

RESEARCH ARTICLE

Highly connected orientations from edge-disjoint rigid subgraphs

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Abstract

We give an affirmative answer to a long-standing conjecture of Thomassen, stating that every sufficiently highly connected graph has a k -vertex-connected orientation. We prove that a connectivity of order $O(k^2)$ suffices. As a key tool, we show that for every pair of positive integers d and t , every $(t \cdot h(d))$ -connected graph contains t edge-disjoint d -rigid (in particular, d -connected) spanning subgraphs, where $h(d) = 10d(d + 1)$. This also implies a positive answer to the conjecture of Kriesell that every sufficiently highly connected graph G contains a spanning tree T such that $G - E(T)$ is k -connected.

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1. Introduction

It follows from a classical theorem of Nash-Williams that every $2k$ -edge-connected graph has a k -arc-connected orientation. In 1985, Thomassen asked whether a similar statement is true for vertex-connectivity.

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Conjecture 1.1 [18, Conjecture 10]. *For every positive integer k , there exists a (smallest) integer $f(k)$ such that every $f(k)$ -connected graph has a k -connected orientation.*

Conjecture 1.1 has a long history. It is well-known that if $f(k)$ exists, then $f(k) \geq 2k$. Thomassen, together with Jackson, also posed the stronger conjecture that $f(k) = 2k$ ([18, Conjecture 11]), and later Frank gave the even stronger conjecture that a graph has a k -connected orientation if and only if it remains $(2k - 2j)$ -edge-connected after the deletion of any set of $j < k$ vertices ([5, Conjecture 7.8]).

The $k = 1$ case of Frank's conjecture is the well-known theorem of Robbins [16]; this implies that $f(1) = 2$. The $k = 2$ case of Conjecture 1.1 was proved by the second author [9] by showing that $f(2) \leq 18$. Subsequently, Thomassen [19] proved the $k = 2$ case of Frank's conjecture, hence establishing that $f(2) = 4$. However, Durand de Gevigney [3] recently disproved Frank's conjecture for $k \geq 3$. He also showed that for such k , deciding whether a graph has a k -connected orientation is NP-hard.

In this paper, we give an affirmative answer to Conjecture 1.1, for all k , by showing that $f(k) = O(k^2)$.

Theorem 1.2. *Every $(320 \cdot k^2)$ -connected graph has a k -connected orientation.*

The bound on $f(k)$ given by Theorem 1.2 is probably far from being tight. In particular, it is still open whether $f(k) = 2k$ holds.

The key new tool in our proof is a packing theorem for highly connected graphs (Theorem 1.6 below) that is interesting on its own right. Our original motivation for investigating such packing questions came from the following conjecture of Kriesell from 2003.

Conjecture 1.3 (See, e.g., [12, Problem 444]). *For every positive integer k , there exists a (smallest) integer $g(k)$ such that every $g(k)$ -connected graph G contains a spanning tree T for which $G - E(T)$ is k -connected.*

As with Conjecture 1.1, the edge-connected version of Conjecture 1.3 is classical: it follows from a well-known theorem of Nash-Williams [13] and Tutte [20] that every $(2k + 2)$ -edge-connected graph G contains a spanning tree T such that $G - E(T)$ is k -edge-connected. In particular, we have $g(1) = 4$. The $k = 2$ case of Conjecture 1.3 was answered by the second author. In fact, we have the following 'packing theorem' for 2-rigid graphs. (Definitions are given in the next section.)

Theorem 1.4 [9, Theorem 3.1]. *Every $6t$ -connected graph contains t edge-disjoint 2-rigid (and hence 2-connected) spanning subgraphs. In particular, $g(2) \leq 12$.*

This bound was subsequently improved to $g(2) \leq 8$ in [2], where the authors proved an analogous packing result for the union of the 2-dimensional generic rigidity matroid and the graphic matroid.

The $k = 3$ case was settled in a similar fashion by the first three authors. In this case, the underlying matroid was the \mathcal{C}_2^1 -cofactor matroid, which is conjectured to be the same as the 3-dimensional generic rigidity matroid.

Theorem 1.5 [6, Theorem 5.11]. *Every $12t$ -connected graph contains t edge-disjoint \mathcal{C}_2^1 -rigid (and hence 3-connected) spanning subgraphs. In particular, $g(3) \leq 24$.*

Our second main result is a similar packing result for the d -dimensional generic rigidity matroid.

Theorem 1.6. *Every $(t \cdot 10d(d + 1))$ -connected graph contains t edge-disjoint d -rigid (and hence d -connected) spanning subgraphs.*

The existence of a constant $h(d)$ such that every $(t \cdot h(d))$ -connected graph contains t edge-disjoint d -connected spanning subgraphs was conjectured by the first three authors [6, Conjecture 5.10]. The bound in Theorem 1.6 is almost certainly not tight. In fact, we believe that every $t \cdot d(d + 1)$ -connected graph contains t edge disjoint d -rigid spanning subgraphs, and hence that $h(d) \leq d(d + 1)$. For $t = 1$, this was recently proved by the fourth author.

Theorem 1.7 [21, Theorem 1.1]. *Every $d(d + 1)$ -connected graph is d -rigid.*

We show that for packing d -rigid spanning subgraphs, the bound $t \cdot d(d+1)$ would be optimal (Lemma 6.1).

We prove Theorem 1.6 in Section 3, and in Section 4, we use it to derive Theorem 1.2. In Section 5, we further investigate the conjecture of Kriesell. Theorem 1.6 implies that Conjecture 1.3 is true with $g(d) \leq 20d(d+1)$. To improve upon this result, we adapt the proof technique of Theorem 1.7 in [21] to the union of the d -dimensional rigidity matroid and the graphic matroid. This leads to the following bound.

Theorem 1.8. *Every $(d^2 + 3d + 5)$ -connected graph G contains edge-disjoint spanning subgraphs G_0 and T such that G_0 is d -rigid and T is a tree. In particular, $g(d) \leq d^2 + 3d + 5$.*

The bound given by Theorem 1.8 is still not optimal. We believe that every $(d(d+1) + 2)$ -connected graph contains edge-disjoint copies of a d -rigid spanning subgraph and a spanning tree. Again, this bound would be tight (Lemma 6.1).

An immediate corollary of Theorem 1.8 is that if G is sufficiently highly connected, then for each pair $s, t \in V(G)$, there exists a path P from s to t in G such that $G - E(P)$ is k -connected. The existence of such paths was verified earlier in [10], assuming that G is $(1600k^4 + k + 2)$ -connected. With Theorem 1.8, the connectivity requirement can be substantially weakened.

2. Preliminaries

We start by setting some notation. Throughout the paper, we only consider simple graphs – that is, graphs without loops and parallel edges. For a graph G , we let $V(G)$ and $E(G)$ denote the vertex and edge sets of G , respectively. For a subset $X \subseteq V(G)$, we let $G[X]$ denote the subgraph of G induced by X , and we let $i_G(X) = |E(G[X])|$ be the number of edges induced by X in G . We use $K(X)$ to denote the complete graph on vertex set X , and similarly, K_n denotes the complete graph on n vertices. Given a vertex $v \in V(G)$, $N_G(v)$ is the set of neighbors of v and $\deg_G(v) = |N_G(v)|$ is the degree of v in G . Given a positive integer k , we say that a connected graph is k -connected if it has at least $k+1$ vertices and it remains connected after the removal of any set of fewer than k vertices.

For a directed graph D and a vertex $v \in V(D)$, we use $N_D^-(v)$ and $N_D^+(v)$ to denote the set of in-neighbors and out-neighbors of v , respectively. (A vertex $v \in V - X$ is an *in-neighbor* of the vertex set X in the directed graph D if there is an edge vx of D with $x \in X$. Out-neighbors are defined analogously.) We let $\rho_D(v) = |N_D^-(v)|$ and $\delta_D(v) = |N_D^+(v)|$ denote the in-degree and out-degree, respectively, of v in D . A directed graph is *strongly connected* if it contains a directed u - v path for every pair of vertices u and v , and it is k -connected, for a positive integer k , if it has at least $k+1$ vertices and it remains strongly connected after the removal of any set of fewer than k vertices.

2.1. Rigidity matroids

Next, we recall the relevant definitions and facts from rigidity theory. For completeness, we start by giving the geometric definition of the generic d -dimensional rigidity matroid. In fact, we will only use some combinatorial properties of this matroid, which we collect below. We assume familiarity with the basic notions of matroid theory; the standard reference is [15]. For a more thorough introduction to rigidity theory, see, for example, [17, 22].

Let $G = (V, E)$ be a graph and let d be a positive integer. A (d -dimensional) *realization* of G is a pair (G, p) , where $p : V(G) \rightarrow \mathbb{R}^d$ maps the vertices of G to d -dimensional Euclidean space. A realization is *generic* if its coordinates do not satisfy any nonzero polynomial with rational coefficients. We identify the space of all d -dimensional realizations with $(\mathbb{R}^d)^V$, and we define the *measurement map* $m_{d,G} : (\mathbb{R}^d)^V \rightarrow \mathbb{R}^E$ by

$$m_{d,G}(p) = \left(\|p(u) - p(v)\|^2 \right)_{uv \in E}.$$

The *rigidity matrix* $R(G, p)$ of a realization (G, p) is the Jacobian of $m_{d,G}$ evaluated at the point p . This is a matrix whose rows are indexed by the edges of G , and hence, the row matroid of $R(G, p)$ can be viewed as a matroid on ground set E . It is known that this row matroid is the same for every generic d -dimensional realization of G . Thus, we define the *d -dimensional rigidity matroid* of G , denoted by $\mathcal{R}_d(G)$, to be the row matroid of $R(G, p)$ for some generic d -dimensional realization (G, p) . We let r_d denote the rank function of \mathcal{R}_d . Using a slight abuse of terminology, we also use $r_d(G)$ to denote the rank of $\mathcal{R}_d(G)$.

We say that a graph $G = (V, E)$ is *d -rigid* if $r_d(G) = r_d(K(V))$, and *minimally d -rigid* if it is d -rigid but $G - e$ is not, for every $e \in E$. Finally, G is *\mathcal{R}_d -independent* if $r_d(G) = |E|$. In other words, G is d -rigid (resp. minimally d -rigid, \mathcal{R}_d -independent) if and only if E is a spanning set (resp. base, independent set) in $\mathcal{R}_d(K(V))$. It is folklore that a graph is 1-rigid if and only if it is connected. A combinatorial characterization (and an efficient deterministic recognition algorithm) is also available for 2-rigid graphs, but finding such a characterization for d -rigid graphs is a major open question for $d \geq 3$.

As we noted above, we shall only use some well-known combinatorial properties of $\mathcal{R}_d(G)$. These are as follows.

- (a) For $n \geq d$, $r_d(K_n) = dn - \binom{d+1}{2}$.
- (b) Hence, if $G = (V, E)$ is a minimally d -rigid graph on at least d vertices, then $|E| = d|V| - \binom{d+1}{2}$, and

$$i_G(X) \leq d|X| - \binom{d+1}{2} \tag{1}$$

holds for every subset $X \subseteq V$ of vertices with $|X| \geq d$.

- (c) The addition of a vertex of degree d to a graph preserves \mathcal{R}_d -independence as well as d -rigidity.
- (d) If a graph on at least $d + 1$ vertices is d -rigid, then it is d -connected.
- (e) If G_1 and G_2 are d -rigid graphs with at least d vertices in common, then $G_1 \cup G_2$ is also d -rigid.

The (*d -dimensional*) *edge split operation* replaces an edge uv of graph G with a new vertex joined to u and v , as well as to $d - 1$ other vertices of G .

- (f) The d -dimensional edge split operation preserves \mathcal{R}_d -independence as well as d -rigidity.

We note that properties (a)-(f) are shared by all 1-extendable abstract rigidity matroids; see [7, 14]. Since our proofs will only use these properties, our results involving the rigidity matroid remain true for any 1-extendable abstract rigidity matroid. For simplicity, we only give the statements for the generic rigidity matroid.

2.2. Unions of rigidity matroids

We shall also consider unions of rigidity matroids. Let $\mathcal{M}_i = (E, \mathcal{I}_i)$, $i \in \{1, \dots, t\}$ be a collection of matroids on a common ground set E . The *union* of $\mathcal{M}_1, \dots, \mathcal{M}_t$ is the matroid $\mathcal{M} = (E, \mathcal{I})$ whose independent sets are defined by

$$\mathcal{I} = \{I_1 \cup \dots \cup I_t : I_1 \in \mathcal{I}_1, \dots, I_t \in \mathcal{I}_t\}.$$

Let $r_{\mathcal{M}}$ and $r_{\mathcal{M}_i}$, $i \in \{1, \dots, t\}$ denote the rank functions of the respective matroids. It is immediate from the definition of \mathcal{M} that $r_{\mathcal{M}}(E) \leq \sum_{i=1}^t r_{\mathcal{M}_i}(E)$, with equality if and only if E contains disjoint subsets E_1, \dots, E_t such that E_i is a base of \mathcal{M}_i for each $i \in \{1, \dots, t\}$.

Let $G = (V, E)$ be a graph and let t be a positive integer. We shall denote the t -fold union of the d -dimensional rigidity matroid of G by $\mathcal{R}_d^t(G) = (E, r_d^t)$. Analogously to the $t = 1$ case, we define G to be *\mathcal{R}_d^t -rigid* if $r_d^t(G) = r_d^t(K(V))$, and to be *\mathcal{R}_d^t -independent* if $r_d^t(G) = |E|$. Let us define a pair of vertices $\{u, v\}$ to be *\mathcal{R}_d^t -linked in G* if $r_d^t(G + uv) = r_d^t(G)$. Thus, G is \mathcal{R}_d^t -rigid if and only if every pair of vertices of G is \mathcal{R}_d^t -linked in G .

It follows from property (c) above that the addition of a vertex of degree td preserves \mathcal{R}_d^t -independence. Similarly, it follows from property (f) (together with property (c)) that the td -dimensional edge split operation preserves \mathcal{R}_d^t -independence.

Lemma 2.1. *If $n \geq 2td$, then $r_d^t(K_n) = tdn - t \binom{d+1}{2}$. Hence, an \mathcal{R}_d^t -rigid graph on at least $2td$ vertices contains t edge-disjoint d -rigid spanning subgraphs.*

Proof. We show that K_n contains t edge-disjoint d -rigid spanning subgraphs G_1, \dots, G_t . Let us choose disjoint subsets of vertices $V_i^j, i \in \{1, \dots, t\}, j \in \{1, 2\}$, each of size d , and let $X = V(K_n) - \bigcup_{i=1}^t \bigcup_{j=1}^2 V_i^j$. For each $i \in \{1, \dots, t\}$, let G_i consist of the complete graph on $V_i^1 \cup V_i^2$, plus the edge sets

$$\bigcup_{\ell < i} \left((E(V_i^1, V_\ell^1) \cup E(V_i^2, V_\ell^2)) \right) \cup \bigcup_{\ell > i} \left((E(V_i^1, V_\ell^2) \cup E(V_i^2, V_\ell^1)) \right) \cup E(V_i^1, X).$$

Now G_i is d -rigid since it has a spanning subgraph obtained from a complete graph by adding vertices of degree d , and it is easy to see that G_i and G_j are edge-disjoint for $i \neq j$. □

Lemma 2.2. *Let $G = (V, E)$ be a graph and let $v_0 \in V$ be a vertex with $\deg_G(v_0) \geq td + 1$. If $r_d^t(G - v_0) = r_d^t(G) - td$, then every pair of vertices $u, v \in N_G(v_0)$ is \mathcal{R}_d^t -linked in $G - v_0$.*

Proof. Suppose, for a contradiction, that $\{u, v\}$ is not \mathcal{R}_d^t -linked in $G - v_0$ for some pair of vertices $u, v \in N_G(v_0)$. This means that $r_d^t(G - v_0 + uv) = r_d^t(G - v_0) + 1$. Let G_0 be a maximal \mathcal{R}_d^t -independent subgraph of $G - v_0 + uv$; by the previous observation, we must have $uv \in E(G_0)$. We can now obtain an \mathcal{R}_d^t -independent subgraph G' of G with $r_d^t(G') = r_d^t(G_0) + td$ by performing a td -dimensional edge split on the edge uv in G_0 . But this means that

$$r_d^t(G) \geq r_d^t(G') = r_d^t(G_0) + td = r_d^t(G - v_0) + td + 1 = r_d^t(G) + 1,$$

a contradiction. □

2.3. Tools from probability theory

We shall need the following standard result from probability theory. We let $\text{Bin}(n, p)$ denote the binomial distribution with parameters n and p .

Theorem 2.3 (Chernoff bound for the binomial distribution, see, for example, [1, Theorem A.1.13]). *Let $X \sim \text{Bin}(n, p)$. Then for any $0 \leq \eta \leq 1$,*

$$\mathbb{P}(X \leq (1 - \eta)np) \leq e^{-\eta^2 np/2}.$$

We shall also use the following technical lemma.

Lemma 2.4. *Let S be a finite set of size at least d and fix $s \in S$. Let π be a uniformly random ordering of S , and let $f(\pi)$ denote the number of elements of S that precede s in π . Then we have*

$$\mathbb{E}(\min(d, f(\pi))) = d - \frac{1}{2} \cdot \frac{d(d+1)}{|S|}.$$

Proof. Clearly, $f(\pi)$ is uniformly randomly distributed on $\{0, \dots, |S| - 1\}$. Thus, we have

$$\begin{aligned} \mathbb{E}(\min(d, f(\pi))) &= \sum_{i=0}^{|S|-1} \mathbb{P}(f(\pi) = i) \cdot \min(d, i) = \sum_{i=0}^{|S|-1} \frac{1}{|S|} \cdot \min(d, i) \\ &= \frac{1}{|S|} \left(\binom{d+1}{2} + d(|S| - 1 - d) \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{|S|} \left(d(|S| - 1) - \frac{d(d-1)}{2} \right) \\
 &= \frac{1}{|S|} \left(d|S| - \frac{d(d+1)}{2} \right) \\
 &= d - \frac{1}{2} \cdot \frac{d(d+1)}{|S|}.
 \end{aligned}$$

□

3. Packing rigid spanning subgraphs

In this section, we prove Theorem 1.6. If $d \leq 2$ or $t = 1$, then the statement is true by Theorems 1.4 and 1.7. Thus, we will assume that $d \geq 3$ and $t \geq 2$. (The proof also works for $d \leq 2$ or $t = 1$, but the bound we obtain is weaker.)

We remark that the factor 10 is rather arbitrary and a more rigorous analysis of the argument presented here would yield a slightly better constant. For a lower bound on vertex-connectivity required in Theorem 1.6, see Section 6.

The following is the main lemma for our proof of Theorem 1.6.

Lemma 3.1. *Let d, t be integers with $d \geq 3$ and $t \geq 2$, and let $G = (V, E)$ be a graph. If the minimum degree of G is at least $(t \cdot 10d(d + 1))$, then G has a vertex $v_0 \in V$ such that $r_d^t(G - v_0) = r_d^t(G) - td$.*

Proof. Our goal is to construct a maximal \mathcal{R}_d^t -independent subgraph G' of G with minimum degree td . As we will see, the existence of such a subgraph quickly implies the lemma.

Fix an orientation \vec{G} of G such that $\delta_{\vec{G}}(v) \geq \lfloor \deg_G(v)/2 \rfloor$ for each $v \in V$. It is a well-known result that such an orientation exists (see, for example, [4, Theorem 1.3.8]). Then $\delta_{\vec{G}}(v) \geq t \cdot 5d(d + 1)$ for each $v \in V$. For a subset F of E , let \vec{F} denote the corresponding set of oriented edges. Recall that $N_{\vec{F}}^+(v)$ denotes the set of vertices $u \in V$ for which $vu \in \vec{F}$.

Let U be a random subset of V such that each $v \in V$ is in U independently with probability $1/2$. For each $j \in \{1, \dots, t\}$, we recursively define a random subgraph $H_j = (U, F_j)$ of $G[U]$ as follows. Suppose that H_1, \dots, H_{j-1} are already given. Let

$$E_j = E(G[U]) - (F_1 \cup \dots \cup F_{j-1}).$$

Consider a uniformly random ordering π_j of the vertices in U . For each $v \in U$, let $A_j(v) = \{u \in N_{\vec{E}_j}^+(v) : u \text{ precedes } v \text{ in } \pi_j\}$, and fix a subset $B_j(v) \subseteq A_j(v)$ of size $\min(d, |A_j(v)|)$. Finally, let

$$F_j = \bigcup_{v \in U} \{vu : u \in B_j(v)\}.$$

Let $F = \bigcup_{j=1}^t F_j$ and $H = (U, F)$. For each $v \in V - U$, add v to H and connect it with $\min(td, |N_{\vec{G}}^+(v) \cap U|)$ vertices of $N_{\vec{G}}^+(v) \cap U$. Let the resulting graph be denoted by $G_0 = (V, E_0)$.

Claim 3.2. G_0 is \mathcal{R}_d^t -independent.

Proof. For each $j \in \{1, \dots, t\}$, the graph H_j can be obtained by taking a graph on one vertex and then adding new vertices, one at a time, of degree at most d . Hence, H_j is \mathcal{R}_d -independent. It follows that H is \mathcal{R}_d^t -independent. G_0 can be obtained from H by adding new vertices of degree at most td , and thus, G_0 is also \mathcal{R}_d^t -independent. □

Claim 3.3. $\mathbb{E}(|E_0|) \geq \left(td - \frac{1}{4}\right)n$.

Proof. For convenience, we define $n = |V|$ and $k = 5d(d + 1)$. Let $\eta \leq \frac{1}{2}$ be a parameter to be chosen later, and let $r = (1 - \eta)\frac{k}{2}$.

Let us fix a vertex $v \in V$. Using the Chernoff bound (Theorem 2.3), we obtain that

$$\mathbb{P}\left(|N_G^+(v) \cap U| \leq tr\right) = \mathbb{P}\left(|N_G^+(v) \cap U| \leq (1 - \eta)\frac{tk}{2}\right) \leq e^{-\eta^2 tk/4}. \tag{2}$$

Let Q denote the event that $v \in U$ and $|N_G^+(v) \cap U| > tr$. If Q holds, then $|N_{E_j}^+(v)| \geq tr - td > d$ for every $j \in \{1, \dots, t\}$. Note that $\delta_{F_j}(v) = |B_j(v)| = \min(d, |A_j(v)|)$. Thus, it follows from Lemma 2.4 that

$$\mathbb{E}\left(\delta_{F_j}(v) \mid Q\right) \geq d - \frac{1}{2} \cdot \frac{d(d+1)}{tr - td + 1} \geq d - \frac{d(d+1)}{2t(r-d)},$$

and hence,

$$\mathbb{E}\left(\delta_{\bar{F}}(v) \mid Q\right) = \mathbb{E}\left(\sum_{j=1}^t \delta_{F_j}(v) \mid Q\right) = \sum_{j=1}^t \mathbb{E}\left(\delta_{F_j}(v) \mid Q\right) \geq td - \frac{d(d+1)}{2(r-d)}.$$

Equation (2) and the fact that the two sub-events in the definition of Q are independent together imply $\mathbb{P}(Q) \geq \frac{1}{2}(1 - e^{-\eta^2 tk/4})$. Hence,

$$\mathbb{E}(\delta_{\bar{F}}(v)) \geq \mathbb{E}(\delta_{\bar{F}}(v) \mid Q) \cdot \mathbb{P}(Q) \geq \frac{1}{2} \left(1 - e^{-\eta^2 tk/4}\right) \left(td - \frac{d(d+1)}{2(r-d)}\right).$$

After summing over the vertices, we get

$$\mathbb{E}(|F|) = \mathbb{E}\left(\sum_{v \in V} \delta_{\bar{F}}(v)\right) = \sum_{v \in V} \mathbb{E}(\delta_{\bar{F}}(v)) \geq \frac{1}{2} \left(1 - e^{-\eta^2 tk/4}\right) \left(td - \frac{d(d+1)}{2(r-d)}\right) n.$$

Let D denote $E_0 - F$. Then

$$\mathbb{E}\left(\delta_{\bar{D}}(v) \mid v \notin U \wedge |N_G^+(u) \cap U| > tr\right) = td.$$

Thus, by (2), we have

$$\mathbb{E}(\delta_{\bar{D}}(v)) \geq \frac{1}{2} \left(1 - e^{-\eta^2 tk/4}\right) td,$$

and by summing over the vertices, we obtain

$$\mathbb{E}(|D|) \geq \frac{1}{2} \left(1 - e^{-\eta^2 tk/4}\right) tdn.$$

It follows that

$$\begin{aligned} \mathbb{E}(|E_0|) &= \mathbb{E}(|F|) + \mathbb{E}(|D|) \geq \left(1 - e^{-\eta^2 tk/4}\right) \left(td - \frac{d(d+1)}{4(r-d)}\right) n \\ &\geq tdn - \left(e^{-\eta^2 tk/4} \cdot td + \frac{d(d+1)}{4(r-d)}\right) n \\ &\geq tdn - \left(e^{-\frac{\eta^2 tk}{4} + \frac{td}{3}} + \frac{d(d+1)}{4(r-d)}\right) n, \end{aligned}$$

where the last inequality follows from the fact that $td \geq 6$, and thus, $td < e^{td/3}$.

Let us now fix the value of η to be 0.45. With this choice, it is easy to verify that

$$\frac{\eta^2 tk}{4} - \frac{td}{3} = \left(\frac{\eta^2}{4} - \frac{1}{5(d+1) \cdot 3} \right) t \cdot 5d(d+1) \geq \left(\frac{0.45^2}{4} - \frac{1}{60} \right) 120 = 4.075,$$

where for the inequality we used, again, the assumption that $d \geq 3$ and $t \geq 2$. Note that $d \geq 3$ also implies $d \leq \frac{1}{4}d(d+1)$, and hence

$$r - d = 0.55 \cdot \frac{1}{2} \cdot 5 \cdot d(d+1) - d \geq 1.375 d(d+1) - 0.25 d(d+1) = 1.125 d(d+1).$$

Thus,

$$\mathbb{E}(|E_0|) \geq tdn - \left(e^{-4.075} + \frac{1}{4 \cdot 1.125} \right) n \geq \left(td - \frac{1}{4} \right) n,$$

which completes the proof of the claim. □

It follows from Claim 3.3 that

$$\mathbb{E} \left(|E_0| + \frac{|V - U|}{2} \right) = \mathbb{E}(|E_0|) + \frac{1}{4}n \geq tdn.$$

Hence, there exist a set of vertices $U \subseteq V$ and a corresponding spanning subgraph $G_0 = (V, E_0)$ such that

$$|E_0| + \frac{|V - U|}{2} \geq tdn.$$

Since E_0 is \mathcal{R}_d^t -independent by Claim 3.2, we can extend it to a maximal \mathcal{R}_d^t -independent subgraph $G' = (V, E_0 \cup E_1)$ of G by adding a suitable set of edges $E_1 \subseteq E - E_0$. Then $|E_0 \cup E_1| = r_d^t(G)$, and thus,

$$|E_1| = r_d^t(G) - |E_0| \leq tdn - t \binom{d+1}{2} - |E_0| \leq \frac{|V - U|}{2} - t \binom{d+1}{2} < \frac{|V - U|}{2},$$

where in the first inequality, we used the fact that $r_d^t(K_n) = tdn - t \binom{d+1}{2}$, which follows from Lemma 2.1.

Hence, there is some vertex $v_0 \in V - U$ that is not incident to any edge in E_1 . It follows from the construction of G_0 that $d_{G'}(v_0) \leq td$, and thus,

$$r_d^t(G - v_0) \geq r_d^t(G' - v_0) \geq r_d^t(G) - td.$$

However, since $\deg_G(v_0) \geq td$, and the addition of a vertex of degree td preserves \mathcal{R}_d^t -independence, we must have $r_d^t(G) \geq r_d^t(G - v_0) + td$. It follows that $r_d^t(G - v_0) = r_d^t(G) - td$. □

Proof of Theorem 1.6. As we noted before, if $d \geq 2$ or $t = 1$, then the statement follows from Theorems 1.4 and 1.7. Hence, it suffices to prove in the case when $d \geq 3, t \geq 2$. We prove the statement by induction on the number of vertices. Let $c = t \cdot 10d(d+1)$. If $|V| = c + 1$, then G is complete and thus \mathcal{R}_d^t -rigid. It follows from Lemma 2.1 that G contains t edge-disjoint d -rigid spanning subgraphs.

Suppose now that $|V| > c + 1$. By Lemma 3.1, there is some vertex $v_0 \in V$ such that $r_d^t(G - v_0) = r_d^t(G) - td$. Let us consider the graph $G' = G - v_0 + K(N_G(v_0))$. On the one hand, by Lemma 2.2 and the choice of v_0 , every pair of vertices $u, v \in N_G(v_0)$ is \mathcal{R}_d^t -linked in $G - v_0$, and hence, $G - v_0$ is \mathcal{R}_d^t -rigid if and only if G' is. On the other hand, G' is c -connected: indeed, it arises from the c -connected graph $G + K(N_G(v_0))$ by deleting a vertex whose neighbor set is a clique, and it is easy to see that deleting such a vertex preserves c -connectivity (except for the complete graph on $c + 1$ vertices). Hence, by induction, G' is \mathcal{R}_d^t -rigid, and thus so is $G - v_0$.

Since $|V| \geq c + 1 \geq 2td + 1$, Lemma 2.1 implies that $G - v_0$ contains t edge-disjoint d -rigid spanning subgraphs. Adding a vertex of degree at least td to $G - v_0$ corresponds to adding a vertex of degree at least d to each of these subgraphs, an operation that preserves d -rigidity. Since $\deg_G(v_0) \geq c \geq td$, we conclude that G contains t edge-disjoint d -rigid spanning subgraphs, as claimed. \square

4. Highly connected orientations

In this section, we prove Theorem 1.2. The following ‘orientation lemma’ will be a key ingredient in our proof. Given a graph $G = (V, E)$ and a function $g : V \rightarrow \mathbb{Z}_+$, we shall use the notation $g(X) = \sum_{v \in X} g(v)$ for subsets $X \subseteq V$.

Theorem 4.1 (Hakimi [8]). *Let $G = (V, E)$ be a graph and let $g : V \rightarrow \mathbb{Z}_+$ be a function. Then G has an orientation \vec{G} in which $\rho_{\vec{G}}(v) = g(v)$ for all $v \in V$ if and only if*

- (a) $i_G(X) \leq g(X)$ for all nonempty $X \subseteq V$, and
- (b) $|E| = g(V)$ hold.

The same conditions are equivalent to the existence of an orientation in which g specifies the out-degrees.

We shall consider degree-specified orientations of minimally d -rigid graphs. Given a minimally d -rigid graph $G = (V, E)$ with $|V| \geq \binom{d+1}{2}$ and a subset $R \subseteq V$ with $|R| = \binom{d+1}{2}$, we define the in-degree specification function $g_{d,R}$ by putting $g_{d,R}(v) = d$ for all $v \in V - R$ and $g_{d,R}(r) = d - 1$ for all $r \in R$. We say that an orientation \vec{G} of G is a (d, R) -orientation if its in-degrees respect the specification $g_{d,R}$.

Lemma 4.2. *Let $G = (V, E)$ be a minimally d -rigid graph with $|V| \geq \binom{d+1}{2}$ and let $R \subseteq V$ be a set of vertices with $|R| = \binom{d+1}{2}$. Then G has a (d, R) -orientation.*

Proof. We have to verify that the conditions of Theorem 4.1 are satisfied. Let $g = g_{d,R}$. As $|V| \geq \binom{d+1}{2} \geq d$, we have $|E| = d|V| - \binom{d+1}{2} = g(V)$. Now consider a set $X \subseteq V$ with $|X| \geq d$. By (1), we have

$$i_G(X) \leq d|X| - \binom{d+1}{2} \leq d|X| - |R \cap X| = g(X),$$

as required. Next, consider a set $X \subseteq V$ with $|X| \leq d - 1$. Then, since G is simple, we have $i_G(X) \leq \binom{|X|}{2} = \frac{|X|(|X|-1)}{2} \leq |X|(d - 1) \leq g(X)$, which completes the proof. \square

The next lemma provides a lower bound on the number of in-neighbors of certain subsets in (d, R) -orientations, establishing a link between degree-specified orientations and high vertex-connectivity.

Lemma 4.3. *Let d and k be integers with $k \geq 2$ and $d \geq 4k - 4$. Let $G = (V, E)$ be a minimally d -rigid graph with $|V| \geq \binom{d+1}{2}$, $R \subseteq V$ a set of vertices with $|R| = \binom{d+1}{2}$, and let \vec{G} be a (d, R) -orientation of G . Finally, let $X \subseteq V$ be a set of vertices. If $|X \cap R| \leq \binom{d+1}{2}$, then X has at least k in-neighbors in \vec{G} .*

Proof. Let W denote the set of in-neighbors of X in \vec{G} and let $R_X = X \cap R$. First, assume that $|X| \geq d$. We have $\rho_{\vec{G}}(v) = d$ for all $v \in X - R_X$, and $\rho_{\vec{G}}(v) = d - 1$ for all $v \in R_X$. Thus,

$$d|X| - |R_X| = \sum_{v \in X} \rho_{\vec{G}}(v) \leq i_G(X \cup W) \leq d|X \cup W| - \binom{d+1}{2},$$

where the last inequality follows from (1). Hence, $d|W| \geq \binom{d+1}{2} - |R_X| \geq \frac{\binom{d+1}{2}}{2}$, and thus, $|W| \geq \frac{d+1}{4} > k - 1$.

Next, assume that $|X| \leq d - 1$. Note that we have

$$\rho_{\vec{G}}(X) = \left(\sum_{v \in X} \rho_{\vec{G}}(v) \right) - i_G(X) \geq d|X| - |R_X| - \binom{|X|}{2} = |X| \left(d - \frac{|X| - 1}{2} \right) - |R_X|.$$

Since each in-neighbor of X can send at most $|X|$ edges to X in \vec{G} , we also have $|X||W| \geq \rho_{\vec{G}}(X)$. It follows that

$$|W| \geq d - \frac{|X| - 1}{2} - \frac{|R_X|}{|X|} \geq d - \frac{d - 2}{2} - 1 = \frac{d}{2} \geq 2k - 2 \geq k,$$

as desired. □

We are now ready to prove Thomassen’s conjecture.

Proof of Theorem 1.2. Since the $k = 1$ case is settled by the theorem of Robbins, we may assume that $k \geq 2$. Let $d = 4k - 4$. We shall prove that if a graph $G = (V, E)$ has two edge-disjoint minimally d -rigid spanning subgraphs, then G has a k -connected orientation. Note that Theorem 1.6 guarantees the existence of such subgraphs if G is $(320k^2 - 560k + 240)$ -connected.

Suppose that G has two edge-disjoint minimally d -rigid spanning subgraphs G_1 and G_2 . Let us fix a set $R \subseteq V$ of vertices with $|R| = \binom{d+1}{2}$. We construct the orientation \vec{G} of G by defining the orientations of G_1 and G_2 , and then orienting the remaining edges arbitrarily. The orientation of G_1 is chosen to be a (d, R) -orientation. The orientation of G_2 is chosen to be a *reversed* (d, R) -orientation (in other words, its out-degrees respect the (d, R) -specification). By Lemma 4.2, these orientations exist.

It remains to show that the union \vec{G} of these oriented spanning subgraphs is k -connected. Suppose not; then there is a subset $S \subseteq V$ with $|S| \leq k - 1$ for which $\vec{G} - S$ is not strongly connected. This means that there is a set $X \subseteq V$ with $V - X - S \neq \emptyset$ and whose in-neighbors are all in S (and hence, the out-neighbors of the set $V - X - S$ are all in S). If $|X \cap R| \leq \binom{d+1}{2}$, then Lemma 4.3, applied to the (d, R) -orientation of G_1 , gives a contradiction. Otherwise, $|(V - X - S) \cap R| \leq \binom{d+1}{2}$, and we can apply the lemma to the orientation of G_2 to obtain a contradiction. □

5. Removable spanning trees

In this section, we prove Theorem 1.8. Our proof is an adaptation of the proof of Theorem 1.7 from [21] to the union of the generic d -dimensional rigidity matroid and the graphic matroid. The same method can be applied to the t -fold union of the rigidity matroid to show that sufficiently highly connected graphs are \mathcal{R}_d^t -rigid. However, the bound on the required connectivity obtained in this way is quadratic in t , in contrast to the linear bound given by Theorem 1.6.

We recall the following combinatorial lemma from [21]. For a positive integer n , we let $[n] = \{1, \dots, n\}$, and for a set X and a nonnegative integer i , we let $\binom{X}{i}$ denote the family of subsets of X of size i .

Lemma 5.1 [21, Lemma 2.5]. *Let n, r, ℓ, m be nonnegative integers with $\ell + 1 \leq m \leq n - 1$. Suppose that H_1, \dots, H_r are distinct proper subsets of $\{1, \dots, n\}$ with $|H_i \cap H_j| \leq \ell - 2$ for every $1 \leq i < j \leq r$. Then*

$$\left| \left\{ S \in \binom{[n]}{m} : \exists j \in \{1, \dots, r\}, S \subseteq H_j \right\} \right| \leq \binom{n-1}{m}.$$

We also recall the central construction of [21]. Let us fix $D \geq 2$. Let $G = (V, E)$ be a graph and $\pi = (v_1, \dots, v_n)$ an ordering of the vertices of G . Let us define

$$N_{G, \pi}(v_i) = \{u \in N_G(v_i) : u \text{ precedes } v_i \text{ in } \pi\},$$

and let $\deg_{G, \pi}(v_i)$ denote $|N_{G, \pi}(v_i)|$.

We construct a subgraph $G_\pi^D = (V, E_\pi^D)$ of G according to the following rules.

- (a) If $\deg_{G, \pi}(v_i) \leq D$, then in G_π^D , we connect v_i with every vertex of $N_{G, \pi}(v_i)$.
- (b) If $\deg_{G, \pi}(v_i) \geq D + 1$ and $N_{G, \pi}(v_i)$ induces a clique in G , then in G_π^D , we connect v_i with D vertices of $N_{G, \pi}(v_i)$.
- (c) If $\deg_{G, \pi}(v_i) \geq D + 1$ and $N_{G, \pi}(v_i)$ does not induce a clique in G , then in G_π^D , we connect v_i with $D + 1$ vertices of $N_{G, \pi}(v_i)$, including two vertices x and y that are not adjacent in G .

The following lemma is an adaptation of the main technical lemma in [21].

Lemma 5.2. *Let $G = (V, E)$ be a graph and let k_0, ℓ and D be positive integers. Suppose that for each $v \in V$, $\deg_G(v) \geq k_0$, $N_G(v)$ does not induce a clique in G , and that if H_1 and H_2 are the vertex sets of two different maximal cliques of $G[N_G(v)]$, then $|H_1 \cap H_2| \leq \ell - 2$. Let π be a uniformly random ordering of V . Finally, suppose that k_0, ℓ and D satisfy the inequality*

$$k_0^2 + k_0(1 - D(D + 1)) - \ell(\ell + 1) \geq 0.$$

Then $\mathbb{E}(|E_\pi^D|) \geq D|V|$.

Proof. We essentially repeat the proof of [21, Lemma 3.2], which is the special case when $\ell = D$ and $k_0 = D(D + 1)$. Fix $v \in V$, and let k denote $\deg_G(v)$. Lemma 2.4 implies that

$$\mathbb{E}(\min(\deg_{G, \pi}(v), D)) = D - \frac{1}{2} \cdot \frac{D(D + 1)}{k + 1}. \tag{3}$$

Let H_1, \dots, H_r denote the vertex sets of the maximal cliques of $G[N_G(v)]$. For each $i \in \{D + 1, \dots, k\}$, let

$$\begin{aligned} S_i &= \left\{ S \in \binom{N_G(v)}{i} : S \text{ induces a clique in } G \right\} \\ &= \left\{ S \in \binom{N_G(v)}{i} : \exists j \in \{1, \dots, r\}, S \subseteq H_j \right\}. \end{aligned}$$

Then $|S_k| = 0$ and, by Lemma 5.1, $|S_i| \leq \binom{k-1}{i}$ for each $i \in \{\ell + 1, \dots, k - 1\}$.

Let Q denote the event that $\deg_{G, \pi}(v) \geq D + 1$ and $N_{G, \pi}(v)$ does not induce a clique in G . If $\deg_{G, \pi}(v) = i \geq D + 1$, then Q occurs if and only if $N_{G, \pi}(v) \notin S_i$. Hence, for $i \geq \ell + 1$, we have

$$\mathbb{P}(Q | \deg_{G, \pi}(v) = i) = 1 - \frac{|S_i|}{\binom{k}{i}} \geq 1 - \frac{\binom{k-1}{i}}{\binom{k}{i}} = 1 - \frac{k - i}{k} = \frac{i}{k}.$$

It follows that

$$\begin{aligned} \mathbb{P}(Q) &\geq \sum_{i=\ell+1}^k \mathbb{P}(\deg_{G, \pi}(v) = i) \mathbb{P}(Q | \deg_{G, \pi}(v) = i) \\ &\geq \sum_{i=\ell+1}^k \frac{1}{k+1} \frac{i}{k} \\ &= \frac{1}{k(k+1)} \cdot \left(\binom{k+1}{2} - \binom{\ell+1}{2} \right) \\ &= \frac{1}{2} - \frac{1}{2} \cdot \frac{\ell(\ell+1)}{k(k+1)}. \end{aligned} \tag{4}$$

If Q does not occur, then $\deg_{G_\pi^D, \pi}(v) = \min(\deg_{G, \pi}(v), D)$. If Q occurs, then $\deg_{G_\pi^D, \pi}(v) = D + 1 = \min(\deg_{G, \pi}(v), D) + 1$. Hence, by combining (3) and (4), we obtain

$$\begin{aligned} \mathbb{E}(\deg_{G_\pi^D, \pi}(v)) &= \mathbb{E}(\min(\deg_{G, \pi}(v), D)) + \mathbb{P}(Q) \\ &\geq D + \frac{1}{2} - \frac{1}{2} \cdot \left(\frac{D(D+1)}{k+1} + \frac{\ell(\ell+1)}{k(k+1)} \right) \geq D, \end{aligned} \tag{5}$$

where the last inequality follows from the assumption that

$$k_0^2 + k_0(1 - D(D+1)) - \ell(\ell+1) \geq 0.$$

Thus,

$$\mathbb{E}(|E_\pi^D|) = \mathbb{E} \left(\sum_{v \in V} \deg_{G_\pi^D, \pi}(v) \right) = \sum_{v \in V} \mathbb{E}(\deg_{G_\pi^D, \pi}(v)) \geq D|V|. \quad \square$$

For the rest of the section, we let $\mathcal{M}_d(G)$ denote the union of $\mathcal{R}_d(G)$ and $\mathcal{R}_1(G)$ (i.e., the graphic matroid of G), for every graph G . Let $r_{\mathcal{M}_d}(G)$ denote the rank of $\mathcal{M}_d(G)$. We define $G = (V, E)$ to be \mathcal{M}_d -independent if $r_{\mathcal{M}_d}(G) = |E|$, and \mathcal{M}_d -rigid if $r_{\mathcal{M}_d}(G) = r_{\mathcal{M}_d}(K(V))$. A pair of vertices $u, v \in V$ is \mathcal{M}_d -linked in G if $r_{\mathcal{M}_d}(G + uv) = r_{\mathcal{M}_d}(G)$.

Note that a graph is \mathcal{M}_d -independent if and only if it can be written as the edge-disjoint union of an \mathcal{R}_d -independent graph and a forest. It follows that \mathcal{M}_d -independence is preserved by the addition of vertices of degree $d + 1$, as well as under the $(d + 1)$ -dimensional edge split operation.

Lemma 5.3 (Adaptation of [21, Lemma 3.1]). *Let $G = (V, E)$ be a graph, and let $\pi = (v_1, \dots, v_n)$ be an ordering of the vertices of G . Suppose that every \mathcal{M}_d -linked pair in G is adjacent in G . Then G_π^{d+1} is \mathcal{M}_d -independent.*

Proof. We prove that $F_i = G_\pi^{d+1}[\{v_1, \dots, v_i\}]$ is \mathcal{M}_d -independent by induction on i . F_1 is a single vertex, which is \mathcal{M}_d -independent. Let us thus suppose that $2 \leq i \leq n$. If F_i is constructed from F_{i-1} according to rule (a) or (b), then it is obtained from F_{i-1} by the addition of a vertex of degree at most $d + 1$, and hence, F_i is \mathcal{M}_d -independent. Suppose that F_i is constructed from F_{i-1} according to rule (c), and let x, y be vertices as described in the rule. As x, y are nonadjacent in G , they are not \mathcal{M}_d -linked in G , and hence neither in F_{i-1} . This means that $F_{i-1} + xy$ is \mathcal{M}_d -independent. Since F_i is obtained from $F_{i-1} + xy$ by a $(d + 1)$ -dimensional edge split, it follows that F_i is also \mathcal{M}_d -independent, as claimed. \square

We shall also need an analogue of Lemma 2.1 for \mathcal{M}_d -rigid graphs.

Lemma 5.4. *Let a be the smallest integer for which $\binom{a+1}{2} \geq d$. If $n \geq d + a + 2$, then $r_{\mathcal{M}_d}(K_n) = (d + 1)n - \binom{d+1}{2} - 1$. Hence, an \mathcal{M}_d -rigid graph on at least $d + a + 2$ vertices contains edge-disjoint spanning subgraphs G_0 and T such that G_0 is d -rigid and T is a tree.*

Proof. We show that K_n contains edge-disjoint spanning subgraphs G_0 and T such that G_0 is d -rigid and T is a tree. We first assume $n = d + a + 2$. Let us label the vertices of K_n as $\{u_1, \dots, u_{a+1}, v_1, \dots, v_{d+1}\}$, and let us define the integers $t_0 = 0, t_i = \binom{i+1}{2}$ for $i \in \{1, \dots, a - 1\}$, and $t_a = d$.

We construct a spanning subgraph T of K_n by adding an edge between u_i and v_j for each $i \in \{1, \dots, a\}$ and $j \in \{t_{i-1} + 1, \dots, t_i\}$, and then adding the edges $v_{d+1}u_{a+1}$ and $u_i u_{a+1}$ for each $i \in \{1, \dots, a\}$. See Figure 1. It is easy to verify that T is a spanning tree of K_n . Note that for $i \in \{1, \dots, a - 1\}$, u_i has exactly i neighbors in T among v_1, \dots, v_d , while u_a has at most a such neighbors.

Let G_0 be the complement of T in K_n . Then G_0 contains the complete graph on $\{v_1, \dots, v_{d+1}\}$. Moreover, u_1 is adjacent with $\{v_2, \dots, v_{d+1}\}$, and similarly, for each $i \in \{2, \dots, a + 1\}$, the vertex u_i has at least d neighbors among $\{v_1, \dots, v_{d+1}, u_1, \dots, u_{i-1}\}$. It follows that G_0 can be constructed from a complete graph by the addition of vertices of degree d , and hence, it is d -rigid, as required.

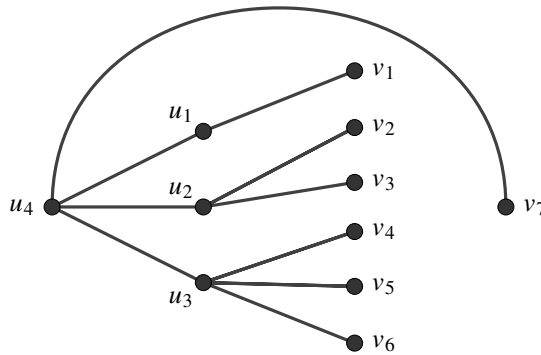


Figure 1. The construction of T in Lemma 5.4 in the case when $a = 3$ and $d = 6$. The complement of T can be obtained from a complete graph K_{d+1} on $d+1$ vertices by successively adding vertices of degree d .

The $n > d + a + 2$ case follows from the observation that for such n , K_n can be constructed from K_{d+a+2} by the addition of vertices of degree at least $d + 1$. □

Lemma 5.5. *Let a be the smallest integer for which $\binom{a+1}{2} \geq d$. If G_1 and G_2 are complete graphs with $|V(G_1) \cap V(G_2)| \geq d + a + 2$, then $G_1 \cup G_2$ is \mathcal{M}_d -rigid.*

Proof. Let $G = G_1 \cup G_2$. By Lemma 5.4, $G_1 \cap G_2$ contains edge-disjoint copies of a spanning tree and d -rigid spanning subgraph. We can use these subgraphs and the fact that each vertex of $V(G) - (V(G_1) \cap V(G_2))$ has at least $d + 1$ neighbors in $V(G_1) \cap V(G_2)$ to construct edge-disjoint copies of a spanning tree and a d -rigid spanning subgraph in G . □

Proof of Theorem 1.8. Since in the $d \in \{1, 2\}$ case we have stronger bounds from the theorem of Nash-Williams [13] and Tutte [20], and Theorem 1.4, respectively, we may suppose that $d \geq 3$. (The proof also works for $d \in \{1, 2\}$, but the bound we obtain is slightly weaker.) Let $c = d^2 + 3d + 5$.

Suppose, for a contradiction, that $G = (V, E)$ is a c -connected graph that is not \mathcal{M}_d -rigid. We may assume that G has the least possible number of vertices among all such graphs. We may also assume that G has the largest number of edges among all such graphs on $|V|$ vertices. Then, for each $v \in V$, $N_G(v)$ does not induce a clique in G , for otherwise deleting v would result in a smaller counterexample. (As we noted before, deleting a vertex whose neighbor set is a clique preserves k -connectivity unless the graph is a complete graph on $k + 1$ vertices.) Furthermore, every \mathcal{M}_d -linked pair is adjacent in G , for otherwise connecting a nonadjacent \mathcal{M}_d -linked pair by an edge would result in a counterexample with more edges. In particular, the \mathcal{M}_d -rigid induced subgraphs of G are complete.

Let $\ell = d + a + 2$, where

$$a = \left\lceil \sqrt{2d + \frac{1}{4}} - \frac{1}{2} \right\rceil.$$

A short calculation shows that $\binom{a+1}{2} \geq d$. Consider a vertex $v \in V$, and let H_1, H_2 be the vertex sets of two different maximal cliques of $G[N_G(v)]$. Then $G[H_1 \cup H_2 \cup \{v\}]$ is non-complete and hence not \mathcal{M}_d -rigid. It follows from Lemma 5.5 that

$$|(H_1 \cup \{v\}) \cap (H_2 \cup \{v\})| \leq \ell - 1,$$

and thus, $|H_1 \cap H_2| \leq \ell - 2$.

Hence, we may apply Lemma 5.2 with $D = d + 1, \ell = d + a + 2$, and $k_0 = d^2 + 3d + 5$. (It is a straightforward, although tedious, calculation to check that these numbers satisfy the condition in the statement of Lemma 5.2, under the condition that $d \geq 3$.) It follows that if π is a uniformly random ordering of V , then $\mathbb{E}(|E_\pi^{d+1}|) \geq (d + 1)|V|$. This implies that there exists some ordering π_0 of V for which $|E_{\pi_0}^{d+1}| \geq (d + 1)|V|$. Moreover, by Lemma 5.3, $G_{\pi_0}^{d+1}$ is \mathcal{M}_d -independent. But this is impossible, since an \mathcal{M}_d -independent graph on at least d vertices can have at most $(d + 1)|V| - \binom{d+1}{2} - 1$ edges.

Hence, every c -connected graph is \mathcal{M}_d -rigid. Combining this with Lemma 5.4 (using the observation that $c + 1 \geq d + a + 2$), we deduce that every c -connected graph contains edge-disjoint copies of a spanning tree and a d -rigid spanning subgraph, as required. \square

6. Concluding remarks

As we noted in the introduction, we believe that the bound in Theorem 1.6 can be replaced by $t \cdot d(d + 1)$. The following lemma shows that this would be best possible. For full generality, we state it for arbitrary unions of rigidity matroids.

Lemma 6.1. *Let d_1, \dots, d_k be a collection of positive integers and define*

$$K = \left(\sum_{i=1}^k d_i(d_i + 1) \right) - 1.$$

There exist infinitely many K -connected graphs G that do not contain edge-disjoint spanning subgraphs G_1, \dots, G_k such that G_i is d_i -rigid for each $i \in \{1, \dots, k\}$.

Proof. The proof follows the construction of Lovász and Yemini [11] for 5-connected graphs that are not 2-rigid. Let $G = (V_0, E_0)$ be a K -regular K -connected graph on $2s$ vertices, where s is a large integer to be determined later. Let $G = (V, E)$ be the graph obtained from G' by splitting every vertex into K vertices of degree one, and then adding a complete graph G_v on the K vertices corresponding to v , for each $v \in V_0$. It is not difficult to verify that G is K -connected.

Let $\mathcal{M} = (E, r)$ denote the union of $\mathcal{R}_{d_i}(G)$ for $i \in \{1, \dots, k\}$. Since we have $E(G) = E_0 \cup \bigcup_{v \in V_0} E(G_v)$, we obtain

$$\begin{aligned} r(E) &\leq |E_0| + \sum_{v \in V} r(E(G_v)) \leq sK + 2s \sum_{i=1}^k \left(d_i K - \binom{d_i + 1}{2} \right) \\ &= 2sK \left(\left(\sum_{i=1}^k d_i \right) + \frac{1}{2} - \frac{1}{2} \cdot \frac{K + 1}{K} \right) \\ &= |V| \left(\left(\sum_{i=1}^k d_i \right) + \frac{1}{2} - \frac{1}{2} \cdot \frac{K + 1}{K} \right). \end{aligned}$$

If s is sufficiently large, then the right-hand side is less than $\sum_{i=1}^k \left(d_i |V| - \binom{d_i + 1}{2} \right)$, and hence, G cannot contain edge-disjoint d_i -rigid spanning subgraphs for $i \in \{1, \dots, k\}$. \square

Finally, we briefly consider the algorithmic aspects of our results. Theorem 1.6 is equivalent to the statement that if a graph G is sufficiently highly connected, then there exist t disjoint bases of the generic d -dimensional rigidity matroid $\mathcal{R}_d(G)$. It is known that one can construct a random matrix that is a linear representation of the (t -fold union of) $\mathcal{R}_d(G)$ with high probability. We can use this fact and one of the several polynomial-time algorithms for matroid partition to obtain a randomized algorithm that finds, in expected polynomial time, a packing of t edge-disjoint d -rigid spanning subgraphs in graphs

satisfying the condition of Theorem 1.6. A similar approach can be used in the case of Theorem 1.8 to find a packing of a d -rigid spanning subgraph and a spanning tree in suitably highly connected graphs.

We note that our proof of Theorem 1.2 is algorithmic in the sense that, given a packing of two $(4k-4)$ -rigid spanning subgraphs in a graph, it can be used to explicitly construct a k -connected orientation of the graph. Combined with the ideas outlined in the previous paragraph, we obtain a randomized polynomial time algorithm for finding a k -connected orientation of a sufficiently highly connected graph.

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