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NUMBER FIELDS WITH DISCRIMINANT $\pm 2^a 3^b$ AND GALOIS GROUP A_n OR S_n

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Abstract

The authors present three-point and four-point covers having bad reduction at 2 and 3 only, with Galois group A_n or S_n for n equal to 9, 10, 12, 18, 28, and 33. By specializing these covers, they obtain number fields ramified at 2 and 3 only, with Galois group A_n or S_n for n equal to 9, 10, 11, 12, 17, 18, 25, 28, 30, and 33.

1. *Introduction*

1.1. *An inverse problem*

The most standard inverse Galois problem asks for a Galois number field $K \subset \mathbb{C}$ with Galois group $\mathrm{Gal}(K/\mathbb{Q})$ isomorphic to a given finite group G. This problem is easy to solve for the symmetric groups S_n, and not too much harder for the alternating groups A_n. For Lie-type groups, the problem is much harder, but it has been solved for many groups by various techniques; see, for example, [8].

The deepest results on number fields pay close attention to how primes ramify. It is therefore natural to modify the standard inverse Galois problem to ask for number fields K which not only have Galois group G, but which also have discriminant divisible by only primes in a given non-empty finite set S. In this setting, symmetric groups and alternating groups suddenly become problematic. In particular, known constructional techniques based on étale cohomology or automorphic forms apply mainly only to Lie-type groups.

More precisely, let $\mathrm{NF}(S, n)$ be the set of Galois number fields $K \subset \mathbb{C}$ with $\mathrm{Gal}(K/\mathbb{Q})$ either A_n or S_n, and all prime factors of the discriminant in S. On the one hand, current constructional techniques have only moderate control over ramifying primes, and it seems quite possible that $\mathrm{NF}(S) := \coprod \mathrm{NF}(S, n)$ would be finite for all S. On the other hand, if we fix two distinct primes p and ℓ, and put $S = \{p, \ell\}$, then one can prove, using modular forms, that there are infinitely many fields with G of the form $\mathrm{PGL}_2(\mathbb{F}_{\ell^n})$ and ramification set within S. From the ABC construction of [10], one can similarly expect that for many such $S = \{p, \ell\}$, there would be infinitely many fields with, say, Galois group of the form $\mathrm{PSp}_{2n}(\mathbb{F}_\ell)$ and ramification within S. Given these analogs, it also seems quite possible that $\mathrm{NF}(S)$ is infinite, at least for S sufficiently large. Thus the nature of the sets $\mathrm{NF}(S, n)$ for S fixed and n increasing is completely unknown.

1.2. *Overview of this paper*

The present work explores this situation computationally, for A_n and S_n fields. As in [3], [4], and [10], we focus exclusively on $S = \{2, 3\}$, so as to keep things manageable.

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Previously, only 152 fields in degrees $1 \leqslant n \leqslant 9$ were known, plus an additional S_{32} field given in [10]. The present paper goes substantially beyond the previous work, as it constructs 23 more fields in degree 9, and then 277 fields in degrees 10 through 33.

As in [10], our main method of constructing number fields is to specialize three-point covers $F : X \longrightarrow \mathbb{P}^1_t$, with t indicating the coordinate on the base curve. The covers that we are concerned with in this paper satisfy the following three logically independent conditions.

$$\text{The monodromy group of } F \text{ is } A_n \text{ or } S_n. \tag{1.1}$$

$$F \text{ is defined over } \mathbb{Q}. \tag{1.2}$$

$$\text{The bad reduction set of } F \text{ is contained in } \{2, 3\}. \tag{1.3}$$

For the purposes of this paper, we say that a three-point cover is 'good' if and only if it satisfies Conditions (1.1)–(1.3). Degrees $n \leqslant 4$ are rather trivial, as Condition (1.3) is then automatic. In degrees at least 5, only two good covers were previously known, denoted here as $6a$ and $9c$. This paper presents nine more good covers, denoted $9d$, $10c$, $12a$, $12b$, $12c$, $12d$, $18a$, $28c$, and $33a$ (the labels $9a$, $9b$, $10a$, $10b$, $28a$, and $28b$ are used in [10] for covers with Lie-type Galois groups). For all eleven A_n or S_n covers, X has genus zero and, moreover, has a rational point. We give each cover in the form $F : \mathbb{P}^1_x \longrightarrow \mathbb{P}^1_t$ by giving t as a rational function of x.

With regard to the cardinality of the sets $\mathrm{NF}(S)$, our work here still leaves the matter wide open. On the one hand, the known part of $\mathrm{NF}(\{2, 3\})$ has roughly tripled in size. On the other hand, our techniques are not general. In discussing the new covers, we treat a number of their aspects in some detail, in the hope that our descriptions might facilitate further progress on the size of $\mathrm{NF}(S, n)$.

1.3. *Contents of the sections*

Section 2 provides some background material on three-point covers, in an effort to make this paper reasonably self-contained. Readers familiar with the theory of three-point covers need only skim over Section 2 to acquaint themselves with our notational conventions. Section 3 is also a background section. It presents Covers $6a$ and $9c$, and explains how Katz's theory of rigid local systems forces them to have bad reduction set exactly $\{2, 3\}$. In contrast, we do not have a theoretical reason to explain why the new covers of this paper have bad reduction set $\{2, 3\}$.

Section 4 presents Covers $9d$ and $10c$. It also explains how our computations have established that $6a$, $9c$, $9d$, and $10c$ are the only genus-zero good covers in degrees $5 \leqslant n \leqslant 11$. Section 5 presents Covers $12a$, $12b$, $12c$, and $12d$. It also explains how $12a$ is naturally a special case of a one-parameter family $12A$ of four-point covers, while $12b$, $12c$, and $12d$ are special cases of another one-parameter family $12BCD$ of four-point covers. Family $12A$ also includes $6a$ and $9c$ as degenerate special cases. Section 6 presents Cover 18a. This is the only one of our eleven three-point covers that appears in isolation, as the others are naturally grouped: ($6a$, $9c$, $12a$), ($12b$, $12c$, $12d$), and ($9d$, $10c$, $28c$, $33a$). Section 7 presents Covers $28c$ and $33a$, and explains how they are members of a discretely indexed family, also containing $9d$ and $10c$. All four covers are doubly exceptional members of this family, in the sense that usually neither Condition (1.2) nor Condition (1.3) is satisfied.

Section 8 gives alternative defining polynomials for most of the covers considered in Sections 3–7. The polynomials given in these previous sections are the natural ones from the point of view of the theory of three-point covers, while the new polynomials have fewer terms.

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Finally, Section 9 describes the fields obtained by specializing the new covers, and shows how our new results extend the results of [3], [4], and [10]. Cover $18a$ yields 82 new fields, while the other eight new three-point covers yield 23 to 36 fields each. Also, $12A$ yields 14 fields beyond those from $12a$, while $12BCD$ yields 1 field beyond those from $12b$, $12c$, and $12d$. The absolute discriminants $2^a 3^b$ of all the known fields with $n \geqslant 9$ satisfy $a \geqslant n$ and $b \geqslant n$. In particular, as was the case in degrees $4 \leqslant n \leqslant 8$, they are all wildly ramified at both 2 and 3.

2. *Background*

2.1. *Partition triples*

In the study of three-point covers, it is natural to consider ordered triples

$$\Lambda = (\lambda_0, \lambda_1, \lambda_\infty) \tag{2.1}$$

of partitions of n, having altogether $n + 2 - 2g$ parts, for g a non-negative integer. All eleven of our examples have $g = 0$. Also, all eleven of our examples have a singleton in λ_∞, meaning a part d that occurs only once. To keep the setting simple, we discuss three-point covers only in this restricted context. The first restriction means that all the covering curves X that we consider will have genus zero. The second means that each of these curves X will, moreover, be isomorphic to the projective line, rather than to a conic in the projective plane without rational points.

2.2. *ABC equation*

An *ABC triple* of degree n over a field $k \subseteq \mathbb{C}$ is a triple $(A(x), B(x), C(x))$ of polynomials in $k[x]$ with $A(x)$ and $B(x)$ of degree n, $C(x)$ of degree $n - d < n$,

$$A(x) + B(x) + C(x) = 0, \tag{2.2}$$

and $A(x)B(x)C(x)$ having exactly $n + 1$ roots in $\mathbb{C}$, not counting multiplicity. The ramification invariant of $(A(x), B(x), C(x))$ is $\Lambda = (\lambda_0, \lambda_1, \lambda_\infty)$, where λ_0, λ_1, and λ_∞ give the multiplicities of the roots of $A(x)$, $B(x)$, and $C(x)$ respectively; here, to account for the degree-drop d of $C(x)$, we consider ∞ as a root of $C(x)$ with multiplicity d. Henceforth, we consider only degree-n *ABC* triples where d is a singleton of λ_∞. Then (Λ, d) is as in the preceding paragraph.

The three-point covers that we are considering in this paper are rational functions

$$F : \mathbb{P}^1_x \longrightarrow \mathbb{P}^1_t$$

arising from *ABC* triples via $F(x) = -A(x)/C(x)$. On the level of function fields over a ground field k, one has $k(\mathbb{P}^1_x) = k(x)$ and $k(\mathbb{P}^1_t) = k(t)$. The equation

$$-\frac{A(x)}{C(x)} = t \tag{2.3}$$

gives the inclusion of $k(t)$ into $k(x)$. There is an algorithm for computing *ABC* triples by means of solving non-linear algebraic equations. The algorithm is described with examples in [8, Chapter I.9]. Our proof here of Proposition 7.1 provides another representative example.

2.3. *Equivalence*

We say that two three-point covers $F, G : \mathbb{P}^1_x \longrightarrow \mathbb{P}^1_t$ are *isomorphic* if there is a fractional linear transformation $H : \mathbb{P}^1_x \longrightarrow \mathbb{P}^1_x$ such that $F \circ H = G$; this fractional linear

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transformation H is not required to preserve the marked point $\infty \in \mathbb{P}^1_x$. More broadly, we say that F and G are *equivalent* if and only if there are an $H : \mathbb{P}^1_x \longrightarrow \mathbb{P}^1_x$ and an $I : \mathbb{P}^1_t \longrightarrow \mathbb{P}^1_t$ such that $I \circ F \circ H = G$. When we say things such as 'there are two old and nine new covers in this paper', it is to be understood that we are speaking of covers up to equivalence.

2.4. *Discriminant*

Rather than working directly with (2.3), we will instead work with polynomials

$$f(t, x) = A(x) + tC(x). \tag{2.4}$$

Then $B(x) = -f(1, x)$. The discriminant of $f(t, x)$ has the form

$$D(t) = \Delta t^{n-\text{length}(\lambda_0)}(t-1)^{n-\text{length}(\lambda_1)} \tag{2.5}$$

for some nonzero Δ in k.

2.5. *Topological description over* $\mathbb{C}$

Three-point covers over $\mathbb{C}$ can be analyzed topologically as follows. Let

$$T = \mathbb{P}^1_t - \{0, 1, \infty\}.$$

Basepoint T by $\star = 1/2$. For $t = 0,1$, let g_t be the class in $\pi_1(T, \star)$ of the circle of radius $1/2$ going counterclockwise about t. To treat the third cusp ∞ on a similar footing, define $g_\infty \in \pi_1(T, \star)$ by

$$g_0 g_1 g_\infty = 1. \tag{2.6}$$

Given a three-point cover $F : X \longrightarrow \mathbb{P}^1_t$, let $X_\star$ be the inverse image of $\star$. Then $\pi_1(T, \star)$ acts naturally on $X_\star$. The partition λ_t can then be recovered as the sizes of the orbits of g_t on $X_\star$. The monodromy group M of F is the subgroup of the symmetric group $\text{Sym}(X_\star)$ generated by the g_t. So our Condition (1.1) requires that the monodromy group be as large as possible, consistent with the parities of the λ_t. Sections 6 and 7 each give sample monodromy calculations, centering on Figures 6.1 and 7.1 respectively.

The topological picture lets one count isomorphism classes of three-point covers over $\mathbb{C}$. Let Λ be given as above. Let $\overline{\Sigma}(\Lambda)$ be the set of $(g_0, g_1, g_\infty) \in S_n^3$ satisfying (2.6) with g_t having orbit partition λ_t. The mass $m(\Lambda)$ of Λ is by definition $|\overline{\Sigma}(\Lambda)|/n!$, and there is a convenient formula expressing $m(\Lambda)$ in terms of a sum indexed by the irreducible characters of S_n; see [**8**, Theorem I.5.8]. Of greater interest to us is the subset $\Sigma(\Lambda)$ of $\overline{\Sigma}(\Lambda)$ consisting of triples (g_0, g_1, g_∞) generating all of A_n or S_n. The action of S_n on $\overline{\Sigma}(\Lambda)$ by simultaneous conjugation restricts to a free action of S_n on $\Sigma(\Lambda)$. The quotient set $J(\Lambda) = \Sigma(\Lambda)/S_n$ naturally indexes isomorphism classes of covers with monodromy group A_n or S_n and ramification-invariant Λ. One has $|J(\Lambda)| \leqslant m(\Lambda)$.

2.6. *Descent and moduli algebras*

A three-point cover over $\mathbb{C}$ canonically descends to a three-point cover over $\overline{\mathbb{Q}}$. Here we are using the fact that our three ramification points are required to be the standard points 0, 1, and ∞. Because of this canonical descent, one has canonically associated to Λ, a moduli algebra $A(\Lambda)$ of degree $|J(\Lambda)|$ over $\mathbb{Q}$. The set of homomorphisms from $A(\Lambda)$ into $\overline{\mathbb{Q}}$ is identified with $J(\Lambda)$. The algebra $A(\Lambda)$ is a product of number fields. Sample computations of $A(\Lambda)$ are given in Sections 4 and 7.

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Covers satisfying our Condition (1.2) correspond to factors $\mathbb{Q}$ of $A(\Lambda)$. The paper [7] systematically examines the algebras $A(\Lambda)$ in certain degree-n cases, where $n \leqslant 13$. The computations there, extended in Section 4 here, suggest that there is a very definite tendency for $A(\Lambda)$ itself to be a field. This makes our Condition (1.2) hard to satisfy in cases beyond $|J(\Lambda)| = 1$.

2.7. *Bad reduction*

A three-point cover X over $\overline{\mathbb{Q}}$ without non-identity automorphisms has a canonically defined set S of bad reduction primes. General bounds

$$S_{\text{local}} \subseteq S \subseteq S_{\text{global}} \tag{2.7}$$

are known. Here, S_{local} is the set of primes dividing the parts of the λ_t. So our Condition (1.3) immediately restricts the possible Λ by requiring that all parts of each λ_t be among $\{1, 2, 3, 4, 6, 8, 9, 12, \dots\}$. Also, S_{global} is the set of primes dividing $|M|$. So in our cases, S_{global} is the set of primes at most n. Again, computations suggest that there is no particular tendency in general for S to be near S_{local}. This makes Condition (1.3) hard to satisfy. In (2.7), the lower bound is relatively elementary, while the upper bound is deeper. Entry points into the literature on (2.7) include [**1**, Section 2] and [**8**, Sections I.10.3 and I.10.4]; both these references refer back to the original work of Grothendieck and Beckmann on the upper bound.

3. *Covers 6a and 9c, and linear rigidity*

3.1. *Two previously known examples*

The two examples of this section are 6*a* and 9*c*:

$$\begin{aligned}
\Lambda_{6a} &= (33, 3111, 42),\\
f_{6a}(t, x) &= (x^2 - 2)^3 + t(3x - 4)^2,\\
f_{6a}(1, x) &= (x - 1)^3(x^3 + 3x^2 - 8),\\
D_{6a}(t) &= 2^{13}3^6 t^4 (t - 1)^2;
\end{aligned}$$

$$\begin{aligned}
\Lambda_{9c} &= (9, 21^7, 81),\\
f_{9c}(t, x) &= x^9 + t(-9x + 8),\\
f_{9c}(1, x) &= (x - 1)^2(x^7 + 2x^6 + 3x^5 + 4x^4 + 5x^3 + 6x^2 + 7x + 8),\\
D_{9c}(t) &= -2^{24}3^{18} t^8 (t - 1).
\end{aligned}$$

Cover 6*a* is discussed, together with a related degree-10 cover, in [**10**, Section 5]. Cover 9*c* is one of a two-parameter family of trinomial covers; it is discussed in [**10**, Section 10]. These two covers visibly satisfy Conditions (1.2) and (1.3) of the introduction: (1.2) because all coefficients are rational, and (1.3) because the numerical coefficient Δ in $D_n(t)$ has the form $\pm 2^a 3^b$. This will be the case for all our new covers as well. Condition (1.1) is also quickly verifiable, either conceptually by monodromy techniques, or computationally by considering primes p and t in $\mathbb{F}_p - \{0, 1\}$ and factoring $f(t, x) \in \mathbb{F}_p[t, x]$ until enough partitions of n have arisen.

In what follows, in order to save space, we will not display $f_n(1, x)$.

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3.2. *Linear rigidity*

Katz's theory of rigid local systems [5] gives a strengthening of $S \subseteq S_{\text{global}}$ for some covers as follows. Let $\rho : M \longrightarrow \mathrm{GL}_r(\overline{\mathbb{F}}_\ell)$ be an irreducible representation of the monodromy group M of a cover. Suppose that ρ is linearly rigid, in the sense that the sum of the centralizer dimensions of $\rho(g_0)$, $\rho(g_1)$, and $\rho(g_\infty)$ is as large as possible, namely r^2+2. Then $S \subseteq S_{\text{local}} \cup \{\ell\}$. If one similarly has linear rigidity for an absolutely irreducible characteristic-zero representation, then, as a formal consequence, one has $S = S_{\text{local}}$. If $\rho(g_t)$ is semisimple – as it always is for a characteristic-zero representation – then its centralizer dimension is the sum of the squares of the multiplicities of its eigenvalues.

The groups $M = A_n$ and $M = S_n$ have a natural n-dimensional representation R into $\mathrm{GL}_n(\mathbb{Z})$ as permutation matrices. If g has cycle type λ, then each part s of λ contributes the s roots of unity of order dividing s to the complex eigenvalues of $R(g)$. The representation R is not irreducible, but is rather of the form $1+\rho$, where 1 is the trivial representation and ρ is absolutely irreducible for $n \geqslant 4$. For $6a$, the sum of the centralizer dimensions is thus calculated as follows.

t	λ_t	Eigenvalues of $R(g_t)$	Cent. dim. of $\rho(g_t)$
0	33	$1, \omega, \overline{\omega};\ 1, \omega, \overline{\omega}$	$1^2+2^2+2^2 = 9$
1	3111	$1, \omega, \overline{\omega};\ 1;\ 1;\ 1$	$3^2+1^2+1^2 = 11$
∞	42	$1, i, -1, -i;\ 1, -1$	$1^2+2^2+1^2+1^2 = 7$
			27

For example, on the line $t = 1$, one has eigenvalue 1 with multiplicity 4 in $R(g_1)$, but multiplicity only $4-1=3$ in $\rho(g_1)$, thus yielding the contribution of 3^2 to the centralizer dimension of $\rho(g_1)$. Since $27 = 5^2+2$, one has linear rigidity in the case of $6a$. In the case of $9c$, the cusps $t = 0$ and $t = \infty$ each contribute 8, while $t = 1$ contributes $7^2+1^2 = 50$. One has $8+8+50 = 66 = 8^2+2$, and thus linear rigidity here too. It is in this sense that $6a$ and $9c$ are forced to have bad reduction at exactly $\{2, 3\}$.

Katz's theory is at its most powerful for groups M that have small-rank linear representations, mainly classical groups over finite fields. When M is A_n or S_n, it seems that linear rigidity is limited to very low degrees and the single two-parameter family $\Lambda = (mr, 21^{n-2}, n)$ for $m+r = n$ and m and r relatively prime, as discussed in [10, Section 10]. To see that Katz's theory is far from applying to the new covers in this paper, define the mobility of an r-dimensional representation ρ to be $((r^2+2)-C)/2$, where C is the sum of the centralizer dimensions of $\rho(g_t)$. Then for our nine new covers, the mobilities are not zero, but rather as follows.

Cover:	$6a$	$9c$	$9d$	$10c$	$12a$	$12b$	$12c$	$12d$	$18a$	$28c$	$33a$
Mobility(Λ)	0	0	4	6	7	4	5	3	8	31	45
Mass(Λ)	1	1	2	2	1.5	1.5	1	2.5	$1\frac{1}{18}$	2	2
$\lvert J(\Lambda)\rvert$	1	1	2	2	1	1	1	1	1	2	2

For ℓ a prime dividing n, the representation ρ of S_n reduces mod ℓ to an irreducible $(n-2)$-dimensional representation ρ' plus the trivial representation 1. However, one can check that the mobility of ρ' is always at least the mobility of ρ, and so one cannot make Katz's theory

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apply to our new covers via this route either. Also, if an A_n or S_n cover has mobility zero, then a consequence of Katz's theory is that the corresponding mass $m(\Lambda)$ is 1. Even this consequence holds for only one of our nine new covers.

4. *A completeness result, and Covers* 9*d and* 10*c*

PROPOSITION 4.1. *In degrees* 5–11*, there are only four equivalence classes of genus-zero good covers: Covers* 6*a and* 9*c from the previous section, and Covers* 9*d and* 10*a described in this section.*

Proof. Since we are working with equivalence classes, we consider only partition triples $\Lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ with $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_\infty$ with respect to some ordering on partitions. Also, we consider only those Λ where all parts of each λ_t are in $\{1, 2, 3, 4, 6, 8, 9\}$. We impose the genus-zero condition that λ_0, λ_1, and λ_∞ have altogether $n+2$ parts. We do not impose the existence of a singleton condition as discussed at the beginning of Section 2. The total number of partition triples to check is then as given in Table 4.1.

For each partition, we compute the group-theoretically defined quantity $|J(\Lambda)|$. The results are summarized in Table 4.1.

If $|J(\Lambda)| = 0$, then there is no further work to be done. If $|J(\Lambda)| = 1$, we compute a defining equation and check whether or not the bad reduction set is within $\{2, 3\}$.

When $|J(\Lambda)| \geqslant 2$, we proceed in one of two ways. The main way is again to compute the defining equations. In this way, one can obtain the degree-$|J(\Lambda)|$ moduli algebra $A(\Lambda)$ over $\mathbb{Q}$. Typically, $A(\Lambda)$ is simply a field, and then again we stop. Exceptionally, $A(\Lambda)$ has some factors of $\mathbb{Q}$. Then we examine the bad reduction for the cover corresponding to each such factor, exactly as in the case $|J(\Lambda)| = 1$. When this approach is beyond computational feasibility for our current programs, we can always find that there are no corresponding good covers by finding a small prime $p \geqslant 5$ for which there are no covers $A(x)+tC(x) \in \mathbb{F}_p[t, x]$ that could be the reduction of a good cover. This reduction method is a very much easier calculation. We persevered with the first method whenever possible, however, because knowledge of the moduli algebras is a key step toward obtaining a full understanding of the situation.

Table 4.1: Information about the search for good covers in degrees $5 \leqslant n \leqslant 11$.

n	Total	$\lvert J(\Lambda)\rvert$:			$S \cap \{5,7,11\}$ for cases with $\lvert J(\Lambda)\rvert = 1$								Max
		$=0$	$=1$	$\geqslant 2$	$\emptyset$	$\{5\}$	$\{7\}$	$\{5,7\}$	$\{11\}$	$\{5,11\}$	$\{7,11\}$	$\{5,7,11\}$	$\lvert J(\Lambda)\rvert$
5	6	1	4	1	0	1							2
6	27	18	3	6	1	2							4
7	34	3	9	22	0	0	6	3					12
8	113	47	6	61	0	2	2	2					24
9	206	28	12	166	1	3	6	2					60
10	488	95	10	383	0	4	4	2					432
11	782	0	10	772	0	0	0	0	2	3	3	2	1064

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In degrees at most 8, one can quickly check that the twenty-two $|J(\Lambda)| = 1$ cases yield only the good cover 6*a*. For example, in degree seven, all nine covers have bad reduction at 7, as indicated by Table 4.1. The 90 cases with $|J(\Lambda)| \geqslant 2$ are all covered in [7]. The moduli algebra fails to be a field only four times, always for the simple conceptual reasons indicated in [7]. Even in these four exceptions, there is no factor of $\mathbb{Q}$.

In degree nine, from the twelve $|J(\Lambda)| = 1$ cases, one gets only 9*c*. The thirty $|J(\Lambda)| \geqslant 2$ cases with one of λ_t corresponding to an involution are in [7]. We succeeded in computing moduli algebras for the 136 remaining Λ with the single exception of $\Lambda = (3321, 3321, 621)$ with $|J(\Lambda)| = 36$. Since there are seven singletons here, but $\mathbb{P}^1(\mathbb{F}_5)$ has only six elements, this case can be excluded without any computation, for if there exists a cover defined over $\mathbb{Q}$, it would have to have bad reduction at 5. The moduli algebra for $(63, 3111111, 81)$ is $\mathbb{Q} \times \mathbb{Q}$, and one factor of $\mathbb{Q}$ corresponds to Cover 9*d*:

$$\begin{aligned} \Lambda_{9d} &= (\underline{6}3, \underline{3}111111, \underline{8}1), \\ f_{9d}(t, x) &= x^6(x-2)^3 + t(3x-2), \\ D_{9d}(t) &= 2^{24}3^9 t^7 (t-1)^2. \end{aligned}$$

The underlined singletons correspond to $x = 0$, $x = 1$, and $x = \infty$, respectively. This normalization convention is needed for Cover 9*d* to fit into the framework of Section 7. None of the other moduli algebras that we computed in degree $n = 9$ has $\mathbb{Q}$ as a factor.

In degree ten, the ten cases with $|J(\Lambda)| = 1$ do not yield a good cover. The three $|J(\Lambda)| \geqslant 2$ cases with one of the λ_t corresponding to an involution and another to an order-three element are given in [7], all with $A(\Lambda)$ a field. Of the remaining 380 cases, we succeeded in computing the moduli algebra $A(\Lambda)$ in 301 of them. We obtain one good cover, Cover 10*c*, coming from one of the factors of $A(82, 31111111, 91) = \mathbb{Q} \times \mathbb{Q}$:

$$\begin{aligned} \Lambda_{10c} &= (\underline{8}2, \underline{3}111111, \underline{9}1), \\ f_{10c}(t, x) &= x^8(2x-3)^2 - t(4x-3), \\ D_{10c}(t) &= 2^{36}3^{18}t^8(t-1)^2. \end{aligned}$$

Here, our underlining convention is exactly as it was for 9*d*. Of the remaining 79 cases, 25 were eliminated because they had seven or more singletons, and the remaining 54 were eliminated by either a mod 5 or a mod 7 computation.

In degree eleven, the ten cases with $|J(\Lambda)| = 1$ do not yield a good cover, as indeed all ten covers have bad reduction at 11. Nor are any good covers obtained from the 772 cases with $|J(\Lambda)| \geqslant 2$. Indeed, 76 cases, including all thirteen cases with $|J(\Lambda)| \geqslant 354$, are eliminated as above by the simple fact that the λ_t together contain at least seven singletons. □

All the moduli algebras considered in the proof of Proposition 4.1 are ramified within $\{2, 3, 5, 7, 11\}$, as a consequence of the upper bound in (2.7). It may happen that a factor of a moduli algebra $A(\Lambda)$ is ramified only within $S = \{2, 3\}$. Certainly this is the case if the factor is $\mathbb{Q}$. As more complicated examples, let $\Lambda_1 = (333, 32211, 621)$ and $\Lambda_2 = (333, 3321, 4311)$. The corresponding moduli algebras $A(\Lambda_1)$ and $A(\Lambda_2)$ each have degree six. To compute them, we use the fact that each Λ_i has four singletons. For Λ_1, we normalize the defining equation by requiring that the singletons 6, 3, and 2 correspond respectively to $x = \infty$, $x = 0$, and $x = 1$. For Λ_2, we normalize by similarly requiring that 4, 3, and 2 correspond to $x = \infty$, $x = 1$, and $x = 0$, respectively.

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There are six possibilities for the x-values corresponding to the singleton 1, and we find them to be the roots of

$$F_1(x) = 7x^6 + 318x^5 + 4515x^4 + 5944x^3 - 336x^2 + 1056x + 160,$$

and

$$F_2(x) = 8x^6 - 3744x^5 + 23409x^4 - 63288x^3 + 161838x^2 - 209952x + 91854.$$

These polynomials are both irreducible, with Galois group A_6 and field discriminants $2^8 3^8$ and $2^{12} 3^8$ respectively. The polynomials

$$f_1(x) = x^6 + 3x^5 + 3x^4 + 2x^3 - 3x^2 - 3x - 1$$

and

$$f_2(x) = x^6 - 8x^3 + 9x^2 - 6$$

have smaller coefficients and define the same sextic fields.

However, the cases Λ_1 and Λ_2 are the only cases that we computed in the course of proving Proposition 4.1, whose moduli algebras contain fields of degree at least 6 and discriminant of the form $\pm 2^a 3^b$. Similarly, we have encountered no such fields in our computations with moduli algebras in higher-degree n. It therefore seems that moduli algebras are not a promising approach for constructing number fields ramified within $\{2, 3\}$.

As a final remark in connection with Λ_1 and Λ_2, note that 5 divides $F_1(0) = 2^5 \cdot 5$ and $F_2(1) = 5^3$. Similarly, note that 7 divides $F_1(\infty) = 7$ and $F_2(0) = 2 \cdot 3^8 \cdot 7$. So, for $i = 1$ and $i = 2$, the six conjugate three-point covers $X_{i,j}$ corresponding to Λ_i have bad reduction at both 5 and 7. In other words, the upper bound in (2.7) is exact. This remark shows another aspect of the difficulty of keeping all ramification within $\{2, 3\}$.

5. *Covers* 12*a*, 12*b*, 12*c*, *and* 12*d as specializations of two four-point covers*

We carried out extensive (but not exhaustive) searches in degree 12, and found four good three-point covers. Having found these covers, we pursued the situation further, and found two one-parameter families of four-point covers:

$$\begin{aligned}
\Lambda_{12A} &= (3333, 222222, 81111, 21^{10}),\\
f_{12A}(s,t,x) &= (3s^2x^4 + 6sx^2 + 16sx + 3)^3\\
&\quad +64st(9s^2x^4 + 8s^2x^3 + 18sx^2 + 72sx + 48s + 9),\\
D_{12A}(s,t) &= -2^{128}3^{44}s^{81}(s-3)^3t^8(t-1)^6\big(64st + (3s-1)^3(s-3)\big);
\end{aligned}$$

$$\begin{aligned}
\Lambda_{12BCD} &= (3333, 33111111, 444, 21^{10}),\\
f_{12BCD}(s,t,x) &= 2\big(18(s+2)^2(s^2-2s-2)x^4 - 36(s+2)s^2(s-1)x^2\\
&\quad - 16(2+s)^2s^2x + 3s^3(s-16)\big)^3\\
&\quad -9(s+2)^3(s-4)\big(6(s-1)(s+2)x^3 - 9s^2x - 4s^2\big)^4t,\\
D_{12BCD}(s,t) &= -2^{63}3^{73}(s+2)^{96}s^{88}(s-4)^{74}t^8(t-1)^4\\
&\quad \times\big((s+2)(s-1)^4(s-4)t - (s^2-2s-2)^3\big).
\end{aligned}$$

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The ramification invariants now have the form $(\lambda_0, \lambda_1, \lambda_\infty, \lambda_{G(s)})$. One always has one-parameter families when dealing with four-point covers because the extra ramification point $t = G(s)$ can move. The map $G : \mathbb{P}^1_s \longrightarrow \mathbb{P}^1_t$ comes from the last printed factor of $D_n(s, t)$ in each case. This map G is itself a three-point cover, with ramification invariant $(31, 22, 31)$ in the case $12A$ and $(33, 411, 411)$ in the case $12BCD$. The covers were calculated by modifying our computational techniques for three-point covers. We still use (2.2), but now we allow $A(x)B(x)C(x)$ to have $n + 3$ roots in $\mathbb{C} \cup \{\infty\}$, rather than $n + 2$.

Our original three-point cover $12a$ is $12A$ specialized at $s = 1/3$. Similarly, $12b$, $12c$, and $12d$ are respectively $12BCD$ specialized at $s = 1$, $s = \infty$, and $s = 2$:

$$\begin{aligned}
\Lambda_{12a} &= (633, 222222, 81111),\\
f_{12a}(t, x) &= (x+1)^6(x^2 - 2x + 9)^3 + 64t(9x^4 + 8x^3 + 54x^2 + 216x + 225),\\
D_{12a}(t, x) &= 2^{143}3^{25}t^9(t-1)^6;
\end{aligned}$$

$$\begin{aligned}
\Lambda_{12b} &= (3333, 33111111, 84),\\
f_{12b}(t, x) &= 2(54x^4 + 16x + 5)^3 - t(9x + 4)^4,\\
D_{12b}(t, x) &= -2^{63}3^{114}t^8(t-1)^4;
\end{aligned}$$

$$\begin{aligned}
\Lambda_{12c} &= (3333, 4311111, 444),\\
f_{12c}(t, x) &= 2(18x^4 - 36x^2 - 16x + 3)^3 - 9t(6x^3 - 9x - 4)^4,\\
D_{12c}(t) &= -2^{63}3^{73}t^8(t-1)^5;
\end{aligned}$$

$$\begin{aligned}
\Lambda_{12d} &= (3333, 3321111, 444),\\
f_{12d}(t, x) &= (36x^4 + 36x^2 + 64x + 21)^3 - 36t(6x^3 - 9x - 4)^4,\\
D_{12d}(t) &= 2^{134}3^{73}t^8(t-1)^5.
\end{aligned}$$

The four-point covers also have other interesting specializations to three-point covers satisfying our reduction condition (1.3). At $s = 3$, Cover $12A$ becomes Cover $10a$ of [**10**, Section 5], with generic Galois group $P\Gamma L_2(9)$, which is in turn connected to Cover $6a$, as explained there. At $s = \infty$, Cover $12A$ becomes Cover $9c$ of Section 3. At $1 \pm 2\sqrt{3}/3$, Cover $12A$ becomes two conjugate covers, each with monodromy group S_{12}. Similarly, Cover $12BCD$ at $s = 1 \pm \sqrt{3}$ becomes two conjugate covers with monodromy group S_{12}. In Section 9, we explain how our two four-point covers each give new number fields beyond those coming from the three-point covers $12a$, $12b$, $12c$, and $12d$.

6. *Cover* $18a$ *and its associated dessin*

We carried out modest searches for good covers in degrees 13–24, and found only one more:

$$\begin{aligned}
\Lambda_{18a} &= (3^6, 2^9, 96111),\\
f_{18a}(t, x) &= 4(x^6 + 12x^4 - 16x^3 + 18x^2 - 24x + 10)^3\\
&\quad + 27(2x-1)^6(4x^3 + 3x^2 + 48x - 32)t,\\
D_{18a}(t) &= -2^{81}3^{177}t^{12}(t-1)^9.
\end{aligned}$$

In general, the dessin of a three-point cover $F : X \longrightarrow \mathbb{P}^1_t$ is the inverse image of $[0, 1]$ in X. In the case $18a$, this inverse image in the x-plane is drawn in Figure 6.1.

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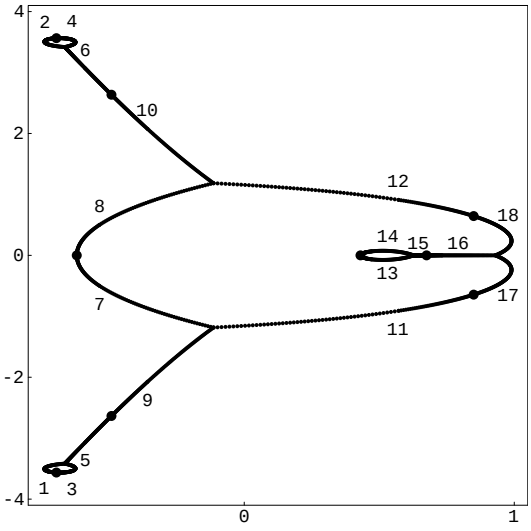

Figure 6.1: The dessin of Cover 18*a*, with the loop 13-14 expanded from its attachment point on its right by a linear factor of 30, to make it clearly visible.

In Figure 6.1, the eighteen points in $X_{18a,\star} = F_{18a}^{-1}(1/2)$ are labelled by a number in $\{1, \ldots, 18\}$ printed nearby. Equally well, these numbers index the eighteen segments in $F_{18a}^{-1}((0,1))$, and we will regard these segments as the objects being permuted by the g_t. The six points in $F_{18a}^{-1}(0)$ are the six triple junctions of the dessin. The nine points in $F_{18a}^{-1}(1)$ would not ordinarily be visible, as they are merely the double junctions; to make them visible, they are drawn as dots.

From the dessin, one can read off the monodromy: g_0 rotates segments counterclockwise about the triple junctions, while g_1 rotates segments about the double junctions. Explicitly,

$$\begin{aligned} g_0 &= (1,3,5)(2,6,4)(7,9,11)(8,12,10)(13,15,14)(16,17,18), \\ g_1 &= (1,3)(2,4)(5,9)(6,10)(7,8)(11,17)(12,18)(13,14)(15,16), \\ g_\infty &= (1,9,8,6,4,10,18,11,5)(7,17,15,14,16,12)(2)(3)(13). \end{aligned}$$

As stated at the end of Section 3, the mass $m(\Lambda_{18a})$ is $1\frac{1}{18}$. Rather than appealing to the mass formula and a sum over the characters of S_{18}, one can confirm geometrically that the part of $m(\Lambda_{18a})$ corresponding to transitive permutation representations is 1, by checking that any dessin belonging to Λ_{18a} is isotopic to the drawn one. The remaining part arises as $(1/6) \cdot (1/3)$. Here the $1/6$ corresponds to the degree-six genus-one cover belonging to $(33, 222, 6)$; it has Galois group C_6. The $1/3$ corresponds to the degree-twelve genus-zero cover belonging to $(3333, 222222, 9111)$; it has Galois group $T233$, of the form $3^4.S_4$.

7. *Covers* 28*c and* 33*a as exceptional members of a family*

It is often natural to consider families of three-point covers indexed by discrete parameters. The best-known example is the doubly indexed family including 9*c* mentioned at the end of Section 3, corresponding to trinomial covers.

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In this section, we consider the family of partition triples

$$\Lambda_{a,b,n} = ((\underline{a}, n-a), (\underline{3}, 1, \ldots, 1), (\underline{b}, n-b)). \tag{7.1}$$

Here, n runs over positive integers while a and b each run over integers in $[1, n-1]$, so that we have a triply indexed family.

7.1. *Defining equations*

Our underlining in (7.1) indicates how we will normalize our defining equation, namely by making the singletons a, 3, and b correspond to $x = 0$, $x = 1$, and $x = \infty$, respectively. Thus we need to look among the irreducible polynomials in $\mathbb{C}[t, x]$ of the form

$$f(t, x) = x^a(x - z)^{n-a} + t\lambda(x - y)^{n-b}, \tag{7.2}$$

and find those such that $(x-1)^3$ divides $f(1, x)$. We are interested only in cases where $d := \mathrm{GCD}(a, n-a, b, n-b)$ is 1, as otherwise the rational function $-x^a(x-z)^{n-a}/(\lambda(x-y)^{n-b})$ in $\mathbb{C}(x)$ is a dth power, and so the associated monodromy group would be in the wreath product $C_d \wr S_{n/d}$, and thus not A_n or S_n.

Proposition 7.1. *Let a and b be positive integers less than n, such that a, $n-a$, b, and $n-b$ are all distinct. Then there are exactly two irreducible polynomials $f(t, x)$ of the form (7.2) satisfying*

$$(x-1)^3 | f(1, x). \tag{7.3}$$

These are

$$f_{a,b,n,\pm}(t, x) = x^a(x - z_\pm)^{n-a} + t\lambda_{a,b,n,\pm}(x - y_\pm)^{n-b},$$

where

$$\Delta = a(n-a)b(n-b), \tag{7.4}$$

$$z_\pm = \frac{ab \pm \sqrt{\Delta}}{a(a+b-n)}, \tag{7.5}$$

$$y_\pm = \frac{ab \mp \sqrt{\Delta}}{an}, \tag{7.6}$$

$$\lambda_\pm = -(1 - y_\pm)^{b-n}(1 - z_\pm)^{n-a}. \tag{7.7}$$

Proof. We will explain how one arrives computationally at the given $f_{a,b,n,\epsilon}$. It will be clear from the process that these are irreducible solutions to (7.3), and that there are no others.

The *ABC* equation (2.2) takes the form

$$x^a(x - z)^{n-a} - (x-1)^3 g(x) = -\lambda(x - y)^{n-b}, \tag{7.8}$$

where $g(x)$ is a degree-$(n-3)$ monic polynomial. The derivative of (7.8) with respect to x is

$$(nx - az)x^{a-1}(x - z)^{n-a-1} - (x-1)^2\big(3g(x) - g'(x) + xg'(x)\big) = -\lambda(n-b)(x - y)^{n-b-1}. \tag{7.9}$$

Both Equations 7.8 and 7.9 have three terms, naturally labelled from left to right by 0, 1, and ∞. Consider the linear combination

$$(n-b) \times (\text{Equation (7.8)}) - (x - y) \times (\text{Equation (7.9)}),$$

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chosen so that the terms from ∞ drop out. Writing the terms from 0 on the left and the terms from 1 on the right and factoring, we obtain

$$\begin{aligned}&\big[(n-b)x(x-z)-(x-y)a(x-z)-(x-y)x(n-a)\big]x^{a-1}(x-z)^{n-a-1}\\&\quad=\big[(n-b)(x-1)g(x)-3(x-y)g(x)-(x-y)(x-1)g'(x))\big](x-1)^2.\end{aligned}\tag{7.10}$$

Since $z=1$ would make $f(t,x)$ reducible, the fact that 1 is a double root of the right-hand side of (7.10) forces 1 to be a double root of the bracketed factor on the left-hand side of (7.10), giving us

$$-bx^2+(ny+az+bz-nz)x-ayz=-b(x-1)^2.\tag{7.11}$$

Here, the left-hand side of (7.11) is the expansion of the bracketed factor on the left-hand side of (7.10); the proportionality factor $-b$ on the right-hand side of (7.11) is forced by comparing the x^2 terms. Comparing the x terms and then the constant terms gives

$$\begin{aligned}ny+az+bz-nz&=2b,\\-ayz&=-b.\end{aligned}$$

Solving this system for y and z yields (7.5) and (7.6). Substituting $x=1$ in (7.8) then yields (7.7). □

7.2. *Discriminant*

The next proposition gives the discriminant of each $f_{a,b,n,\epsilon}(t,x)$ as an element of $\mathbb{Q}(\sqrt{\Delta})[t]$, up to a common sign.

PROPOSITION 7.2. *The discriminant of $f_{a,b,n,\epsilon}(t,x)$ with respect to x is*

$$D_\epsilon=\pm z_\epsilon^n y_\epsilon^{(a-1)(n-b)}(y_\epsilon-z_\epsilon)^{(n-a-1)(n-b)}\lambda_\epsilon^{n-2}a^a(n-a)^{n-a}t^{n-2}(t-1)^2.\tag{7.12}$$

Proof. We know that the discriminant of $f_{a,b,n,\epsilon}(t,x)$ has the form $\Delta_\epsilon t^{n-2}(t-1)^2$ from (2.5). At issue is the determination of Δ_ϵ.

In general, let

$$f(t,x)=\left(\prod_i A_i(x)^{m_i}\right)+tC(x),\tag{7.13}$$

where the $A_i(x)$ are separable polynomials without a common root. Then, up to sign, the discriminant of $f(t,x)$ is

$$\left(\prod_{i<j}\operatorname{Res}(A_i,A_j)^{m_i+m_j}\right)\left(\prod_i \operatorname{disc}(A_i)\operatorname{Res}(A_i,C)^{m_i-1}m_i^{m_i\deg(A_i)}\right)t^{\sum_i(m_i-1)}+\ldots,\tag{7.14}$$

where '...' indicates higher-order terms in t. The required discriminants and resultants are trivial to evaluate, since both our A_i have degree 1, and we obtain (7.12). □

7.3. *Bad reduction*

Let $X_{a,b,n,\pm}$ be the three-point cover given by $f_{a,b,n,\pm}(t,x)=0$. To analyze the bad reduction of these two covers, we use the following formulas, which are elementary consequences of Propositions 7.1 and 7.2:

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$$\begin{aligned}
y_+y_- &= \frac{b(a+b-n)}{an},\\
(y_+ - z_+)(y_- - z_-) &= \frac{(a-b)^2 b}{a(a+b-n)n},\\
(1-y_+)(1-y_-) &= \frac{(a-b)(n-b)}{an},\\
(1-z_+)(1-z_-) &= \frac{(b-a)(n-a)}{a(a+b-n)},\\
D_+D_- &= b^{(2b-n-bn+n^2)}a^{(1+a-n)n}(n-a)^{n(n-a)}\\
&\quad\times(n-b)^{(b-n)(n-2)}(b-a)^{2a+2ab-2n-3an-bn+2n^2}n^n\\
&\quad\times(a+b-n)^{-2a-2ab+n+3an+bn-2n^2}t^{2n-4}(t-1)^4.
\end{aligned} \tag{7.15}$$

From (7.15), we see that both $X_{a,b,n,+}$ and $X_{a,b,n,-}$ have bad reduction within the set of primes S dividing $na(n-a)b(n-b)(b-a)(a+b-n)$.

Generically, Δ is not a perfect square. In this case, $X_{a,b,n,+}$ and $X_{a,b,n,-}$ are conjugate covers, and they each have bad reduction set exactly S. Exceptionally, Δ is a perfect square. Then the bad reduction sets S_+ of $X_{a,b,n+}$ and S_- of $X_{a,b,n,-}$ may differ. Both contain the set S_{local} of primes dividing $3a(n-a)b(n-b)$, by the lower bound of (2.7). Also, $S_+ \cup S_- = S$.

Table 7.1 lists out all the triples (a,b,n) known to us with $n/2 < a < b < n$, $\text{GCD}(a, n-a, b, n-b) = 1$, and $S_{\text{local}} \subseteq \{2,3\}$, the first of these being a normalization condition to avoid repetitions. For each triple (a,b,n), we use Proposition 7.2 to determine how the primes in $S - S_{\text{local}}$ are distributed between S_+ and S_-. The conclusion is that besides $(a,b,n,\epsilon) = (6,8,9,+)$ giving $9d$ and $(a,b,n,\epsilon) = (8,9,10,+)$ giving $10c$, one also has

$$\Lambda_{28c} = ((16,12),(3,1^{25}),(27,1)), \qquad \Lambda_{33a} = ((27,6),(3,1^{30}),(32,1)),$$
$$f_{28c}(t,x) = x^{16}(2x-3)^{12} + t(8x-9), \qquad f_{33a}(t,x) = x^{27}(3x-4)^6 - t(9x-8),$$
$$D_{28c}(t) = -2^{396}3^{81}t^{26}(t-1)^2; \qquad D_{33a}(t) = -2^{160}3^{252}t^{31}(t-1)^2.$$

Table 7.1: Triples (a,b,n) and the bad reduction sets S_+ and S_-. Boldface entries are primes besides 2 and 3 in $S_+ \cup S_-$.

a	$n-a$	b	$n-b$	n	$b-a$	$n-a-b$	$S_+ - \{2,3\}$	$S_- - \{2,3\}$
6	3	8	1	$9 = 3^2$	2	**5**	–	**5**
8	2	9	1	$10 = 2\cdot$**5**	1	**7**	–	**5**, **7**
16	12	27	1	$28 = 2^2$**7**	**11**	$3\cdot$**5**	**5**, **7**, **11**	–
27	6	32	1	$33 = 3\cdot$**11**	**5**	$2\cdot$**13**	–	**5**, **11**, **13**
27	8	32	3	$35 =$ **5**$\cdot$**7**	**5**	$2^3$3	**5**, **7**	**5**
81	64	144	1	$145 =$ **5**$\cdot$**29**	3^2**7**	2^4**5**	**5**, **7**, **29**	**5**
243	16	256	3	$259 =$ **7**$\cdot$**37**	**13**	$2^4$3$\cdot$**5**	**5**, **13**, **37**	**7**
486	27	512	1	$513 = 3^3$**19**	2$\cdot$**13**	**5**$\cdot$**97**	**5**	**13**, **19**, **97**

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(a)

(b)

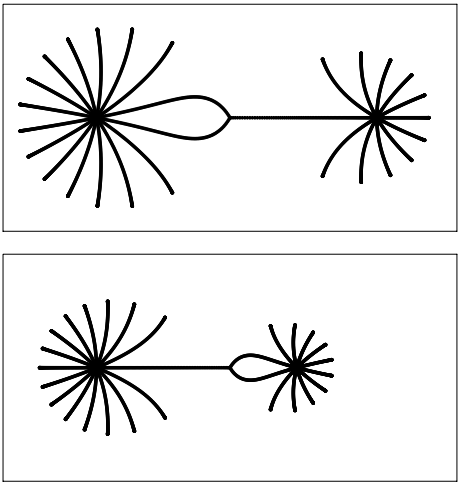

Figure 7.1: (a) The dessin of cover $X_{16,27,28,+}$; (b) The dessin of cover $X_{28c} = X_{16,27,28,-}$.

7.4. *Monodromy*

View $X_{a,b,n,\epsilon,\star} = F^{-1}_{a,b,n,\epsilon}(1/2)$ as indexing the set of edges of the dessin of $X_{a,b,n,\epsilon}$, that is, the set of components of $F^{-1}_{a,b,n,\epsilon}((0,1))$. Figure 7.1 shows the two dessins for $(a,b,n) = (16,27,28)$, and this figure will guide the discussion for the rest of this section. Unlike in Figure 6.1, the real and imaginary coordinates in Figure 7.1 are drawn using the same scale, so that edges leaving a valence-k juncture are separated by the angle $2\pi/k$.

Under the normalization condition $n/2 < a < b < n$, one has

$$0 < y_+ < 1 < y_- < z_- < z_+$$

from the explicit formulas (7.5) and (7.6). In the drawn case $(a,b,n) = (16,27,28)$, these inequalities become

$$0 < 0.803\ldots < 1 < 1.125 < 1.5 < 2.1.$$

Geometrically, $x = 0$ is the left-hand junction, $x = 1$ is the triple junction in the center, and $x = z_\pm$ is the right-hand junction; these junctions belong to $t = 0$, $t = 1$, and $t = 0$ respectively. Also $x = y_\pm$ is 'the center' of the bounded region in the complement to the dessin, $x = \infty$ being 'the center' of the unbounded region. In general, the bounded region has $n - b - 1$ spokes into it, while the unbounded region has $b - 2$ spokes into it; our four (a,b,n) fail to represent the general case in the sense that $n - b - 1$ is always zero.

To describe the monodromy of $X_{a,b,n,\epsilon}$, we will not identify the edge set $X_{a,b,n,\epsilon,\star}$ with $\{1,2,\ldots,n\}$, as we did in the previous section for $X_{18a,\star}$. Rather, we will incorporate some of the structure of the situation into our labelling system. Let μ_k denote the kth roots of unity in $\mathbb{C}$. Let $\mu'_k = \mu_{2k} - \mu_k$. We write

$$X_{a,b,n,+,\star} = \{L(w)\}_{w\in\mu'_a} \coprod \{R(w)\}_{w\in-\mu_{n-a}},$$
$$X_{a,b,n,-,\star} = \{L(w)\}_{w\in\mu_a} \coprod \{R(w)\}_{w\in-\mu'_{n-a}}.$$

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Here, L stands for 'left' and R for 'right'. The label $L(w)$ indicates the segment leaving the left-hand juncture $x = 0$ with tangent direction w. Similarly, $R(w)$ indicates the segment leaving the right-hand juncture z_ϵ with tangent direction w. There is a natural action of complex conjugation σ_{01} on $X_{a,b,n,\epsilon,\star}$, and this is given by

$$\sigma_{01} L(w) = L(\overline{w}),$$
$$\sigma_{01} R(w) = R(\overline{w}).$$

Similarly, the monodromy of $X_{a,b,n,\epsilon}$ is given as follows.

PROPOSITION 7.3. *Normalize by $n/2 < a < b < n$. Then the permutation g_0 acts on the edge set $X_{a,b,n,\epsilon,\star}$ by rotation counterclockwise about the two $t = 0$ vertices, that is, by*

$$g_0 L(w) = L\big(e^{2\pi i/a} w\big),$$
$$g_0 R(w) = R\big(e^{2\pi i/(n-a)} w\big).$$

The permutation g_1 acts on $X_{a,b,n,\epsilon,\star}$ by rotation counterclockwise about the $t = 1$ vertices, that is, by the three cycle

$$g_1 = \begin{cases} \big(R(-1),\, L\big(e^{2\pi i(n-b)/(2a)}\big),\, L\big(e^{2\pi i(b-n)/(2a)}\big)\big) & \text{if } \epsilon = +, \\ \big(L(1),\, R\big(e^{2\pi i(a-b)/(2n-2a)}\big),\, R\big(e^{2\pi i(b-a)/(2n-2a)}\big)\big) & \text{if } \epsilon = -. \end{cases}$$ □

8. *Polynomials with fewer terms*

8.1. *Width in general*

Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

be a degree-n polynomial over a ground field k. The width w of $f(x)$ is the number of monomials appearing in $f(x)$. So w is the number of nonzero values of a_j, and one has $w \leqslant n + 1$.

Let L be a degree-n extension field of k. Then the width w of L is the smallest width of a polynomial $f(x) \in k[x]$ satisfying $L = k[x]/f(x)$. So if k has characteristic zero, one has $w \leqslant n$, as one can always choose $f(x)$ with $a_{n-1} = 0$.

For characteristic-zero fields L of degree $n = 5$ (Hermite, 1861) or $n = 6$ (Joubert, 1867), one always has $w \leqslant n - 1$ because both a_{n-1} and a_{n-3} can be made simultaneously zero; see [6]. Otherwise, there do not seem to be general results, but it can be expected that typical degree-n number fields would have width quite near n.

In general, polynomials $f(t, x) = A(x) + tC(x)$ defining three-point covers often have small width. Suppose that λ_0 has a singleton e and λ_∞ has a singleton d. Then we can take $A(x) = x^e \tilde{A}(x)$ and $C(x)$ of degree $n - d$. Then, if $e + d = n + 2 + j$ with $j \geqslant 0$, one has the vanishing coefficients $a_{n-d+1}, \ldots, a_{e-1}$, so that the width of $f(t, x) \in \mathbb{Q}[t, x]$ is at most $n - j$.

8.2. *Width in our setting*

Table 8.1 gives the widths of the polynomials f_n of the previous sections. In f_{6a}, one has $a_5 = a_3 = 0$, reflecting the general fact about sextics cited above. For the degree-12 polynomials and f_{18}, one has only $a_{n-1} = 0$. For $9c$, $9d$, $10c$, $28c$, and $33a$, interior coefficients vanish, as described in the previous paragraph.

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We see no theoretical reason to expect *a priori* that our extensions of $\mathbb{Q}(t)$ and $\mathbb{Q}(t, s)$ would have smaller width than the width of their natural defining polynomials, $f_n(t, x)$ or $f_n(s, t, x)$. In this section, however, we present polynomials $g_n(t, y)$ and $g_n(s, t, y)$ that define the same extensions but have smaller width, as indicated by Table 8.1. We expect that in all cases the width of g_n is the width of the extension; that is, the g_n cannot be improved upon in the sense of width.

The existence of narrower polynomials was first suggested in our study of the individual number fields of the next section. We applied the PARI command *polredabs* [9] to replace a given irreducible $f_n(\tau, x)$ with a monic polynomial $h_{n,\tau}(x) \in \mathbb{Z}[x]$ having the property that $h_{n,\tau}(x)$ minimizes the root sum $\sum |\alpha_i|^2$ among all monic polynomials in $\mathbb{Z}[x]$ defining the same root field as $f_n(\tau, x)$. The behavior of our largest case $n = 33a$ was typical. Of the 23 rational numbers τ considered in the next section, the widths of $h_{33a,\tau}(x)$ were 7 (eleven times), 12, 13, 33 (nine times), and 34. Moreover, the eleven cases of width 7 all had the same form, namely

$$h_{33a,\tau}(x) = x^{33} + a_{20}x^{20} + a_{13}x^{13} + a_7x^7 + a_6x^6 + a_1x + a_0.$$

In general, we then found $g_n(t, y) \in \mathbb{Z}[t, y]$ such that for all τ in question there is a $c_\tau \in \mathbb{Q}^\times$ with $g_n(\tau, y) = c_\tau^{-n} h_{n,\tau}(c_\tau y)$. The interpolation process in the case of the two four-point covers was similar, but more involved. In each case, we algebraically confirmed the correctness of our new polynomial by finding an element y in $\mathbb{Q}(x)$ or $\mathbb{Q}(s, x)$ with minimal polynomial $g_n(t, y)$ or $g_n(s, t, y)$.

8.3. *The narrower polynomials*

Our narrower polynomials $g_n(t, y)$ and $g_n(s, t, y)$ and their discriminants are displayed in the rest of this section. We also give y as a function of x, and intersperse several comments.

$$\begin{aligned} g_{6a}(t, y) &= 2y^6 + 3ty^4 + 4ty^3 - t^2(t - 1), \\ y &= \frac{(x^2 - 2)(x - 1)}{3x - 4}, \\ d_{6a}(t) &= 2^{11}3^6t^{10}(t - 1)^2(4t - 1)^2; \end{aligned}$$

$$\begin{aligned} g_{9d}(t, y) &= y^9 - 12t^2y^3 + 27t^3y + 16t^3, \\ y &= x^2(x - 2), \\ d_{9d}(t) &= 2^{24}3^{27}t^{25}(t - 1)^2. \end{aligned}$$

For $12A$, $12BCD$, and $18a$, we use the abbreviation $u = t - 1$; for $12BCD$, we also use the abbreviation $r = s + 2$. Also, the quantity $c_k(t)$ in a discriminant formula represents an irreducible polynomial in $\mathbb{Z}[t]$ of degree k, playing a role similar to $c_1(t) = 4t - 1$ for $6a$. Similarly, $c_{j,k}(s, t)$ represents a polynomial of degree j in s and degree k in t.

Table 8.1: Widths of the polynomials f_n of the previous sections and the polynomials g_n of this section.

n :	$6a$	$9c$	$9d$	$10c$	$12A$	$12a$	$12BCD$	$12b$	$12c$	$12d$	$18a$	$28c$	$33a$
width(f_n) :	5	3	6	5	12	12	12	12	12	12	18	15	9
width(g_n) :	4	3	4	5	7	7	9	6	9	9	8	6	7

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$$\begin{aligned}
g_{12A}(s,t,y) &= -243s^3(s-3)y^{12} + 486s^3uty^{10} + 432s^3u^2ty^9 + 729s^2u^2t^2y^8 \\
&\quad +216s^2u^3t^2y^6 + 216su^4t^3y^4 + 16u^6t^4, \\
y &= \frac{3s^2x^4 + 6sx^2 + 16sx + 3}{16s} \\
&\quad \times \frac{3s^3x^6 + 9s^2x^4 + 24s^2x^3 + 9sx^2 + 24sx + 32s + 3}{9s^2x^4 + 8s^2x^3 + 18sx^2 + 72sx + 48s + 9}, \\
d_{12A}(s,t) &= -2^{68}3^{64}s^{33}(s-3)t^{44}(t-1)^{66}\big(64st + (s-3)(3s-1)^3\big)c_{3,3}(s,t)^2;
\end{aligned}$$

$$g_{12a}(t,y) = g_{12A}(1/3,t,y);$$

$$\begin{aligned}
g_{12BCD}(s,t,y) &= 27t^4u^2y^{12} - 216r^2st^3uy^9 - 162r^2s^2t^3uy^8 \\
&\quad -36r^3s^2t^2\big((s-4)t - (s+20)\big)y^6 - 216r^4s^3t^2y^5 + 243r^4s^4t^2y^4 \\
&\quad -16r^5s^3(13s-28)ty^3 + 108r^5s^4(s-4)ty^2 + 12r^6s^4(s-4)^2, \\
y &= \frac{\big(6r(s-1)x^3 - 9s^2x - 4s^2\big)^2}{18r^2(s^2-2s-2)x^4 - 36rs^2(s-1)x^2 - 16r^2s^2x + 3s^3(s-16)} \\
&\quad \times \frac{3(s-4)r^2}{18rx^2 + 6rsx + s(-8+5s)}, \\
d_{12BCD}(s,t) &= -2^{67}3^{47}(s-4)^4s^{44}(s+2)^{66}(t-1)^{10}t^{44} \\
&\quad \times\big((s-4)(s-1)^4(s+2)t - (s^2-2s-2)^3\big)c_{6,4}(s,t)^2;
\end{aligned}$$

$$\begin{aligned}
g_{12b}(t,x) &= 32y^{12} - 256ty^9 + 768t^2y^6 - 486t^3y^4 - 160t^3y^3 - t^4, \\
y &= \frac{54x^4 + 16x + 5}{9x + 4}, \\
d_{12b}(t) &= -2^{79}3^{60}t^{44}(t-1)^4;
\end{aligned}$$

$$g_{12c}(t,y) = \text{Coefficient}\big(g_{12BCD}(s,t,sy), s^{12}\big);$$

$$g_{12d}(t,y) = g_{12BCD}(2,t,y).$$

In general, both the numerator and the denominator of the rational function y divide the product $A(x)B(x)C(x)$ or $A(s,x)B(s,x)C(s,x)$. Cover 18a provides a representative example. Write $y = f_9(x)/(6f_1(x)f_6(x))$. Then $A(x) = 4f_6(x)^3$, $B(x) = f_9(x)^2$, and $f_1(x)|C(x)$.

$$\begin{aligned}
g_{18a}(t,y) &= 1728t^6y^{18} - 576t^4u^3y^{12} - 512t^3u^5y^9 - 432t^3u^5y^8 \\
&\quad +152t^2u^6y^6 + 32tu^8y^3 - 21tu^8y^2 + 2u^9, \\
y &= \frac{2x^9 + 36x^7 - 48x^6 + 162x^5 - 360x^4 + 330x^3 - 153x + 56}{6(2x-1)(x^6 + 12x^4 - 16x^3 + 18x^2 - 24x + 10)}, \\
d_{18a}(t) &= -2^{137}3^{51}t^{102}(t-1)^{153}c_6(t)^2.
\end{aligned}$$

The coefficients of each y^k in all our polynomials are simpler than one might expect. Put $g_{9c}(t,y) = f_{9c}(t,y)$ and $g_{10c}(t,y) = f_{10c}(t,y)$. Then in the cases $n = 9c$, $n = 9d$, $n = 10c$, $n = 12b$, $n = 28c$, and $n = 33a$, all the coefficients of $g_n(t,y)$ have the form ct^m.

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Moreover, except for $160 = 2^5 5$ appearing as a c in $12b$ and $408240 = 2^4 3^6 5^1 7$ appearing as a c in $33a$, all these c are of the form $\pm 2^a 3^b$.

$$\begin{aligned} g_{28c}(t, y) &= y^{28} - 216ty^{14} - 864ty^{13} - 729ty^{12} - 512t^2y - 432t^2, \\ y &= x(2x - 3), \\ d_{28c}(t) &= -2^{132}3^{81}t^{50}(t-1)^2c_2(t)^2; \end{aligned}$$

$$\begin{aligned} g_{33a}(t, y) &= y^{33} - 1728t^2y^{20} - 55296t^3y^{13} + 408240t^4y^7 - 262144t^4y^6 \\ &\quad -531441t^5y + 442368t^5, \\ y &= -x^4(3x - 4), \\ d_{33a}(t) &= -2^{160}3^{102}t^{157}(t-1)^2c_3(t)^2. \end{aligned}$$

9. *Summary of the number fields constructed*

For each of our nine new polynomials $f_n(t, x)$, one can substitute rational numbers $\tau \neq 0, 1$ for t to obtain separable polynomials in $\mathbb{Q}[x]$. If one chooses these specialization points suitably, then the root algebra $\mathbb{Q}[x]/f_n(\tau, x)$ has algebra discriminant of the form $\pm 2^a 3^b$, and thus the splitting field K is also ramified only within $\{2, 3\}$.

9.1. *Specialization points*

We use the notation of [10]. Let $T^*_{\infty,\infty,\infty}$ consist of 2, 3, 4, and 9 and their orbits under $S_3 = \langle t \mapsto 1/t, t \mapsto 1 - t \rangle$. Explicitly, with rows indicating the orbits, we have

$$T^*_{\infty,\infty,\infty} = \left\{ \begin{array}{cccccc} -1, & 1/2, & 2, & & & \\ -2, & -1/2, & 1/3, & 2/3, & 3/2, & 3, \\ -3, & -1/3, & 1/4, & 3/4, & 4/3, & 4, \\ -8, & -1/8, & 1/9, & 8/9, & 9/8, & 9 \end{array} \right\}.$$

Then certainly the τ in $T^*_{\infty,\infty,\infty}$ work for all nine polynomials, as even the polynomial discriminant of $f_n(\tau, x)$ has the form $\pm 2^a 3^b$. Further, for a given cover X_n, let p, q and r be the least common multiples of the parts of λ_0, λ_1, and λ_∞, respectively. Then there is a larger set $T^*_{p,q,r}$ for which there may be extraneous primes in the polynomial discriminant of $f_n(\tau, x)$, but not in the algebra discriminant.

9.2. *Galois groups of specializations*

The nine displayed formulas for $D_n(t)$ come into play when we study the Galois groups of specializations. Fix a cover X_n, and write $D_n(t) = (-1)^s 2^a 3^b t^c (t-1)^d$. Then, as stated in (2.5), c and d are determined as n minus the length of λ_0 and λ_1 respectively. In particular, the parities of c and λ_0 agree, and similarly the parities of d and λ_1 agree. So $(c, d) \equiv (0, 0)$ (2) exactly in the three cases where the monodromy group is A_n, namely $10c$, $12b$ and $28c$.

The parities of the exponents s, a, and b cannot be predicted directly from the partition triple Λ_n. Our summarizing Table 9.1 refines the distinction A_n vs. S_n by sorting fields according to their discriminant class $d \in \{-6, -3, -2, -1, 1, 2, 3, 6\}$. The case $d = 1$, corresponding to A_n, is printed in bold. In the remaining cases, $\mathbb{Q}(\sqrt{d})$ is the unique quadratic subfield of the splitting field of $f_n(\tau, x)$.

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In [10] we pointed out that the trinomial cover X_{9c} has the property that for all 35 specialization points in $T^*_{9,2,8}$, the Galois group is as large as possible, meaning either A_9 or S_9, according to whether or not the discriminant class is 1. The covers $10c$, $12c$, $28c$, and $33a$ all share this property of generic specialization.

In contrast, in the cases shown below, the polynomial $f_n(\tau, x)$ factors as a linear factor times a degree-$(n-1)$ factor.

n	τ				
$12a$	-1	-24	-48		
$12b$	$125/128$				
$12d$	$1/9$				
$18a$	$4/3$	-8	$125/27$	$2 \cdot 53^3/3^6$	$-505^3/2^{27}3$

In these cases, the Galois group is always S_{n-1}, except for $n = 18a$ and $\tau = -505^3/2^{27}3$, where it is A_{17}.

Table 9.1: Summary of the known fields $K \subset \mathbb{C}$ with $\mathrm{Gal}(K/\mathbb{Q})$ either A_n or S_n and discriminant of the form $\pm 2^a 3^b$.

n	-6	-3	-2	-1	1	2	3	6	s		Comments
3	1	4	1	1	**1**			1	1	3	[3] (complete)
4	3	6	3	3	**1**			7	2	0, 4	[3] (complete)
5	1					2	1	1	3	1	[3] (complete)
6	1	1		2	**4**	13	4	6	0, 4	2	[3] (complete)
7	4	1	1	4					1		[4] (complete); 1, 1 from $9c$, $10c$ at t=1
8	10	4	4	7	**1**	2	1	1	2	0, 4	2 from [10, §11,12]; all from [2]
9	4	3	8	4	**13**	9	10	8	3	1	35, 22, 1, 1 from $9c$, $9d$, $18a$, [10, §12]
10				1	**23**	1			0	2	23 A_{10}'s from $10c$; 2 S_{10}'s from $12A$
11	2		2	1					1		3, 1, 1 from $12a$, $12b$, $12d$
12	9	17	38	7	**12**	11	11	13	2	0, 4	33, 26, 24, 22, 12, 1 from $12a$, $12b$, $12c$, $12d$, $12A$, $12BCD$
17	1		1	1	**1**			1	3	1	All 5 from $18a$
18	5	15	5	4	**15**	10	10	12	4	2	All 76 from $18a$
25					**1**					1	From $28c$ at $t = 1$
28		23							2		All 23 from $28c$
30							1			2	From $33a$ at $t = 1$
32	1								2		From [10, §12]
33	4	4	4	4	**1**	4	2		3	1	All 23 from $33a$

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There are exactly two other exceptional specializations, these being to transitive subgroups of S_n. Cover $9d$ at $\tau = 4/3$ has Galois group $\mathrm{SL}_2(8).3$ and Cover $12d$ at $\tau = 11^3/5^4 2$ has Galois group $T258 = [3^4 : 2]S_4$ of order $2^4 3^5 = 3888$. One can also specialize our polynomials at the cusps $\tau = 0$, $\tau = 1$, and $\tau = \infty$. The cases yielding nonsolvable fields of degree at least 7 are $\tau = 1$ for $9c$, $10c$, $18a$, $28c$, and $33a$.

9.3. *More fields from the four-point covers*

One can also specialize $12A$ at points beyond those considered in $12a$, and $12BCD$ at points beyond those considered in $12b$, $12c$, and $12d$. At $s = 1$ and $s = -1$, the fourth ramification point of $12A$ has the very low height 4, as one has

$$D_{12A}(1, t) = 2^{135} 3^{44} t^8 (t-1)^6 (4t-1),$$

and

$$D_{12A}(-1, t) = 2^{140} 3^{44} t^8 (t-1)^6 (t-4).$$

For the first case, we extract τ from $T^*_{3,2,8}$ with $(4\tau - 1)$ a perfect square, obtaining

$$T_1 = \{-2, -1/2, -1/8, 1/3, 1/2, 1/4, 3/4\}.$$

For the second case, we extract τ from $T^*_{3,2,8}$ with $(\tau - 4)$ a perfect square, obtaining

$$T_{-1} = \{-8, -2, -1/2, 4/3, 2, 3, 4\}.$$

In fact, all these specialization points are in $T^*_{\infty,\infty,\infty}$, which explains the relation $\tau \in T_{-1}$ if and only if $\tau^{-1} \in T_1$. From Cover $12BCD$, we found only one extra field, coming from $(s, t) = (-1/2, 1/9)$. The cases $\tau = 1/4$ and $\tau = 4$ for $12A$ give cuspidal specializations, with Galois group S_{10}. Otherwise all these specializations have Galois group A_{12} or S_{12}.

9.4. *Real places of number fields*

When setting up inverse Galois problems, one often focuses on the Galois group G first, and the behavior of complex conjugation $\sigma \in G$ second. Table 9.1 gives the possible values of s, where σ has cycle type $2^r 1^s$. The first column corresponds to s arising from fields with negative discriminant, in which case $s \equiv n - 2$ (4). The second column corresponds to fields with positive discriminant, in which case $s \equiv n$ (4).

For a given three-point cover X_n, the number s of real roots of the defining polynomial $f_n(\tau, x)$ depends only on the interval $(-\infty, 0)$, $(0, 1)$, or $(1, \infty)$ containing τ. Moreover, if λ_t has only odd parts, like our $\lambda_1 = 31^{n-3}$, the two intervals with endpoint t yield the same s. So specializing three-point covers does not lend itself to producing a wide range of s. In fact, the only ambiguity for s on Table 9.1 in degrees $n \geqslant 9$ is for $n = 12$ with $d > 0$. In this case, $s = 0$ if and only if the field comes from $12a$ or the new part of $12A$ or $12BCD$. For d equal to 1, 2, 3, and 6, there are 6, 7, 6, and 5 such fields, respectively.

9.5. *Distinctness of fields*

We are considering many pairs (n, τ) yielding an A_n or S_n Galois field $K_{n,\tau} \subset \mathbb{C}$ with $n \geqslant 7$. One might have expected 'accidental' repetitions, such as $K_{27b,-48} = K_{27d,32/81}$ for the Galois group $W(E_6)$, discussed in [**10**, Section 9]. However, in every case here, if $(n, \tau) \neq (n', \tau')$, then the resulting fields $K_{n,\tau}$ and $K_{n',\tau'}$ are distinct.

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