])} \\ &= Q_{X \cap Y}. \end{aligned}$$

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For the proof of Lemma 3.1(vi), pick a foundation set $\{X_i := \bigcup_{j=1}^{m_i} \alpha_{ij}\mathbb{F}_\theta^+\}_{i=1}^n$ of $\mathcal{J}(\mathbb{F}_\theta^+)$. Notice that $\{\alpha_{ij}\mathbb{F}_\theta^+ : 1 \le i \le n, 1 \le j \le m_i\}$ is also a foundation set of $\mathcal{J}(\mathbb{F}_\theta^+)$. By Lemma 3.2, $\{\alpha_{ij} : 1 \le i \le n, 1 \le j \le m_i\}$ is exhaustive. Then it follows from [RSY04, Proposition B.1]) that

$$\prod_{i=1}^{n}(1 - Q_{X_i}) = \prod_{i=1}^{n}\prod_{j=1}^{m_i}(1 - v_{\alpha_{ij}}v_{\alpha_{ij}}^*) = 0.$$

By the universal property of $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie G)$, there exists a homomorphism $\pi\colon \mathcal{Q}(\mathbb{F}_\theta^+ \bowtie G) \to \mathcal{A}$ such that $\pi(t_\mu) = T_\mu, \pi(s_g) = S_g, \pi(q_X) = Q_X$ for all $\mu \in \mathbb{F}_\theta^+, g \in G, X \in \mathcal{J}(\mathbb{F}_\theta^+)$.

Conversely, let

$$V_\mu := t_\mu,\ U_g := s_g \quad \text{for all} \quad \mu \in \mathbb{F}_\theta^+,\ g \in G.$$

Clearly, V_μ is an isometry and U_g is a unitary. We verify that V is a $*$-representation of $\mathbb{F}_\theta^+$. Obviously, we only need to show that $\sum_{\mu\in(\mathbb{F}_\theta^+)^{e_i}} V_\mu V_\mu^* = 1$ for all $1 \le i \le k$. To this end, let $1 \le i \le k$. For distinct $\mu, \nu \in (\mathbb{F}_\theta^+)^{e_i}$, by Lemma 3.1(ii), we get $t_\mu t_\mu^* = q_{\mu\mathbb{F}_\theta^+}$ and $t_\nu t_\nu^* = q_{\nu\mathbb{F}_\theta^+}$. By Lemma 3.1(v), we have $t_\mu t_\mu^* t_\nu t_\nu^* = 0$. Since $\{\mu F_\theta^+\}_{\mu\in(\mathbb{F}_\theta^+)^{e_i}}$ is a foundation set of $\mathcal{J}(\mathbb{F}_\theta^+)$, we have

$$\begin{aligned}
1 - \sum_{\mu\in(\mathbb{F}_\theta^+)^{e_i}} V_\mu V_\mu^* &= 1 - \sum_{\mu\in(\mathbb{F}_\theta^+)^{e_i}} t_\mu t_\mu^* \\
&= \prod_{\mu\in(\mathbb{F}_\theta^+)^{e_i}} (1 - t_\mu t_\mu^*) \\
&= \prod_{\mu\in(\mathbb{F}_\theta^+)^{e_i}} (1 - q_{\mu\mathbb{F}_\theta^+}) \quad \text{(by Lemma 3.1(vi))} \\
&= 0.
\end{aligned}$$

Thus, by the universal property of $\mathcal{A}$, there exists a homomorphism $\rho\colon \mathcal{A} \to \mathcal{Q}(\mathbb{F}_\theta^+ \bowtie G)$ such that $\rho(v_\mu) = V_\mu, \rho(u_g) = U_g$ for all $\mu \in F_\theta^+, g \in G$.

It remains to show that π and ρ are inverses of each other. For this, let $X := \bigcup_{i=1}^n \alpha_i\mathbb{F}_\theta^+ \in \mathcal{J}(\mathbb{F}_\theta^+)$. Denote by $F := (\mathbb{F}_\theta^+)^{d(\alpha_1)} \setminus \{\alpha_i\}_{i=1}^n$. Then $\{\alpha_i\mathbb{F}_\theta^+, \alpha\mathbb{F}_\theta^+ : 1 \le i \le n, \alpha \in F\}$ and $\{X, \alpha\mathbb{F}_\theta^+ : \alpha \in F\}$ are foundation sets of $\mathcal{J}(\mathbb{F}_\theta^+)$. By Conditions (v)–(vi) of Lemma 3.1, we have

$$\begin{aligned}
\prod_{i=1}^{n}\left(1 - q_{\alpha_i\mathbb{F}_\theta^+}\right)\prod_{\alpha\in F}\left(1 - q_{\alpha\mathbb{F}_\theta^+}\right) &= 1 - \sum_{i=1}^{n} q_{\alpha_i\mathbb{F}_\theta^+} - \sum_{\alpha\in F} q_{\alpha\mathbb{F}_\theta^+} = 0, \\
\left(1 - q_X\right)\prod_{\alpha\in F}\left(1 - q_{\alpha\mathbb{F}_\theta^+}\right) &= 1 - q_X - \sum_{\alpha\in F} q_{\alpha\mathbb{F}_\theta^+} = 0.
\end{aligned}$$

So $\rho \circ \pi(q_X) = q_X$. Then it is easy to see that $\rho \circ \pi = \mathrm{id}, \pi \circ \rho = \mathrm{id}$. Therefore, we are done. ∎

Remark 3.4 Theorem 3.3 is an analogue of [BRRW14, Theorem 5.2]. However, since a single-vertex k-graph $\mathbb{F}_\theta^+$ is not necessarily right LCM in general (also see Proposition 4.7), the assumptions of [BRRW14, Theorem 5.2] are not satisfied in our case. So here one cannot apply [BRRW14, Theorem 5.2].

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Remark 3.5 If G is trivial, Theorem 3.3 implies that the boundary quotient C*-algebra $\mathcal{Q}(\mathbb{F}_\theta^+)$ is isomorphic to the graph C*-algebra $\mathcal{O}_\theta$ of $\mathbb{F}_\theta^+$. We should also mention that the C*-algebra $\mathrm{C}^*(\mathbb{F}_\theta^+)$ in [DSY08, DSY10] is really $\mathcal{Q}(\mathbb{F}_\theta^+)$ here, instead of the full C*-algebra of (the semigroup) $\mathbb{F}_\theta^+$. (To avoid confusion, the notation $\mathcal{O}_\theta$ was first used in [Yan10].)

4 An Application to the Standard Products of Odometers

Applying the main result in Section 3 to the standard product of k odometers, we further simplify the presentation of its boundary quotient C*-algebra. Our result, loosely speaking, says that the boundary quotient C*-algebra in this case is generated by a unitary representation of a group and a $*$-representation of a single-vertex k-graph, which are compatible with the odometer actions.

For our purpose, we first generalize [BRRW14, Proposition 3.10] to higher dimensional cases.

Proposition 4.1 **(and Definition)** *Let G be a group and let*

$$\mathbb{F}_\theta^+ = \left\langle x_{\mathfrak{s}}^i : \mathfrak{s} \in [n_i],\ 1 \le i \le k;\ x_{\mathfrak{s}}^i x_{\mathfrak{t}}^j = x_{\mathfrak{t}'}^j x_{\mathfrak{s}'}^i \text{ whenever } \theta_{ij}(\mathfrak{s},\mathfrak{t}) = (\mathfrak{t}',\mathfrak{s}') \right\rangle$$

be a single-vertex k-graph. Suppose that G acts self-similarly on each $\mathbb{F}_{n_i}^+$ $(1 \le i \le k)$. Then the action and restriction maps $G \times X_i \to X_i, (g, x_{\mathfrak{s}}^i) \mapsto g \cdot x_{\mathfrak{s}}^i$ and $G \times X_i \to G, (g, x_{\mathfrak{s}}^i) \mapsto g|_{x_{\mathfrak{s}}^i}$ can be extended to $G \times \mathbb{F}_\theta^+ \to \mathbb{F}_\theta^+, (g,\mu) \mapsto g \cdot \mu$ and $G \times \mathbb{F}_\theta^+ \to G, (g,\mu) \mapsto g|_\mu$ satisfying conditions (B1)–(B8) *in Definition 2.9, if and only if*

$$(g \cdot x_{\mathfrak{s}}^i)(g|_{x_{\mathfrak{s}}^i} \cdot x_{\mathfrak{t}}^j) = (g \cdot x_{\mathfrak{t}'}^j)(g|_{x_{\mathfrak{t}'}^j} \cdot x_{\mathfrak{s}'}^i) \tag{4.1}$$

for all generators g of G and $\theta_{ij}(x_{\mathfrak{s}}^i, x_{\mathfrak{t}}^j) = (x_{\mathfrak{t}'}^j, x_{\mathfrak{s}'}^i)$ $(1 \le i < j \le k)$.

The induced Zappa–Szép product $\mathbb{F}_\theta^+ \bowtie G$ is called the product of self-similar actions $\{(G, [n_i])\}_{i=1}^k$.

Proof "Only if": If $\theta_{ij}(x_{\mathfrak{s}}^i, x_{\mathfrak{t}}^j) = (x_{\mathfrak{t}'}^j, x_{\mathfrak{s}'}^i)$, then $x_{\mathfrak{s}}^i x_{\mathfrak{t}}^j = x_{\mathfrak{t}'}^j x_{\mathfrak{s}'}^i$. So from (B4) and (B6), one has

$$(g \cdot x_{\mathfrak{s}}^i)(g|_{x_{\mathfrak{s}}^i} \cdot x_{\mathfrak{t}}^j) = g \cdot (x_{\mathfrak{s}}^i x_{\mathfrak{t}}^j) = g \cdot (x_{\mathfrak{t}'}^j x_{\mathfrak{s}'}^i) = (g \cdot x_{\mathfrak{t}'}^j)(g|_{x_{\mathfrak{t}'}^j} \cdot x_{\mathfrak{s}'}^i),$$
$$g|_{x_{\mathfrak{s}}^i}|_{x_{\mathfrak{t}}^j} = g|_{x_{\mathfrak{s}}^i x_{\mathfrak{t}}^j} = g|_{x_{\mathfrak{t}'}^j x_{\mathfrak{s}'}^i} = g|_{x_{\mathfrak{t}'}^j}|_{x_{\mathfrak{s}'}^i},$$

for all $g \in G$. In particular, (4.1) holds true.

"If": In fact, for $g \in G$ and $u_i \in \mathbb{F}_{n_i}^+$ $(1 \le i \le k)$, define

$$g \cdot (x_{u_1}^1 x_{u_2}^2 \cdots x_{u_k}^k) := (g \cdot x_{u_1}^1)(g|_{x_{u_1}^1} \cdot x_{u_2}^2) \cdots (g|_{x_{u_1}^1}|_{x_{u_2}^2} \cdots |_{x_{u_{k-1}}^{k-1}} \cdot x_{u_k}^k),$$
$$g|_{x_{u_1}^1 x_{u_2}^2 \cdots x_{u_k}^k} := g|_{x_{u_1}^1}|_{x_{u_2}^2} \cdots |_{x_{u_k}^k}.$$

Notice that using (B2) and (B8) one can easily see that (4.1) holds true for all $g \in G$.

Here we only check condition (B4) in Definition 2.9, the others being similar. Clearly, it suffices to verify

$$g \cdot (x_{\mathfrak{s}_1}^{i_1} \cdots x_{\mathfrak{s}_k}^{i_k}) = (g \cdot x_{\mathfrak{s}_1}^{i_1})(g|_{x_{\mathfrak{s}_1}^{i_1}} \cdot x_{\mathfrak{s}_2}^{i_2}) \cdots (g|_{x_{\mathfrak{s}_1}^{i_1}}|_{x_{\mathfrak{s}_2}^{i_2}} \cdots |_{x_{\mathfrak{s}_{k-1}}^{i_{k-1}}} \cdot x_{\mathfrak{s}_k}^{i_k}),$$

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where all i_ns are distinct and $\mathfrak{s}_j \in [m_{i_j}]$. But this follows from the facts that (B2) and (B8) hold true on each $(G, [m_i])$, and that each word $x_{\mathfrak{s}_1}^{i_1} \cdots x_{\mathfrak{s}_k}^{i_k}$ can be obtained from $x_{\mathfrak{t}_1}^1 \cdots x_{\mathfrak{t}_k}^k$ after finite steps by switching the super indices i_n and i_{n+1} with $i_n > i_{n+1}$ only one at a time. ■

Remark 4.2 To see that the above proposition is a generalization of [BRRW14, Proposition 3.10], let $k = 2$, $\theta := \theta_{12}$, $x := x_{\mathfrak{s}}^1$, $y := x_{\mathfrak{t}}^2$, $x_{\mathfrak{s}'}^1 = \theta_X(x_{\mathfrak{s}}, y_{\mathfrak{t}})$ and $y_{\mathfrak{t}'}^2 = \theta_Y(x_{\mathfrak{s}}, y_{\mathfrak{t}})$. Then

$$\begin{aligned}
&\left\{\begin{aligned} &g \cdot \theta_Y(x,y) = \theta_Y(g \cdot x, g|_x \cdot y), \\ &g_{\theta_Y(x,y)} \cdot \theta_X(x,y) = \theta_X(g \cdot x, g|_x \cdot y) \end{aligned}\right\} \\
&\qquad \Leftrightarrow (g \cdot \theta_Y(x,y))(g_{\theta_Y(x,y)} \cdot \theta_X(x,y)) = \theta_Y(g \cdot x, g|_x \cdot y)\theta_X(g \cdot x, g|_x \cdot y) \\
&\qquad\quad \text{(by the unique factorization property of } \mathbb{F}_\theta^+) \\
&\qquad \Leftrightarrow g \cdot (\theta_Y(x,y)\theta_X(x,y)) = (g \cdot x)(g|_x \cdot y) \\
&\qquad \Leftrightarrow g \cdot (xy) = (g \cdot x)(g|_x \cdot y).
\end{aligned}$$

Example 4.3 Let $n_i = n$ for all $1 \le i \le k$ and $\theta_{ij}(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{s}, \mathfrak{t})$ for all $1 \le i < j \le k$. Then it is easy to check that $\mathbb{F}_\theta^+$ is a k-graph (also see [DY09$_2$]). Let G be an *arbitrary* group self-similarly acting on each $\mathbb{F}_n^+$ in the same way. So if

$$g \cdot e_{\mathfrak{s}}^i = e_{\mathfrak{s}_1}^i, \quad g|_{e_{\mathfrak{s}}^i} = h, \quad h \cdot e_{\mathfrak{t}}^j = e_{\mathfrak{t}_1}^j, \quad h|_{e_{\mathfrak{t}}^j} = h_1,$$

then

$$g \cdot e_{\mathfrak{s}}^j = e_{\mathfrak{s}_1}^j, \quad g|_{e_{\mathfrak{s}}^j} = h, \quad h \cdot e_{\mathfrak{t}}^i = e_{\mathfrak{t}_1}^i, \quad h|_{e_{\mathfrak{t}}^i} = h_1.$$

Thus,

$$\begin{aligned}
(g \cdot e_{\mathfrak{s}}^i)(g|_{e_{\mathfrak{s}}^i} \cdot e_{\mathfrak{t}}^j) = e_{\mathfrak{s}_1}^i e_{\mathfrak{t}_1}^j = e_{\mathfrak{s}_1}^j e_{\mathfrak{t}_1}^i = (g \cdot e_{\mathfrak{s}}^j)(g|_{e_{\mathfrak{t}}^j} \cdot e_{\mathfrak{s}}^i), \\
g|_{e_{\mathfrak{s}}^i}|_{e_{\mathfrak{t}}^j} = h|_{e_{\mathfrak{t}}^j} = h_1 = h|_{e_{\mathfrak{t}}^i} = g|_{e_{\mathfrak{s}}^j}|_{e_{\mathfrak{t}}^i}.
\end{aligned}$$

It follows from Proposition 4.1 that one obtains the product $\mathbb{F}_\theta^+ \bowtie G$ of self-similar actions $(G, [n])$.

It is worth mentioning that the above $\mathbb{F}_\theta^+$ is not a right LCM at all (as $\mathbb{F}_\theta^+$ is periodic), and so in this case $\mathbb{F}_\theta^+ \bowtie G$ is not right LCM.

In the sequel, we exhibit a class of products of self-similar actions satisfying all conditions in Theorem 3.3, which plays a vital role in this paper.

Example 4.4 Let $n_1, \ldots, n_k$ be k positive integers. For each $1 \le i \le k$, let $X_i := \{x_{\mathfrak{s}}^i : \mathfrak{s} \in [n_i]\}$, and let $\mathbb{Z}$ act on each X_i as an odometer (see Example 2.12). For $1 \le i < j \le k$, let $\theta_{ij}\colon X_i \times X_j \to X_j \times X_i$ be a bijection defined by

$$\theta_{ij}(x_{\mathfrak{s}}^i, x_{\mathfrak{t}}^j) = (x_{\mathfrak{t}'}^j, x_{\mathfrak{s}'}^i) \text{ if } \mathfrak{s} + \mathfrak{t}n_i = \mathfrak{t}' + \mathfrak{s}'n_j \ \big(\mathfrak{s}, \mathfrak{s}' \in [n_i], \mathfrak{t}, \mathfrak{t}' \in [n_j]\big). \tag{4.2}$$

Let

$$\mathbb{F}_\theta^+ = \big\langle x_{\mathfrak{t}}^i : \mathfrak{t} \in [n_i],\ 1 \le i \le k;\ \theta_{ij} \text{ in (4.2)},\ 1 \le i < j \le k \big\rangle.$$

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One can easily check that $\mathbb{F}_\theta^+$ satisfies the cubic condition, and so $\mathbb{F}_\theta^+$ is indeed a single-vertex k-graph. Moreover, relation (4.1) is satisfied. Then applying Proposition 4.1 gives a Zappa–Szép product $\mathbb{F}_\theta^+ \bowtie \mathbb{Z}$, which is the product of odometers $\{(\mathbb{Z}, [n_i])\}_{i=1}^k$.

Remark 4.5 By induction, one can check that, given $\mu \in \mathbb{F}_\theta^+$ and $l \in \mathbb{Z}$, there exists $l' \in \mathbb{Z}$ such that $l'|_\mu = l$. So the restriction map satisfies the condition required in Theorem 3.3.

Definition 4.6 The Zappa–Szép product $\mathbb{F}_\theta^+ \bowtie \mathbb{Z}$ given in Example 4.4 is called the *standard product of odometers* $\{(\mathbb{Z}, [n_i])\}_{i=1}^k$.

The following proposition is a generalization of a result from [BRRW14].

Proposition 4.7 *Keep the same notation as in Example 4.4. Then the following statements are equivalent:*

(i) *the n_is are pairwise coprime;*
(ii) *for $1 \le i < j \le k$, given $(\mathfrak{s}, \mathfrak{t}') \in [n_i] \times [n_j]$, there exists a unique pair $(\mathfrak{s}', \mathfrak{t}) \in [n_i] \times [n_j]$ such that $x_\mathfrak{s}^i x_\mathfrak{t}^j = x_{\mathfrak{t}'}^j x_{\mathfrak{s}'}^i$;*
(iii) *any two elements $\mu, \nu \in \mathbb{F}_\theta^+$ having a right common multiple have a unique right least common multiple with degree $d(\mu) \vee d(\nu)$;*
(iv) *$\mathbb{F}_\theta^+$ is right LCM.*

Proof (i)⇒(ii). Fix $1 \le i < j \le k$ and $(\mathfrak{s}, \mathfrak{t}') \in [n_i] \times [n_j]$. Assume that $(\mathfrak{s}', \mathfrak{t})$, $(\mathfrak{s}'', \mathfrak{t}'') \in [n_i] \times [n_j]$ such that $x_\mathfrak{s}^i x_\mathfrak{t}^j = x_{\mathfrak{t}'}^j x_{\mathfrak{s}'}^i$ and $x_\mathfrak{s}^i x_{\mathfrak{t}''}^j = x_{\mathfrak{t}'}^j x_{\mathfrak{s}''}^i$. Then $\mathfrak{s}+\mathfrak{t}n_i = \mathfrak{t}'+\mathfrak{s}'n_j$ and $\mathfrak{s} + \mathfrak{t}''n_i = \mathfrak{t}' + \mathfrak{s}''n_j$. So $(\mathfrak{t} - \mathfrak{t}'')n_i = (\mathfrak{s}' - \mathfrak{s}'')n_j$. Since n_i and n_j are coprime, $\mathfrak{t} = \mathfrak{t}''$ and $\mathfrak{s}' = \mathfrak{s}''$.

(ii)⇒(iii) and (iii)⇒(iv). The proofs are straightforward.

(iv)⇒(i). To the contrary, suppose that there exist $1 \le i < j \le k$ such that n_i and n_j are not coprime. Let $l := \gcd(n_i, n_j)$. Then $l > 1$. By the definition of θ_{ij} in (4.2), we have

$$x_0^i x_{n_j/l}^j = x_0^j x_{n_i/l}^i \quad \text{and} \quad x_0^i x_0^j = x_0^j x_0^i.$$

We deduce that x_0^i and x_0^j have right common multiples, but they do not have a right least common multiple. This contradicts the assumption that $\mathbb{F}_\theta^+$ is right LCM. Therefore, n_is are pairwise coprime. ∎

Remark 4.8 In order to include more examples, let us emphasize again that the n_is are arbitrary positive integers. As shown in Proposition 4.7, $\mathbb{F}_\theta^+$ is right LCM if and only if n_i are pairwise coprime. Therefore, [BRRW14, Theorem 5.2] only applies to the case where the n_is are pairwise coprime. However, with the aid of Theorem 3.3, we are still able to simplify $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ without any conditions for the n_is.

Theorem 4.9 *Let $\mathbb{F}_\theta^+ \bowtie \mathbb{Z}$ be the standard product of odometers $\{(\mathbb{Z}, [n_i])\}_{i=1}^k$. Then $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ is isomorphic to the universal C*-algebra $\mathcal{A}$ generated by a unitary f and a family of isometries $\{g_{x_\mathfrak{s}^i} : \mathfrak{s} \in [n_i], 1 \le i \le k\}$ satisfying*

(i) $\sum_{\mathfrak{s} \in [n_i]} g_{x_\mathfrak{s}^i} g_{x_\mathfrak{s}^i}^* = 1$ *for all* $1 \le i \le k$;

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(ii) *for* $1 \le i \le k$, $f g_{x_{\mathfrak{s}}^i} = \begin{cases} g_{x_{\mathfrak{s}+1}^i} & \text{if } 0 \le \mathfrak{s} < n_i - 1 \\ g_{x_0^i} f & \text{if } \mathfrak{s} = n_i - 1; \end{cases}$

(iii) $g_{x_{\mathfrak{s}}^i} g_{x_{\mathfrak{t}}^i} = g_{x_{\mathfrak{t}'}^j} g_{x_{\mathfrak{s}'}^i}$ *whenever* $\theta_{ij}(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{t}', \mathfrak{s}')$ *for all* $1 \le i < j \le k$, $\mathfrak{s}, \mathfrak{s}' \in [n_i]$ *and* $\mathfrak{t}, \mathfrak{t}' \in [n_j]$.

Proof We adopt the characterization of $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ from Theorem 3.3. Let $\{t_\mu, s_N : \mu \in \mathbb{F}_\theta^+, N \in \mathbb{Z}\}$ be the generators of $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$.

For $N \in \mathbb{Z}$, define $S_N := f^N$. Clearly, S is a unitary representation of $\mathbb{Z}$ in $\mathcal{A}$. Define $T_\varnothing := 1$, and $T_{x_{\mathfrak{s}}^i} := g_{x_{\mathfrak{s}}^i}$ for $\mathfrak{s} \in [n_i]$ $(1 \le i \le k)$. By (iii), for any word $w = u_1 \cdots u_k \in \mathbb{F}_\theta^+$ with $u_i \in [n_i]$, we can define an isometry $T_w := g_{x_{u_1}^1} \cdots g_{x_{u_k}^k}$. So this yields an isometric representation of $\mathbb{F}_\theta^+$ in $\mathcal{A}$. Then it follows from (i) that T is a $*$-representation of $\mathbb{F}_\theta^+$.

For $1 \le i \le k$ and $\mathfrak{s} \in [n_i]$, (ii) implies that

$$S_1 T_{x_{\mathfrak{s}}^i} = f g_{x_{\mathfrak{s}}^i} = \begin{cases} g_{x_{\mathfrak{s}+1}^i} & \text{if } 0 \le \mathfrak{s} < n_i - 1, \\ g_{x_0^i} f & \text{if } \mathfrak{s} = n_i - 1; \end{cases} = T_{1 \cdot x_{\mathfrak{s}}^i} S_{1|_{x_{\mathfrak{s}}^i}}.$$

Then one can easily check that Eq. (3.1) holds true. By the universal property of $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$, there exists a homomorphism $\varphi: \mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z}) \to \mathcal{A}$ such that $\varphi(s_N) = S_N$ and $\varphi(t_\mu) = T_\mu$ for all $N \in \mathbb{Z}$ and $\mu \in \mathbb{F}_\theta^+$.

Conversely, define $F := s_1$ and $G_{x_{\mathfrak{s}}^i} := t_{x_{\mathfrak{s}}^i}$ for $\mathfrak{s} \in [n_i]$ $(1 \le i \le k)$. Since t is a $*$-representation of $\mathbb{F}_\theta^+$, (i) and (iii) automatically hold true. For $1 \le i \le k$ and $\mathfrak{s} \in [n_i]$, it follows from (3.1) that

$$F G_{x_{\mathfrak{s}}^i} = s_1 t_{x_{\mathfrak{s}}^i} = t_{1 \cdot x_{\mathfrak{s}}^i} s_{1|_{x_{\mathfrak{s}}^i}} = \begin{cases} G_{x_{\mathfrak{s}+1}^i} & \text{if } 0 \le \mathfrak{s} < n_i - 1, \\ G_{x_0^i} f & \text{if } \mathfrak{s} = n_i - 1, \end{cases}$$

which implies (ii). By the universal property of $\mathcal{A}$, there exists a homomorphism $\pi: \mathcal{A} \to \mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ such that $\pi(f) = F$, $\pi(g_{x_{\mathfrak{s}}^i}) = G_{x_{\mathfrak{s}}^i}$ for all $1 \le i \le k$ and $\mathfrak{s} \in [n_i]$.

It now follows easily that $\pi \circ \varphi = \mathrm{id}$, $\varphi \circ \pi = \mathrm{id}$. Therefore, we are done. ∎

The following properties will be used later.

Corollary 4.10 *Keep the same notation as in Theorem 4.9. Then*

(i) $g_{x_0^i} g_{x_0^j} = g_{x_0^j} g_{x_0^i}$ *for all* $1 \le i < j \le k$;

(ii) $f^{\mathfrak{s}} g_{x_0^i} = g_{x_{\mathfrak{s}}^i}$ *for all* $1 \le i \le k$ *and* $\mathfrak{s} \in [n_i]$;

(iii) $f^{n_i^l N} g_{x_0^i}^l = g_{x_0^i}^l f^N$ *for all* $1 \le i \le k$, $l \ge 0$, $N \in \mathbb{Z}$.

Proof The proofs of (i) and (ii) follow directly from Theorem 4.9.

Clearly, the identities of (iii) hold trivially when either $l = 0$ or $N = 0$. So we can assume that $l \ge 1$ and $N \ne 0$. Since f is a unitary, it suffices to verify them for $N > 0$. Also, it is easy to see that one only needs to show $f^{n_i^l} g_{x_0^i}^l = g_{x_0^i}^l f$, and we do it by induction. Property (ii) of Theorem 4.9 gives $f^{n_i} g_{x_0^i} = g_{x_0^i} f$. Suppose that $f^{n_i^l} g_{x_0^i}^l = g_{x_0^i}^l f$ holds for $l \ge 1$. Then

$$f^{n_i^{l+1}} g_{x_0^i}^{l+1} = g_{x_0^i}^l f^{n_i} g_{x_0^i} = g_{x_0^i}^{l+1} f.$$

This finishes the proof. ∎

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5 $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ via Topological k-graphs

In this main section, we first construct a class of topological k-graphs $\{\Lambda_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^k\}$, which is a higher-dimensional generalization of a class of topological graphs $\{E_{n,1} : n \in \mathbb{N}\}$ studied by Katsura [Kat08]. By Yamashita's construction in [Yam09], there is a product system $X(\Lambda_{\mathbf{n}})$ over $\mathbb{N}^k$. The first main result here shows that the associated Cuntz–Pimsner C*-algebra $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ of $X(\Lambda_{\mathbf{n}})$ is isomorphic to the boundary quotient C*-algebra $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ of the standard product of k odometers (Theorem 5.4). Then, motivated by and with the aid of some results in [Cun08, Kat08, Yam09], we prove our second main theorem (Theorem 5.12) in this section: $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is simple if and only if $\{\ln n_i : 1 \le i \le k\}$ is rationally independent, and is also purely infinite in these cases. The nuclearity of $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is obtained by applying some results in [CLSV11, Yee07]. By [Tu99], $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ satisfies the UCT as well. Combining these two theorems gives a very clear picture on $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ (Theorem 5.13). At the end of this section, we also provide some relations between $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ and the C*-algebra $\mathcal{Q}_{\mathbb{N}}$ introduced by Cuntz [Cun08].

From now on, we only consider the standard product $\mathbb{F}_\theta^+ \rtimes \mathbb{Z}$ of the odometers $\{(\mathbb{Z}, [n_i])\}_{i=1}^k$. For our convenience, we use the notation

$$\mathbf{1} := (1, \dots, 1), \quad \mathbf{n} := (n_1, \dots, n_k), \quad \mathbf{n}^p := \prod_{i=1}^k n_i^{p_i} \ (p \in \mathbb{N}^k).$$

5.1 Realizing $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ as Topological k-graph C*-algebras

In this subsection, we first construct a class of topological k-graphs, whose C*-algebras will be shown to be isomorphic to $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$.

Definition 5.1 Let $\Lambda_{\mathbf{n}} := \bigsqcup_{p \in \mathbb{N}^k} \mathbb{T}$ be a topological k-graph constructed as follows: $\Lambda_{\mathbf{n}}^0 := \mathbb{T} \times \{0\}$. Given $(z, p) \in \Lambda_{\mathbf{n}}$, let

$$r(z,p) := (z,0), \quad s(z,p) := (z^{\mathbf{n}^p}, 0), \quad d(z,p) := p.$$

For $(z,p), (w,q) \in \Lambda_{\mathbf{n}}$ with $s(z,p) = r(w,q)$, define

$$(z,p) \cdot (w,q) := (z, p+q).$$

One can also describe $\Lambda_{\mathbf{n}}$ as follows:

$$\Lambda_{\mathbf{n}}^{e_i} := \mathbb{T}, \quad r(z,e_i) := (z,0), \quad s(z,e_i) := (z^{n_i}, 0) \ (z \in \mathbb{T}, 1 \le i \le k).$$

The commuting squares of $\Lambda_{\mathbf{n}}$ are given by

$$(z,e_i)(z^{n_i}, e_j) = (z,e_j)(z^{n_j}, e_i) \text{ for all } z \in \mathbb{T} \text{ and } 1 \le i \ne j \le k.$$

Thus it is not hard to see that the graph $\Lambda_{\mathbf{n}}$ is a k-dimensional generalization of Katsura's topological graph $E_{n,1}$ in [Kat08], which can also be obtained as $\circlearrowright \bullet \times_{n,1} \mathbb{T}$. In fact, let Λ be the single-vertex k-graph with one edge for each degree e_i. Then one could think of $\Lambda_{\mathbf{n}}$ as $\Lambda \times_{\mathbf{n},\mathbf{1}} \mathbb{T}$.

Remark 5.2 $\Lambda_{\mathbf{n}}$ is indeed a topological k-graph (for $k \ge 1$). In fact, it suffices to verify that $\Lambda_{\mathbf{n}}$ satisfies the cubic condition for $k \ge 3$. To this end, consider $\lambda =$

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$(z, e_i)(z^{n_i}, e_j)(z^{n_i n_j}, e_\mathfrak{k})$ of degree $(1,1,1)$. Then

$$\begin{aligned}\lambda &= (z, e_j)(z^{n_j}, e_i)(z^{n_i n_j}, e_\mathfrak{k}) = (z, e_j)(z^{n_j}, e_\mathfrak{k})(z^{n_j n_\mathfrak{k}}, e_i)\\ &= (z, e_\mathfrak{k})(z^{n_\mathfrak{k}}, e_j)(z^{n_j n_\mathfrak{k}}, e_i)\\ &= (z, e_i)(z^{n_j}, e_\mathfrak{k})(z^{n_i n_\mathfrak{k}}, e_j) = (z, e_\mathfrak{k})(z^{n_\mathfrak{k}}, e_i)(z^{n_i n_\mathfrak{k}}, e_j).\end{aligned}$$

This says exactly that the cubic condition holds true.

Lemma 5.3 *There is a Cuntz–Pimsner covariant representation of $X(\Lambda_\mathbf{n})$ in $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$.*

Proof In the sequel, we adopt the characterization of $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ from Theorem 4.9.

Let $\iota\colon \Lambda_\mathbf{n}^0 \to \mathbb{C}$ be the embedding map. Since f is a unitary in $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$, there exists a homomorphism $\psi_0\colon \mathrm{C}(\Lambda^0) \to \mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ such that $\psi_0(\iota) = f$.

Fix $0 \neq p \in \mathbb{N}^k$. For $l \in \mathbb{Z}$, define $\gamma_l\colon \Lambda_\mathbf{n}^p \to \mathbb{C}$ by

$$\gamma_l(z, p) := z^l \quad \text{for all} \quad z \in \mathbb{T}.$$

Let $\mathcal{F} := \{\gamma_l\}_{l\in\mathbb{Z}}$ and $\mathcal{G} := \{\iota\}$. It is straightforward to see that $\|\gamma\|_{X_p(\Lambda_\mathbf{n})} \le \sqrt{\mathbf{n}^p}\|\gamma\|_{\sup}$ for all $\gamma \in X_p(\Lambda_\mathbf{n})$. By the Stone–Weierstrass theorem, the linear span of $\mathcal{F}$ is dense in $X_p(\Lambda_\mathbf{n})$. It is also straightforward to see that $\mathcal{G}$ generates $\mathrm{C}(\Lambda^0)$. Furthermore, $\mathcal{G}\cdot\mathcal{F} \subset \mathcal{F}$.

Step 1. We construct a linear map $\psi_p\colon X_p(\Lambda_\mathbf{n}) \to \mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ such that (ψ_p, ψ_0) is a representation of $X_p(\Lambda_\mathbf{n})$. Let $i_0 := \min\{1 \le i \le k : p_i \neq 0\}$ and $\mathbf{g}_0^p := \prod_{i=i_0+1}^k g_{x_0^i}^{p_i}$. Clearly, $\mathcal{F}$ is linear independent. Define a linear map $\psi_p\colon \operatorname{span}\mathcal{F} \to \mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ by

$$\psi_p(\gamma_{\mathfrak{s}+n_{i_0}l}) = \mathbf{n}^{\frac{p}{2}} g_{x_\mathfrak{s}^{i_0}} f^l g_{x_0^{i_0}}^{p_{i_0}-1}\mathbf{g}_0^p \quad \text{for all} \quad s \in [n_{i_0}] \text{ and } l \in \mathbb{Z}. \tag{5.1}$$

Then we have

$$\psi_p(\iota\cdot\gamma_{\mathfrak{s}+n_{i_0}l}) = \psi_0(\iota)\psi_p(\gamma_{\mathfrak{s}+n_{i_0}l}) \quad \text{for all} \quad \mathfrak{s} \in [n_{i_0}] \text{ and } l \in \mathbb{Z}.$$

This is done by the following calculations: For $0 \le \mathfrak{s} < n_{i_0} - 1$ and $l \in \mathbb{Z}$,

$$\begin{aligned}\psi_p(\iota\cdot\gamma_{\mathfrak{s}+n_{i_0}l}) &= \psi_p(\gamma_{\mathfrak{s}+1+n_{i_0}l}) = \mathbf{n}^{\frac{p}{2}} g_{x_{\mathfrak{s}+1}^{i_0}} f^l g_{x_0^{i_0}}^{p_{i_0}-1}\mathbf{g}_0^p\\ &= \mathbf{n}^{\frac{p}{2}} f g_{x_\mathfrak{s}^{i_0}} f^l g_{x_0^{i_0}}^{p_{i_0}-1}\mathbf{g}_0^p = \psi_0(\iota)\psi_p(\gamma_{\mathfrak{s}+n_{i_0}l}),\end{aligned}$$

and by Corollary 4.10,

$$\begin{aligned}\psi_p(\iota\cdot\gamma_{n_{i_0}-1+n_{i_0}l}) &= \psi_p(\gamma_{n_{i_0}(l+1)}) = \mathbf{n}^{\frac{p}{2}} g_{x_0^{i_0}} f^{l+1} g_{x_0^{i_0}}^{p_{i_0}-1}\mathbf{g}_0^p\\ &= \mathbf{n}^{\frac{p}{2}} f g_{x_{n_{i_0}-1}^{i_0}} f^l g_{x_0^{i_0}}^{p_{i_0}-1}\mathbf{g}_0^p = \psi_0(\iota)\psi_p(\gamma_{n_{i_0}-1+n_{i_0}l}).\end{aligned}$$

Now for $\mathfrak{s}, \mathfrak{s}' \in [n_{i_0}]$ and $l, l' \ge 0$ with $\mathfrak{s} + n_{i_0}l, \mathfrak{s}' + n_{i_0}l' \in [\mathbf{n}^p]$, we claim that

$$\psi_p(\gamma_{\mathfrak{s}+n_{i_0}l+\mathbf{n}^p m})^*\psi_p(\gamma_{\mathfrak{s}'+n_{i_0}l'+\mathbf{n}^p m'}) = \psi_0\big(\langle\gamma_{\mathfrak{s}+n_{i_0}l+\mathbf{n}^p m}, \gamma_{\mathfrak{s}'+n_{i_0}l'+\mathbf{n}^p m'}\rangle_{\mathrm{C}(\Lambda_\mathbf{n}^0)}\big) \tag{5.2}$$

for all $m, m' \in \mathbb{Z}$.

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On the one hand, by a direct calculation, one has

$$\begin{aligned}
&\psi_0\Big(\big\langle \gamma_{\mathfrak{s}+n_{i_0}l+\mathbf{n}^p m}, \gamma_{\mathfrak{s}'+n_{i_0}l'+\mathbf{n}^p m'}\big\rangle_{\mathrm{C}(\Lambda_{\mathbf{n}}^0)}(z,0)\Big)\\
&\quad= \psi_0\Big(\sum_{w^{\mathbf{n}^p}=z} w^{\mathfrak{s}'-\mathfrak{s}+n_{i_0}(l'-l)+\mathbf{n}^p(m'-m)}\Big)\\
&\quad= \delta_{\mathfrak{s},\mathfrak{s}'}\delta_{l,l'}\mathbf{n}^p\psi_0(z^{m'-m})\\
&\quad= \delta_{\mathfrak{s},\mathfrak{s}'}\delta_{l,l'}\mathbf{n}^p f^{m'-m}.
\end{aligned}$$

On the other hand, repeatedly applying Corollary 4.10(iii), we have

$$\begin{aligned}
&\psi_p(\gamma_{\mathfrak{s}+n_{i_0}l+\mathbf{n}^p m})^*\psi_p(\gamma_{\mathfrak{s}'+n_{i_0}l'+\mathbf{n}^p m'})\\
&\quad\overset{(5.1)}{=} \mathbf{n}^p\Big(\mathbf{g}_0^p\Big)^*\Big(g_{x_0^{i_0}}^{p_{i_0}-1}\Big)^*\Big(f^{l+\frac{\mathbf{n}^p}{n_{i_0}}m}\Big)^*\Big(g_{x_{\mathfrak{s}}^{i_0}}\Big)^* g_{x_{\mathfrak{s}'}^{i_0}} f^{l'+\frac{\mathbf{n}^p}{n_{i_0}}m'} g_{x_0^{i_0}}^{p_{i_0}-1}\mathbf{g}_0^p\\
&\quad= \delta_{\mathfrak{s},\mathfrak{s}'}\mathbf{n}^p\Big(\mathbf{g}_0^p\Big)^*\Big(g_{x_0^{i_0}}^{p_{i_0}-1}\Big)^* f^{l'-l+\frac{\mathbf{n}^p}{n_{i_0}}(m'-m)} g_{x_0^{i_0}}^{p_{i_0}-1}\mathbf{g}_0^p \ (\text{as } g_{x_{\mathfrak{s}}^{i_0}}^* g_{x_{\mathfrak{s}'}^{i_0}} = \delta_{\mathfrak{s},\mathfrak{s}'})\\
&\quad= \delta_{\mathfrak{s},\mathfrak{s}'}\mathbf{n}^p\Big(\mathbf{g}_0^p\Big)^*\Big(g_{x_0^{i_0}}^{p_{i_0}-1}\Big)^* f^{l'-l} g_{x_0^{i_0}}^{p_{i_0}-1} f^{\check{\mathbf{n}}^{\check{p}}(m'-m)}\mathbf{g}_0^p\\
&\quad= \delta_{\mathfrak{s},\mathfrak{s}'}\mathbf{n}^p\Big(\mathbf{g}_0^p\Big)^*\Big(g_{x_0^{i_0}}^{p_{i_0}-1}\Big)^* f^{l'-l} g_{x_0^{i_0}}^{p_{i_0}-1}\mathbf{g}_0^p f^{m'-m},
\end{aligned}\tag{5.3}$$

where $\check{\mathbf{n}}^{\check{p}} := \prod_{i\neq i_0} n_i^{p_i} = \prod_{i=i_0+1}^{k} n_i^{p_i}$ as $p_i = 0$ for all $i < i_0$.

If $l = l'$, then it follows from (5.3) that

$$\psi_p(\gamma_{\mathfrak{s}+n_{i_0}l+\mathbf{n}^p m})^*\psi_p(\gamma_{\mathfrak{s}'+n_{i_0}l'+\mathbf{n}^p m'}) = \delta_{\mathfrak{s},\mathfrak{s}'}\mathbf{n}^p f^{m'-m}.$$

If $l \neq l'$, then repeatedly applying Corollary 4.10 to (5.3), we obtain

$$\psi_p(\gamma_{\mathfrak{s}+n_{i_0}l+\mathbf{n}^p m})^*\psi_p(\gamma_{\mathfrak{s}'+n_{i_0}l'+\mathbf{n}^p m'}) = 0.$$

Thus,

$$\psi_p(\gamma_{\mathfrak{s}+n_{i_0}l+\mathbf{n}^p m})^*\psi_p(\gamma_{\mathfrak{s}'+n_{i_0}l'+\mathbf{n}^p m'}) = \delta_{\mathfrak{s},\mathfrak{s}'}\delta_{l,l'}\mathbf{n}^p f^{m'-m}.$$

Therefore, we prove (5.2).

By Lemma 2.5, ψ_p can be uniquely extended to a bounded linear map on $X_p(\Lambda_{\mathbf{n}})$, which is still denoted by ψ_p. From above, we have shown that (ψ_p, ψ_0) is a representation of $X_p(\Lambda_{\mathbf{n}})$.

Step 2. We show that $\{(\psi_p, \psi_0) : p \in \mathbb{N}^k\}$ satisfies condition (T2) of Definition 2.2. Fix $p, q \in \mathbb{N}^k$. Let $i_0 := \min\{1 \le i \le k : p_i \neq 0\}$ and $i_0' := \min\{1 \le i \le k : q_i \neq 0\}$. Without loss of generality, let us assume that $i_0 \le i_0'$. Repeatedly applying Corollary

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4.10 yields

(5.4)
$$\begin{aligned}
&\psi_p(\gamma_{\mathfrak{s}+n_{i_0}l})\psi_q(\gamma_{\mathfrak{s}'+n_{i_0'}l'})\\
&\overset{(5.1)}{=} \mathbf{n}^{\frac{p+q}{2}} g_{x_{\mathfrak{s}}^{i_0}} f^{l}\big(g_{x_0^{i_0}}\big)^{p_{i_0}-1}\mathbf{g}_0^{p} g_{x_{\mathfrak{s}'}^{i_0'}} f^{l'}\big(g_{x_0^{i_0'}}\big)^{q_{i_0'}-1}\mathbf{g}_0^{q}\\
&= \mathbf{n}^{\frac{p+q}{2}} g_{x_{\mathfrak{s}}^{i_0}} f^{l}\big(g_{x_0^{i_0}}\big)^{p_{i_0}-1}\mathbf{g}_0^{p} f^{\mathfrak{s}'} g_{x_0^{i_0'}} f^{l'}\big(g_{x_0^{i_0'}}\big)^{q_{i_0'}-1}\mathbf{g}_0^{q}\\
&= \mathbf{n}^{\frac{p+q}{2}} g_{x_{\mathfrak{s}}^{i_0}} f^{l}\big(g_{x_0^{i_0}}\big)^{p_{i_0}-1} f^{(\prod_{i=i_0+1}^{k} n_i^{p_i})\mathfrak{s}'}\mathbf{g}_0^{p} g_{x_0^{i_0'}} f^{l'}\big(g_{x_0^{i_0'}}\big)^{q_{i_0'}-1}\mathbf{g}_0^{q}\\
&= \mathbf{n}^{\frac{p+q}{2}} g_{x_{\mathfrak{s}}^{i_0}} f^{l}\big(g_{x_0^{i_0}}\big)^{p_{i_0}-1} f^{(\prod_{i=i_0+1}^{k} n_i^{p_i})(\mathfrak{s}'+n_{i_0'}l')}\mathbf{g}_0^{p}\big(g_{x_0^{i_0'}}\big)^{q_{i_0'}}\mathbf{g}_0^{q}\\
&= \mathbf{n}^{\frac{p+q}{2}} g_{x_{\mathfrak{s}}^{i_0}} f^{l+(\prod_{i=i_0+1}^{k} n_i^{p_i})n_{i_0}^{p_{i_0}-1}(\mathfrak{s}'+n_{i_0'}l')}\big(g_{x_0^{i_0}}\big)^{p_{i_0}-1}\mathbf{g}_0^{p}\big(g_{x_0^{i_0'}}\big)^{q_{i_0'}}\mathbf{g}_0^{q}.
\end{aligned}$$

Notice that

$$\mathbf{g}_0^{p+q} = \begin{cases} \mathbf{g}_0^{p}\mathbf{g}_0^{q} & \text{if } i_0 = i_0',\\ \mathbf{g}_0^{p}\big(g_{x_0^{i_0'}}\big)^{q_{i_0'}}\mathbf{g}_0^{q} & \text{if } i_0 < i_0'. \end{cases}$$

Then continuing (5.4) gives

$$\begin{aligned}
&\psi_p(\gamma_{\mathfrak{s}+n_{i_0}l})\psi_q(\gamma_{\mathfrak{s}'+n_{i_0'}l'})\\
&= \begin{cases} \mathbf{n}^{\frac{p+q}{2}} g_{x_{\mathfrak{s}}^{i_0}} f^{l+(\prod_{i=i_0+1}^{k} n_i^{p_i})n_{i_0}^{p_{i_0}-1}(\mathfrak{s}'+n_{i_0'}l')}\big(g_{x_0^{i_0}}\big)^{p_{i_0}+q_{i_0}-1}\mathbf{g}_0^{p+q} & \text{if } i_0 = i_0',\\ \mathbf{n}^{\frac{p+q}{2}} g_{x_{\mathfrak{s}}^{i_0}} f^{l+\frac{\mathbf{n}^p}{n_{i_0}}(\mathfrak{s}'+n_{i_0'}l')}\big(g_{x_0^{i_0}}\big)^{p_{i_0}-1}\mathbf{g}_0^{p+q} & \text{if } i_0 < i_0', \end{cases}\\
&= \psi_{p+q}\Big(\gamma_{\mathfrak{s}+n_{i_0}\left(l+\frac{\mathbf{n}^p}{n_{i_0}}(\mathfrak{s}'+n_{i_0'}l')\right)}\Big)\\
&= \psi_{p+q}\big(\gamma_{\mathfrak{s}+n_{i_0}l}\diamond\gamma_{\mathfrak{s}'+n_{i_0'}l'}\big).
\end{aligned}$$

Here, the last "=" above holds true due to the following:

$$\begin{aligned}
\gamma_{\mathfrak{s}+n_{i_0}l}\diamond\gamma_{\mathfrak{s}'+n_{i_0'}l'}\big(z,p+q\big) &= \gamma_{\mathfrak{s}+n_{i_0}l}\diamond\gamma_{\mathfrak{s}'+n_{i_0'}l'}\big((z,p)(z^{\mathbf{n}^p},q)\big)\\
&= \gamma_{\mathfrak{s}+n_{i_0}l}\big((z,p)\big)\gamma_{\mathfrak{s}'+n_{i_0'}l'}\big((z^{\mathbf{n}^p},q)\big)\\
&= z^{\mathfrak{s}+n_{i_0}\left(l+\frac{\mathbf{n}^p}{n_{i_0}}(\mathfrak{s}'+n_{i_0'}l')\right)}\\
&= \gamma_{\mathfrak{s}+n_{i_0}\left(l+\frac{\mathbf{n}^p}{n_{i_0}}(\mathfrak{s}'+n_{i_0'}l')\right)}\big((z,p+q)\big).
\end{aligned}$$

Thus far, we have finished the proof of **Step 2**.

Therefore, by piecing the $\{\psi_p\}_{p\in\mathbb{N}^k}$ together we get a representation $\psi\colon X(\Lambda_{\mathbf{n}})\to\mathcal{Q}(\mathbb{F}_\theta^+\bowtie\mathbb{Z})$.

Step 3. We prove that ψ is Cuntz–Pimsner covariant. By Lemma 2.6, it suffices to show that (ψ_{e_i},ψ_0) $(1\le i\le k)$ are Cuntz–Pimsner covariant. Notice that a simple calculation shows that

$$\Big\langle\frac{\gamma^{m}}{\sqrt{\mathbf{n}^p}},\frac{\gamma^{m'}}{\sqrt{\mathbf{n}^p}}\Big\rangle = \begin{cases} \iota^{N} & \text{if } m'-m=\mathbf{n}^pN \text{ for } N\in\mathbb{Z},\\ 0 & \text{if } m'-m\notin\mathbf{n}^p\mathbb{Z}. \end{cases}$$

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Then one can obtain that

$$\phi_{e_i}(\iota) = \sum_{\mathfrak{s}\in[n_i]} \Theta_{\frac{\gamma_{\mathfrak{s}+1}}{\sqrt{n_i}},\frac{\gamma_{\mathfrak{s}}}{\sqrt{n_i}}}.$$

Hence

$$\begin{aligned}
\psi_{e_i}^{(1)}(\phi_{e_i}(\iota)) &= \sum_{\mathfrak{s}\in[n_i]} \psi_{e_i}\Big(\frac{\gamma_{\mathfrak{s}+1}}{\sqrt{n_i}}\Big)\psi_{e_i}\Big(\frac{\gamma_{\mathfrak{s}}}{\sqrt{n_i}}\Big)^* \\
&= \sum_{\mathfrak{s}\in[n_i]} \psi_0(\iota)\psi_{e_i}\Big(\frac{\gamma_{\mathfrak{s}}}{\sqrt{n_i}}\Big)\psi_{e_i}\Big(\frac{\gamma_{\mathfrak{s}}}{\sqrt{n_i}}\Big)^* \\
&= \sum_{\mathfrak{s}\in[n_i]} f g_{x_{\mathfrak{s}}^i} g_{x_{\mathfrak{s}}^i}^* \\
&= f \quad \text{(by Theorem 4.9(i))} \\
&= \psi_0(\iota).
\end{aligned}$$

By Lemma 2.5, (ψ_{e_i}, ψ_0) is Cuntz–Pimsner covariant for $1 \le i \le k$. ∎

The following theorem is inspired by [Kat08].

Theorem 5.4 *Let $\Lambda_{\mathbf{n}}$ be the topological k-graph constructed in Definition 5.1, and $X(\Lambda_{\mathbf{n}})$ be the product system associated to $\Lambda_{\mathbf{n}}$. Then $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ is isomorphic to $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$.*

Proof As before, denote by $\iota\colon \Lambda_{\mathbf{n}}^0 \to \mathbb{C}$ the embedding map. To simplify our writing, denote by $\jmath\colon X(\Lambda_{\mathbf{n}}) \to \mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ the universal Cuntz–Pimsner covariant representation of $X(\Lambda_{\mathbf{n}})$ satisfying that $\jmath$ generates $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$. Let $\psi\colon X(\Lambda_{\mathbf{n}}) \to \mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ be the Cuntz–Pimsner covariant representation constructed in the proof of Lemma 5.3. Then there exists a unital homomorphism $\varphi\colon \mathcal{O}_{X(\Lambda_{\mathbf{n}})} \to \mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ such that $\varphi \circ \jmath = \psi$.

Conversely, define

$$I := \jmath_0(1_{C(\Lambda_{\mathbf{n}}^0)}) \quad \text{and} \quad F := \jmath_0(\iota).$$

Then I is the identity of $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ and F is a unitary in $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$. For $1 \le i \le k$ and $\mathfrak{s} \in [n_i]$, let $\xi_{\mathfrak{s}}^i\colon \Lambda_{\mathbf{n}}^{e_i} \to \mathbb{C}$ be the function $\xi_{\mathfrak{s}}^i(z, e_i) := z^{\mathfrak{s}}/\sqrt{n_i}$ for all $z \in \mathbb{T}$, and define

$$G_{x_{\mathfrak{s}}^i} := \jmath_{e_i}(\xi_{\mathfrak{s}}^i);$$

For $(z, 0) \in \Lambda_{\mathbf{n}}^0$, we have

$$\begin{aligned}
\langle \xi_{\mathfrak{s}}^i, \xi_{\mathfrak{s}}^i \rangle_{C(\Lambda_{\mathbf{n}}^0)}(z, 0) &= \sum_{\{(w,e_i)\in\Lambda_{\mathbf{n}}^{e_i} : w^{n_i} = z\}} |\xi_{\mathfrak{s}}^i(w, e_i)|^2 \\
&= \sum_{\{(w,e_i)\in\Lambda_{\mathbf{n}}^{e_i} : w^{n_i} = z\}} \frac{1}{n_i} = 1.
\end{aligned}$$

So

$$G_{x_{\mathfrak{s}}^i}^* G_{x_{\mathfrak{s}}^i} = \jmath_0(\langle \xi_{\mathfrak{s}}^i, \xi_{\mathfrak{s}}^i \rangle) = I.$$

Hence $G_{x_{\mathfrak{s}}^i}$ is an isometry in $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$.

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For $x \in \mathrm{C}(\Lambda_{\mathbf{n}}^{e_i})$ and $(z, e_i) \in \Lambda_{\mathbf{n}}^{e_i}$, we have

$$\begin{aligned}\sum_{\mathfrak{s}\in[n_i]} \Theta_{\xi_{\mathfrak{s}}^i,\xi_{\mathfrak{s}}^i}(x)(z,e_i) &= \sum_{\mathfrak{s}\in[n_i]} \xi_{\mathfrak{s}}^i(z,e_i)\langle \xi_{\mathfrak{s}}^i, x\rangle_{\mathrm{C}(\Lambda_{\mathbf{n}}^0)}(z^n,0)\\ &= \sum_{\mathfrak{s}\in[n_i]} \frac{z^{\mathfrak{s}}}{\sqrt{n_i}}\Big(\sum_{\{w\in\mathbb{T}:w^{n_i}=z^{n_i}\}} \frac{\overline{w}^{\mathfrak{s}}}{\sqrt{n}} x(w,e_i)\Big)\\ &= \frac{1}{n_i}\sum_{\{w\in\mathbb{T}:w^{n_i}=z^{n_i}\}}\sum_{\mathfrak{s}\in[n_i]} z^{\mathfrak{s}}\overline{w}^{\mathfrak{s}} x(w,e_i)\\ &= x(z,e_i)\\ &= \phi_{e_i}(1_{\mathrm{C}(\Lambda_{\mathbf{n}}^0)})(x)(z,e_i),\end{aligned}$$

where the above 4th "=" holds true because $\sum_{\mathfrak{s}\in[n_i]} z^{\mathfrak{s}}\overline{w}^{\mathfrak{s}} = 0$ unless $w = z$. So

$$\sum_{\mathfrak{s}\in[n_i]} \Theta_{\xi_{\mathfrak{s}}^i,\xi_{\mathfrak{s}}^i} = \phi_{e_i}(1_{\mathrm{C}(\Lambda_{\mathbf{n}}^0)}).$$

Since $\jmath$ is Cuntz–Pimsner covariant, we obtain

$$\begin{aligned}\sum_{\mathfrak{s}\in[n_i]} G_{x_{\mathfrak{s}}^i} G_{x_{\mathfrak{s}}^i}^* &= \sum_{\mathfrak{s}\in[n_i]} \jmath_{e_i}(\xi_{\mathfrak{s}}^i)(\jmath_{e_i}(\xi_{\mathfrak{s}}^i))^* = \jmath_{e_i}^{(1)}\Big(\sum_{\mathfrak{s}\in[n_i]} \Theta_{\xi_{\mathfrak{s}}^i,\xi_{\mathfrak{s}}^i}\Big)\\ &= \jmath_{e_i}^{(1)}\Big(\phi_{e_i}(1_{\mathrm{C}(\Lambda_{\mathbf{n}}^0)})\Big) = \jmath_0(1_{\mathrm{C}(\Lambda_{\mathbf{n}}^0)}) = I.\end{aligned}$$

For $0 \le \mathfrak{s} < n_i - 1$ and $(z, e_i) \in \Lambda_{\mathbf{n}}^{e_i}$, we have

$$(\iota\cdot\xi_{\mathfrak{s}}^i)(z,e_i) = \frac{zz^{\mathfrak{s}}}{\sqrt{n_i}} = \frac{z^{\mathfrak{s}+1}}{\sqrt{n_i}} = \xi_{\mathfrak{s}+1}(z,e_i).$$

So

$$FG_{x_{\mathfrak{s}}^i} = G_{x_{\mathfrak{s}+1}^i}.$$

For $\mathfrak{s} = n_i - 1$, we compute that

$$(\iota\cdot\xi_{n_i-1}^i)(z,e_i) = \iota(z,0)\xi_{n_i-1}^i(z,e_i) = \frac{z^{n_i}}{\sqrt{n_i}} = \xi_0^i(z,e_i)\iota(z^{n_i},0) = (\xi_0^i\cdot\iota)(z,e_i).$$

So

$$FG_{x_{n_i-1}^i} = G_{x_0^i}F.$$

Observe that

$$(z, e_i + e_j) = (z,e_i)(z^{n_i},e_j) = (z,e_j)(z^{n_j},e_i)$$

for all $z \in \mathbb{T}$ and $1 \le i < j \le k$. Then for $\mathfrak{s},\mathfrak{s}' \in [n_i]$ and $\mathfrak{t},\mathfrak{t}' \in [n_j]$ satisfying that $\mathfrak{s} + \mathfrak{t}n_i = \mathfrak{t}' + \mathfrak{s}'n_j$, we have

$$\begin{aligned}(\xi_{\mathfrak{s}}^i \diamond \xi_{\mathfrak{t}}^j)(z,(e_i+e_j)) &= \xi_{\mathfrak{s}}^i(z,e_i)\xi_{\mathfrak{t}}^j(z^{n_i},e_j) = \frac{z^{\mathfrak{s}}}{\sqrt{n_i}}\frac{z^{\mathfrak{t}n_i}}{\sqrt{n_j}} = \frac{z^{\mathfrak{t}'}}{\sqrt{n_j}}\frac{z^{\mathfrak{s}'n_j}}{\sqrt{n_i}}\\ &= \xi_{\mathfrak{t}'}^j(z,e_j)\xi_{\mathfrak{s}'}^i(z^{n_j},e_i) = (\xi_{\mathfrak{t}'}^j \diamond \xi_{\mathfrak{s}'}^i)(z,(e_j+e_i)).\end{aligned}$$

So $\xi_{\mathfrak{s}}^i \diamond \xi_{\mathfrak{t}}^j = \xi_{\mathfrak{t}'}^j \diamond \xi_{\mathfrak{s}'}^i$. By condition (T2) of Definition 2.2, one has

$$G_{x_{\mathfrak{s}}^i} G_{x_{\mathfrak{t}}^j} = G_{x_{\mathfrak{t}'}^j} G_{x_{\mathfrak{s}'}^i}.$$

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Therefore, $\{F, G_{x_{\mathfrak{s}}^i} : \mathfrak{s} \in [n_i], 1 \le i \le k\}$ satisfy conditions (i)–(iii) of Theorem 4.9. By Theorem 4.9, there exists a unital homomorphism $\pi\colon \mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z}) \to \mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ such that $\pi(f) = F$ and $\pi(g_{x_{\mathfrak{s}}^i}) = G_{x_{\mathfrak{s}}^i}$ for all $\mathfrak{s} \in [n_i]$ and $1 \le i \le k$.

Since

$$\mathrm{C}^*\big(\jmath(X(\Lambda_{\mathbf{n}}))\big) = \mathrm{C}^*\Big(\big\{\jmath_0(X(\Lambda_{\mathbf{n}})_0), \jmath_{e_i}(X(\Lambda_{\mathbf{n}})_{e_i}) : 1 \le i \le k\big\}\Big),$$

it is straightforward to see that $\pi \circ \varphi = \mathrm{id}$, $\varphi \circ \pi = \mathrm{id}$. Therefore, we are done. ∎

5.2 Nuclearity, Simplicity, and Pure Infiniteness of $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$

In this subsection, we investigate the conditions under which $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ is nuclear, simple, and purely infinite.

It is necessary to recall the following definitions from [Yam09].

Definition 5.5 A topological k-graph Λ is said to satisfy *Condition* (A) if for any $v \in \Lambda^0$ and for any open neighborhood V of v, there exist $v' \in V$ and $\mu \in v'\Lambda^\infty$ such that $\sigma^p(\mu) \ne \sigma^q(\mu)$ whenever $p \ne q \in \mathbb{N}^k$.

Definition 5.6 Let Λ be a regular topological k-graph. For $v \in \Lambda^0$ and for $\mu \in v\Lambda^\infty$, denote by

$$\mathrm{Orb}^+(v) := r(s^{-1}(v)) \quad \text{and} \quad \mathrm{Orb}(v, \mu) := \bigcup_{n \in \mathbb{N}^k} \mathrm{Orb}^+(\mu(n, n)).$$

Definition 5.7 Let Λ be a regular topological k-graph, and let V be a nonempty precompact open subset of Λ^0. Then V is said to be *contracting* if there exist finitely many nonempty open subsets $U_i \subset \Lambda^{p_i}$, where $i = 1, \dots, l$, $p_i \in \mathbb{N}^k \setminus \{0\}$, such that

(i) $r(U_i) \subset V$ for all $1 \le i \le l$;
(ii) $\mu(0, p_i \wedge p_j) \ne \nu(0, p_i \wedge p_j)$ for all $1 \le i \ne j \le l$, $\mu \in U_i$, $\nu \in U_j$;
(iii) $\overline{V} \subsetneq \bigcup_{i=1}^l s(U_i)$.

Furthermore, Λ is said to be *contracting* if there exists $v \in \Lambda^0$ such that $\overline{\mathrm{Orb}^+(v)} = \Lambda^0$ and any open neighborhood of v contains an open contracting set.

Remark 5.8 In order to pursue the simplicity condition of $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$, we wish to apply [Yam09, Theorems 4.7]. However, it was pointed out by Nicolai Stammeier that there is a flaw in the proof of [Yam09, Theorems 4.7]. Fortunately, we are able to provide it an alternative proof when $1 \le k < \infty$ (see Theorem 5.9 and its proof) by invoking the work of Brown, Clark, Farthing, and Sims [BCFS14] and the work of Yeend [Yee07].

For this, we need to exploit the groupoid C*-algebra technique, which can be referred to [Ren80]. In the sequel, we give a very sketchy introduction to the boundary path groupoid arising from a regular topological k-graph (see [Yee07]).

Let $1 \le k < \infty$ and Λ be a regular topological k-graph. By recalling the construction of [Yee07], we get the set of *boundary path* $\partial\Lambda = \Lambda^\infty$, which is endowed with the topology generated by the basic open sets $Z(U) := \{x \in \Lambda^\infty : x(0, n) \in U\}$ where U is an open subset of Λ^n for some $n \in \mathbb{N}^k$. The *boundary path groupoid* $\mathcal{G}_\Lambda$ of Λ is defined by $\mathcal{G}_\Lambda = \{(x, p - q, y) \in \partial\Lambda \times \mathbb{Z}^k \times \partial\Lambda : \sigma^p(x) = \sigma^q(x)\}$, which is endowed with the

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topology generated by the basic open sets $Z(U,V) := \{(x, p-q, y) \in U \times \mathbb{Z}^k \times V : \sigma^p(x) = \sigma^q(x)\}$ where U and V are open in Λ^p and Λ^q, respectively.

Theorem 5.9 ([Yam09, Theorems 4.7 and 4.13]) *Let $1 \le k < \infty$. Let Λ be a regular topological k-graph. Suppose Λ satisfies Condition* (A) *and $\overline{\mathrm{Orb}(v,\mu)} = \Lambda^0$ for any $v \in \Lambda^0$ and $\mu \in v\Lambda^\infty$. Then $\mathcal{O}_{X(\Lambda)}$ is simple. Furthermore, suppose that Λ is contracting. Then $\mathcal{O}_{X(\Lambda)}$ is purely infinite.*

Proof [Yee07, Theorem 6.8] yields that $\mathcal{G}_\Lambda$ is amenable. So $\mathrm{C}^*(\mathcal{G}_\Lambda) = \mathrm{C}_r^*(\mathcal{G}_\Lambda)$ is nuclear. By [CLSV11, Theorem 5.20] and [SY10, Corollary 5.2], $\mathrm{C}^*(\mathcal{G}_\Lambda) \cong \mathcal{O}_{X(\Lambda)}$. So we must show that $\mathrm{C}^*(\mathcal{G}_\Lambda)$ is simple.

By [Yee07, Theorem 5.2], $\mathcal{G}_\Lambda$ is topologically principal. Let D be a non-empty open invariant subset of $\mathcal{G}_\Lambda^0$. Suppose $D \neq \mathcal{G}_\Lambda^0$, for a contradiction. Then there exists $x \in \mathcal{G}_\Lambda^0 \setminus D$, and so $\overline{\mathrm{Orb}(x(0,0),x)} = \Lambda^0$ by our assumption. Since D is open, take $n \in \mathbb{N}^k$ and a non-empty open subset U of Λ^n satisfying $Z(U) \subset D$. Then there exist $\mu, \nu \in \Lambda, m \in \mathbb{N}^k$ such that $\mu \in U, s(\mu) = r(\nu), s(\nu) = x(m,m)$. So $y := \mu\nu\sigma^m(x) \in D$ and $(x, m-(n+d(\nu)), y) \in \mathcal{G}_\Lambda$. Since D is invariant, one has $x \in D$. This is a contradiction. Hence, $D = \mathcal{G}_\Lambda^0$. Therefore, $\mathcal{G}_\Lambda$ is minimal. By [BCFS14, Theorem 5.1], $\mathrm{C}^*(\mathcal{G}_\Lambda)$ is simple. ∎

One of the referees kindly informed us that Theorem 5.9 can be also obtained from [RSWY09, Theorem 5.3 and Proposition 5.8].

Before giving our main results, we need two lemmas.

Lemma 5.10 *Let $\Lambda_{\mathbf{n}}$ be the topological k-graph constructed in Definition 5.1. If $\{\ln n_i\}_{1\le i\le k}$ is rationally independent, then $\Lambda_{\mathbf{n}}$ satisfies Condition* (A).

Proof Since $\{\ln n_i\}_{1\le i\le k}$ is rationally independent, we have $\mathbf{n}^p \neq \mathbf{n}^q$ for all $p \neq q \in \mathbb{N}^k$. Fix $(z,0) \in \Lambda_{\mathbf{n}}^0$ and an open neighborhood V of z. Pick up $w \in V$ such that $w = e^{2\pi i\theta}$ with $\theta \in (0,1) \setminus \mathbb{Q}$. Notice that, for any $l_1, l_2 \in \mathbb{Z}, w^{l_1} = w^{l_2}$ if and only if $l_1 = l_2$. Let μ be the unique infinite path in $(w,0)\Lambda_{\mathbf{n}}^\infty$ such that $\mu(p,q) = (w^{\mathbf{n}^p}, q-p)$ for all $p \le q \in \mathbb{N}^k$. For $p \neq q \in \mathbb{N}^k$, since $\mathbf{n}^p \neq \mathbf{n}^q$, we have $\sigma^p(\mu)(0,0) \neq \sigma^q(\mu)(0,0)$ and so $\sigma^p(\mu) \neq \sigma^q(\mu)$. Therefore, $\Lambda_{\mathbf{n}}$ satisfies Condition (A). ∎

Lemma 5.11 *Let $\mathbb{F}_\theta^+ \rtimes \mathbb{Z}$ and $\mathbb{F}_\alpha^+ \rtimes \mathbb{Z}$ be two standard products of the odometers $\{(\mathbb{Z}, [n_i])\}_{i=1}^{\mathfrak{k}}$ and $\{(\mathbb{Z}, [m_j]\}_{j=1}^{\ell}$, respectively. Suppose that $1 \le \mathfrak{k} \le \ell \le \infty$ and that $n_i = m_i$ for all $i = 1, \dots, \mathfrak{k}$. Then there is a unital embedding from $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ into $\mathcal{O}_{X(\Lambda_{\mathbf{m}})}$. Hence there exists a unital embedding from $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ into $\mathcal{Q}(\mathbb{F}_\alpha^+ \rtimes \mathbb{Z})$.*

Proof Denote by $\imath: X(\Lambda_{\mathbf{n}}) \to \mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ and $\jmath: X(\Lambda_{\mathbf{m}}) \to \mathcal{O}_{X(\Lambda_{\mathbf{m}})}$ the universal Cuntz–Pimsner covariant representations of $X(\Lambda_{\mathbf{n}})$ and $X(\Lambda_{\mathbf{m}})$, respectively. We realize $\mathbb{N}^{\mathfrak{k}}$ as a subsemigroup of $\mathbb{N}^\ell$ by $p \mapsto (p,0)$. For $p \in \mathbb{N}^{\mathfrak{k}}$, we also realize $X(\Lambda_{\mathbf{n}})_p$ as $X(\Lambda_{\mathbf{m}})_p$ as they are isomorphic as C*-correspondences over $\mathrm{C}(\mathbb{T})$. For $p \in \mathbb{N}^{\mathfrak{k}}$, define $\psi_p: X(\Lambda_{\mathbf{m}}) \to \mathcal{O}_{X(\Lambda_{\mathbf{m}})}$ to be $\jmath_p$. By piecing $\{\psi_p\}_{p\in\mathbb{N}^{\mathfrak{k}}}$ together, one obtains a Cuntz–Pimsner covariant representation of $X(\Lambda_{\mathbf{n}})$. Let $h: \mathcal{O}_{X(\Lambda_{\mathbf{n}})} \to \mathcal{O}_{X(\Lambda_{\mathbf{m}})}$ be the unital homomorphism induced from the universal property of $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$. By [SY10,

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Theorem 4.1, Corollary 5.2], $h|_{\iota_0(X(\Lambda_{\mathbf{n}}))}$ is injective. Let γ be the gauge action for $\jmath$. Then by restriction, γ induces a gauge action for ψ. Invoking Theorem 2.4 yields that h is injective. The second statement follows from Theorem 5.4. ∎

Theorem 5.12 *Let $\Lambda_{\mathbf{n}}$ be the topological k-graph constructed in Definition 5.1.*

(i) $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ *is nuclear and satisfies the UCT.*
(ii) *Then $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is simple if and only if $\{\ln n_i\}_{1\le i\le k}$ is rationally independent.*
(iii) *If $\{\ln n_i\}_{1\le i\le k}$ is rationally independent, then $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is purely infinite.*
(iv) $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ *is a unital UCT Kirchberg algebra if and only if $\{\ln n_i\}_{1\le i\le k}$ is rationally independent.*

Proof (i) First of all, suppose $k \neq \infty$. From the proof of Theorem 5.9, we obtain that $\mathcal{G}_{\Lambda_{\mathbf{n}}}$ is amenable and $\mathrm{C}^*(\mathcal{G}_{\Lambda_{\mathbf{n}}}) \cong \mathcal{O}_{X(\Lambda_{\mathbf{n}})}$. Therefore, $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is nuclear and also satisfies the UCT due to [Tu99].

Now suppose that $k = \infty$. By Lemma 5.11, we obtain an increasing sequence $\{\mathcal{A}_i := \mathcal{O}_{X(\Lambda_{(n_1,\dots,n_i)})}\}_{i=1}^{\infty}$ of unital C*-subalgebras of $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ such that $\bigcup_{i=1}^{\infty}\mathcal{A}_i$ is dense in $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$. Since each $\mathcal{A}_i$ is nuclear by the preceding paragraph, we deduce that $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is nuclear and satisfies the UCT.

(ii) The proof of "If": Suppose that $k \neq \infty$. By Lemma 5.10, $\Lambda_{\mathbf{n}}$ satisfies Condition (A).

Fix $(z,0) \in \Lambda_{\mathbf{n}}^0$, and let μ be the unique infinite path in $(z,0)\Lambda_{\mathbf{n}}^{\infty}$. For $p \in \mathbb{N}^k$ and $w \in \mathbb{T}$ such that $w^{\mathbf{n}^p} = z$, we have $r(w,p) = (w,0), s(w,p) = (w^{\mathbf{n}^p},0) = (z,0)$. So $(w,0) \in \mathrm{Orb}^+((z,0))$. Let $(z',0) \in \Lambda_{\mathbf{n}}^0$ and $\epsilon > 0$. Then we can always find $p \in \mathbb{N}^k$ with $\mathbf{n}^p$ large enough so that the distance between z' and one of $\mathbf{n}^p$-th roots of z is less than ϵ. Hence $\mathrm{Orb}^+((z,0))$ is dense in $\Lambda_{\mathbf{n}}^0$. Since $\mathrm{Orb}^+((z,0)) \subset \mathrm{Orb}((z,0),\mu)$, clearly $\mathrm{Orb}((z,0),\mu)$ is dense in $\Lambda_{\mathbf{n}}^0$ as well. Therefore by Theorem 5.9, $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is simple.

Now suppose that $k = \infty$. By Lemma 5.11, we obtain an increasing sequence $\{\mathcal{A}_i := \mathcal{O}_{X(\Lambda_{(n_1,\dots,n_i)})}\}_{i=1}^{\infty}$ of unital C*-subalgebras of $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ such that $\bigcup_{i=1}^{\infty}\mathcal{A}_i$ is dense in $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$. Since each $\mathcal{A}_i$ is simple by the above argument, we deduce that $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is simple.

The proof of "Only if": We must show that the rational dependence of $\{\ln n_i\}_{1\le i\le k}$ implies that $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is not simple; equivalently, $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ is not simple by Theorem 5.4. Now suppose that $\{\ln n_i\}_{1\le i\le k}$ is rationally dependent. Then there exist $p \neq q \in \mathbb{N}^k$ such that $\mathbf{n}^p = \mathbf{n}^q$. Let $A := \{1 \le i \le k : p_i < q_i\}$ and $B := \{1 \le i \le k : p_i > q_i\}$. We can assume that $A \neq \varnothing$. Then $\prod_{i\in A} n_i^{q_i-p_i} = \prod_{i\in B} n_i^{p_i-q_i}$. Inspired by [Cun08], in what follows, we construct a representation of $\mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z})$ on $\ell^2(\mathbb{Z})$. To this end, let $\{\delta_m\}_{m\in\mathbb{Z}}$ denote the standard orthonormal basis of $\ell^2(\mathbb{Z})$. Define

$$F(\delta_m) := \delta_{m+1}\ (m \in \mathbb{Z}),$$
$$G_{x_{\mathfrak{s}}^i}(\delta_m) := \delta_{\mathfrak{s}+n_i m}\ (m \in \mathbb{Z}, \mathfrak{s} \in [n_i], 1 \le i \le k).$$

Then F is a unitary and $G_{x_{\mathfrak{s}}^i}$s are isometries. Some calculations show that $\{F, G_{x_{\mathfrak{s}}^i} : \mathfrak{s} \in [n_i], 1 \le i \le k\}$ satisfy conditions (i)–(iii) of Theorem 4.9. By Theorem 4.9, there exists a nonzero homomorphism $\pi\colon \mathcal{Q}(\mathbb{F}_\theta^+ \rtimes \mathbb{Z}) \to B(\ell^2(\mathbb{Z}))$ such that $\pi(f) = F$, $\pi(g_{x_{\mathfrak{s}}^i}) = G_{x_{\mathfrak{s}}^i}$ for all $\mathfrak{s} \in [n_i], 1 \le i \le k$. Since $\prod_{i\in A} n_i^{q_i-p_i} = \prod_{i\in B} n_i^{p_i-q_i}$,

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one has $\prod_{i\in A} G_{x_0^i}^{q_i-p_i} = \prod_{i\in B} G_{x_0^i}^{p_i-q_i}$. Suppose that $\prod_{i\in A} g_{x_0^i}^{q_i-p_i} = \prod_{i\in B} g_{x_0^i}^{p_i-q_i}$ for a contradiction. Let $\gamma\colon \prod_{i=1}^k \mathbb{T} \to \mathrm{Aut}(\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z}))$ be the gauge action induced from the universal property of $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$. By Theorem 5.4,

$$0 = \gamma_z\Big(\prod_{i\in A} g_{x_0^i}^{q_i-p_i} - \prod_{i\in B} g_{x_0^i}^{p_i-q_i}\Big) = \prod_{i\in A} z_i^{q_i-p_i} \prod_{i\in A} g_{x_0^i}^{q_i-p_i} - \prod_{i\in B} g_{x_0^i}^{p_i-q_i}$$

for all $z \in \prod_{i=1}^k \mathbb{T}$ such that $z_i = 1$ whenever $i \in B$. Since $p_i < q_i$ for $i \in A \neq \emptyset$, we deduce that $\prod_{i\in A} g_{x_0^i}^{q_i-p_i} = 0$, which is impossible. So

$$0 \neq \prod_{i\in A} g_{x_0^i}^{q_i-p_i} - \prod_{i\in B} g_{x_0^i}^{p_i-q_i} \in \ker(\pi).$$

Thus, $\ker(\pi)$ is a nontrivial closed two-sided ideal in $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$, implying that $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ is not simple.

(iii) Suppose that $k < \infty$. Then $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is simple from (ii). As shown in (ii), $\mathrm{Orb}^+((1,0))$ is dense in $\Lambda_{\mathbf{n}}^0$. Fix an open neighborhood U of 1. Then there exists $\delta > 0$ such that $n_1\delta$ is small enough (say $< 1/4$) and $V := \{e^{2\pi i\theta} : \theta \in (-\delta, \delta)\} \subset U$. Denote by $U_1 := V \times \{e_1\}$. It is straightforward to see that $r(U_1) \subset V \times \{0\}$ and $\overline{V \times \{0\}} \subsetneq s(U_1)$. So $V \times \{0\}$ is contracting. Hence, $\Lambda_{\mathbf{n}}$ is contracting. By Theorem 5.9, $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is purely infinite.

Now suppose that $k = \infty$. By Lemma 5.11, we obtain an increasing sequence $\{\mathcal{A}_i := \mathcal{O}_{X(\Lambda_{(n_1,\dots,n_i)})}\}_{i=1}^\infty$ of unital C*-subalgebras of $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ such that $\bigcup_{i=1}^\infty \mathcal{A}_i$ is dense in $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$. Since each $\mathcal{A}_i$ is simple and purely infinite by the above paragraph, we deduce that $\mathcal{O}_{X(\Lambda_{\mathbf{n}})}$ is simple and purely infinite.

(iv) This now easily follows from (i)–(iii). ∎

As an immediate consequence of Theorems 5.4 and 5.12, one has the following theorem.

Theorem 5.13 *Let $\mathbb{F}_\theta^+ \bowtie \mathbb{Z}$ be the standard product of odometers $\{(\mathbb{Z}, [n_i])\}_{=1}^k$. Then*

(i) *$\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ is nuclear;*

(ii) *$\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ is a unital UCT Kirchberg algebra $\Leftrightarrow$ $\{\ln n_i\}_{1\le i\le k}$ is rationally independent $\Leftrightarrow$ $\mathcal{O}_\theta$ is simple $\Leftrightarrow$ $\mathbb{F}_\theta^+$ is aperiodic.*

Proof By Theorems 5.4 and 5.12, it remains to show that $\mathbb{F}_\theta^+$ is aperiodic $\Leftrightarrow$ $\mathcal{O}_\theta$ is simple $\Leftrightarrow$ $\{\ln n_i\}_{1\le i\le k}$ is rationally independent.

$\mathbb{F}_\theta^+$ is aperiodic $\Rightarrow$ $\mathcal{O}_\theta$ is simple: If $k \neq \infty$, then this follows from [DY09$_2$, Corollary 8.6]. If $k = \infty$, then there is an increasing sequence $\{\mathcal{A}_i\}_{i=1}^\infty$ of C*-subalgebras of $\mathcal{O}_\theta$ such that each $\mathcal{A}_i$ is the C*-algebra of an aperiodic single-vertex finite-rank graph and the union of $\{\mathcal{A}_i\}_{i=1}^\infty$ is dense in $\mathcal{O}_\theta$. So $\mathcal{O}_\theta$ is simple.

$\mathcal{O}_\theta$ is simple $\Rightarrow$ $\{\ln n_i\}_{1\le i\le k}$ is rationally independent: We prove its contraposition. Suppose that $\{\ln n_i\}_{1\le i\le k}$ is rationally dependent. Notice that $\{g_{x_{\mathfrak{s}}^i} : \mathfrak{s} \in [n_i], 1 \le i \le k\}$ is a Cuntz-Krieger $\mathbb{F}_\theta^+$-family in $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$. Then there is a homomorphism $\rho\colon \mathcal{O}_\theta \to \mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ induced from the universal property of $\mathcal{O}_\theta$. Let $\pi\colon \mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z}) \to B(\ell^2(\mathbb{Z}))$ be the nonzero homomorphism given in the proof of Theorem 5.12. Since the kernel of $\pi \circ \rho$ is a nontrivial closed two-sided ideal of $\mathcal{O}_\theta$, $\mathcal{O}_\theta$ is not simple.

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$\{\ln n_i\}_{1\le i\le k}$ is rationally independent $\Rightarrow \mathbb{F}_\theta^+$ is aperiodic: If $k \neq \infty$, then this follows from [DY09$_2$, Theorem 3.4, Definition 3.6 and Theorem 7.1]. If $k = \infty$ and if $\mathbb{F}_\theta^+$ is periodic, then there exists $1 \le l < \infty$ such that the l-graph $\mathbb{F}_\alpha^+$ determined by $\{n_i\}_{i=1}^l$ is periodic as well ([Yan15]). So $\{\ln n_i\}_{1\le i\le l}$ is rationally dependent. Hence, $\{\ln n_i\}_{1\le i\le k}$ is also rationally dependent. Therefore, we are done. ∎

As an immediate consequence of Theorem 5.13, the boundary quotient C*-algebra left in [BRRW14] is now well understood.

Corollary 5.14 *Let $\mathbb{F}_\theta^+ \bowtie \mathbb{Z}$ be the standard product of 2-odometers over n-letter and m-letter alphabets with $\gcd(n,m) = 1$. Then $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ is a unital UCT Kirchberg algebra.*

Remark 5.15 (1) By Theorem 5.13, when $\{\ln n_i\}_{1\le i\le k}$ is rationally independent, the C*-algebras $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ are, due to the celebrated Kirchberg–Phillips classification ([Phi00]), classifiable by K-theory.

(2) When $\{n_i\}_{i=1}^k \subset \mathbb{N}\setminus\{0,1\}$ is a pairwise coprime set, it is also shown in [KOQ14, Stam15], by different approaches, that $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ is a unital UCT Kirchberg algebra. In this case, Barlak–Omland–Stammeier in [BOS15] investigated the K-theory of $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ and obtained a complete classification for $k \le 2$.

As an extreme case, let $k = 1$ in Theorem 5.13. Then we obtain that the boundary quotient C*-algebra $\mathcal{Q}(\mathbb{F}_n^+ \bowtie \mathbb{Z})$ of the odometer action on a n-letter alphabet with $n \ge 2$ is nuclear, simple, and purely infinite. Recall from Theorem 4.9 that $\mathcal{Q}(\mathbb{F}_n^+ \bowtie \mathbb{Z})$ is the universal C*-algebra generated by a unitary f and n isometries g_{x_i} $(i \in [n])$ such that

(a) $$\sum_{i\in[n]} g_{x_i} g_{x_i}^* = 1,$$

(b) $$f g_{x_i} = \begin{cases} g_{x_{i+1}} & \text{if } 0 \le i < n-1, \\ g_{x_0} f & \text{if } i = n-1. \end{cases}$$

Also, given $n \ge 2$, the *n-adic ring C*-algebra $\mathcal{Q}_n$ of the integers* is the universal C*-algebra generated by a unitary u and an isometry s satisfying

(5.5) $$\sum_{i\in[n]} u^i s (u^i s)^* = 1 \quad \text{and} \quad u^n s = su.$$

Corollary 5.16 *There is an isomorphism $\pi\colon \mathcal{Q}(\mathbb{F}_n^+ \bowtie \mathbb{Z}) \to \mathcal{Q}_n$ such that $\pi(f) = u$ and $\pi(g_{x_i}) = u^i s$ for all $i \in [n]$.*

Proof From (5.5) it is easy to check that $\{\pi(f), \pi(g_{x_i}) : i \in [n]\}$ satisfies conditions (a) and (b) above. So by the universal property of $\mathcal{Q}(\mathbb{F}_n^+ \bowtie \mathbb{Z})$, π can be extended to isomorphism as $\mathcal{Q}(\mathbb{F}_n^+ \bowtie \mathbb{Z})$ is simple by Theorem 5.13. ∎

It turns out that $\mathcal{Q}_n$ is isomorphic to the graph C*-algebra $\mathcal{O}(E_{n,1})$ of the topological graph $E_{n,1}$ of Katsura studied in [Kat08], where it is shown that $\mathcal{Q}_n$ is nuclear, simple, and purely infinite. So we recover this result here. Also let us remark that $\mathcal{Q}_2$ was systematically studied by Larsen–Li in [LL12].

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Example 5.17 Consider the standard product $\mathbb{F}_\theta^+ \bowtie \mathbb{Z}$ of the odometers $(\mathbb{Z}, [n_i])$ with $n_i = n$ for all $1 \le i \le k < \infty$. To make our example more interesting, let $n \ge 2$ and $k \ge 2$. This is a special case of Example 4.3 with $G = \mathbb{Z}$ and the self-similar action being the odometer. By Theorem 5.13, $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ is not simple.

5.3 Relations Between $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ and $\mathcal{Q}_\mathbb{N}$

Cuntz [Cun08, Definition 3.1] defined $\mathcal{Q}_\mathbb{N}$ to be the universal C*-algebra generated by a unitary u and a family of isometries $\{s_n\}_{n\in\mathbb{N}^\times}$ satisfying

$$s_n s_m = s_{nm}, \quad u^n s_n = s_n u, \quad \sum_{t=0}^{n-1} u^t s_n s_n^* u^{-t} = 1 \quad \text{for all} \quad n, m \in \mathbb{N}^\times.$$

In what follows, we discuss some relations between $\mathcal{Q}_\mathbb{N}$ and the boundary quotient C*-algebra $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ of the standard product of odometers $\{(\mathbb{Z}, [n_i])\}_{i=1}^k$. For this, define

$$F := u, \quad G_{x_\mathfrak{t}^i} := u^\mathfrak{t} s_{n_i} \quad (\mathfrak{t} \in [n_i],\ 1 \le i \le k).$$

A simple calculation shows that $\{F, G_{x_\mathfrak{t}^i} : \mathfrak{t} \in [n_i], 1 \le i \le k\}$ satisfy conditions (i)–(iii) of Theorem 4.9. By Theorem 4.9, there exists a homomorphism

$$\rho: \mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z}) \to \mathcal{Q}_\mathbb{N} \tag{5.6}$$

such that $\rho(f) = F$ and $\rho(g_{x_\mathfrak{t}^i}) = G_{x_\mathfrak{t}^i}$ for all $\mathfrak{t} \in [n_i], 1 \le i \le k$.

If $k = \infty$ and $\{n_i\}_{i=1}^\infty$ is the set of all prime numbers, then ρ is an isomorphism by Theorem 5.13. Thus, one has the following corollary.

Corollary 5.18 *If $k = \infty$ and $\{n_i\}_{i=1}^\infty$ is the set of all prime numbers, then $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z}) \cong \mathcal{Q}_\mathbb{N}$.*

Let us finish this paper by characterizing when the above homomorphism ρ is injective.

Theorem 5.19 *The homomorphism ρ in (5.6) is injective if and only if $\{\ln n_i\}_{1\le i\le k}$ is rationally independent.*

Proof If $\{\ln n_i\}_{1\le i\le k}$ is rationally independent, then $\mathcal{Q}(\mathbb{F}_\theta^+ \bowtie \mathbb{Z})$ is simple by Theorem 5.13. So ρ is injective.

Conversely, suppose that $\{\ln n_i\}_{1\le i\le k}$ is rationally dependent. Then there exist $p \ne q \in \mathbb{N}^k$ such that $\mathbf{n}^p = \mathbf{n}^q$. It is straightforward to see that

$$\rho\Big(\prod_{i=1}^k g_{x_0^i}^{p_i}\Big) = \rho\Big(\prod_{i=1}^k g_{x_0^i}^{q_i}\Big).$$

Hence, ρ is not injective. ∎

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references [ABLS16, BOS15, Stam16] to our attention. We are also very grateful to the referees for their careful reading and valuable comments/suggestions.

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