COMPOSITIO MATHEMATICA

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EST. 1865

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Abstract

Let K be a finite extension of $\mathbb{Q}_p$ and let $\bar{\rho}$ be a continuous, absolutely irreducible representation of its absolute Galois group with values in a finite field of characteristic p. We prove that the Galois representations that become crystalline of a fixed regular weight after an abelian extension are Zariski-dense in the generic fiber of the universal deformation ring of $\bar{\rho}$. In fact we deduce this from a similar density result for the space of trianguline representations. This uses an embedding of eigenvarieties for unitary groups into the spaces of trianguline representations as well as the corresponding density claim for eigenvarieties as a global input.

1. Introduction

The density of crystalline representations in the generic fiber of a local deformation ring plays an important role in the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and was proven by Colmez [Co08] and Kisin [Ki10] for two-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. This density statement was generalized by Nakamura [Na10] and Chenevier [Ch13] to the case of two-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ for finite extensions K of $\mathbb{Q}_p$ and to the case of d-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ respectively, and the general case was finally treated in [Na11].

In this paper we prove a slightly different density result in the generic fiber of a local deformation ring. The above density statements make heavy use of the fact that the Hodge–Tate weights of the crystalline representations may vary arbitrarily. In contrast to this case, we fix the Hodge–Tate weights but vary the level, or, more specifically, we allow finite (abelian) ramification and allow the representation to be potentially crystalline (more precisely, crystabelline).

Note that this density statement is of a different nature than the density of crystalline representations. The density of crystalline representations holds true in the rigid generic fiber $(\mathrm{Spf}\, R_{\bar{r}})^{\mathrm{rig}}$ of the universal deformation ring $R_{\bar{r}}$ of a given residual $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-representation $\bar{r}$. In contrast to this result, the density of potentially crystalline representations of fixed weight only holds true in the 'algebraic' generic fiber $\mathrm{Spec}(R_{\bar{r}}[1/p])$, as the set of representations with fixed (generalized) Hodge–Tate weights is Zariski-closed in the rigid generic fiber $(\mathrm{Spf}\, R_{\bar{r}})^{\mathrm{rig}}$.

In the special case of two-dimensional potentially Barsotti–Tate representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ our result gives a positive answer to a question of Colmez [Co08].

A proof of this result (for two-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$), using the p-adic local Langlands correspondence, was announced previously by Emerton and Paškūnas. Our approach does not make use of such a correspondence and works in all dimensions and for arbitrary finite extensions of $\mathbb{Q}_p$. We were motivated by the case of two-dimensional potentially Barsotti–Tate representations and possible applications to patching techniques, as for these representations an automorphy lifting theorem is known [Ki09].

More precisely, our results are as follows. Let K be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{G}_K = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ denote its absolute Galois group. Fix an absolutely irreducible continuous representation $\bar{r} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ with values in a finite extension $\mathbb{F}$ of $\mathbb{F}_p$. As the representation is assumed to be absolutely irreducible, the universal deformation ring $R_{\bar{r}}$ of $\bar{r}$ exists.

THEOREM 1.1. *Assume that $p \nmid 2d$ and $\bar{r} \not\cong \bar{r} \otimes \varepsilon$, where ε is the cyclotomic character. Let $\mathbf{k} = (k_{i,\sigma}) \in \prod_{\sigma:K \hookrightarrow \bar{\mathbb{Q}}_p} \mathbb{Z}^d$ be a regular weight. Then the representations that are crystabelline of labeled Hodge–Tate weight $\mathbf{k}$ are Zariski-dense in $\mathrm{Spec}\, R_{\bar{r}}[1/p]$.*

Let us briefly comment on the assumption that the residual representation $\bar{r}$ is absolutely irreducible. In the paper we need to identify the deformation space of the representation $\bar{r}$ with the deformation space of the associated pseudo-character, as the global families of Galois representations that we use usually are pseudo-characters. This identification works for absolutely irreducible representations. Even though it is known for some reducible representations as well, we restricted ourselves to the case of irreducible residual representations. Using framed deformations as in [BHS14] it is possible to resolve this issue anyway. Moreover, we can always assure that the we can globalize the situation in the case of an absolutely irreducible representation, which is essential to our method.

Similarly to the proof of density of crystalline representations, we use a so-called *space of trianguline representations $X(\bar{r})$*. This space should be seen as a local Galois-theoretic counterpart of an eigenvariety of Iwahori level. Indeed, it was shown in [He12b] that certain eigenvarieties embed into a space of trianguline representations in the case $K = \mathbb{Q}_p$. This result is generalized to the case of an arbitrary extension K of $\mathbb{Q}_p$ in § 3.2 below. In fact we prove the following density result for eigenvarieties which might be of independent interest.

Let E be an imaginary quadratic extension of a totally real field F such that $[F : \mathbb{Q}]$ is even and let G be a definite unitary group over F which is quasi-split at all finite places. Let Y be an eigenvariety for a certain set of automorphic representations of $G(\mathbb{A}_F)$ as in [Ch09, 3] which comes along with a Galois pseudo-character interpolating the Galois representations attached to the automorphic representations at the classical points of Y. Given an absolutely irreducible residual representation $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to \mathrm{GL}_d(\mathbb{F})$ there is an open and closed subspace $Y_{\bar{\rho}} \subset Y$ where the pseudo-character reduces to (the pseudo-character attached to) $\bar{\rho}$ modulo p. This gives rise to a map $Y_{\bar{\rho}} \to (\mathrm{Spf}\, R_{\bar{\rho}})^{\mathrm{rig}}$ to the rigid generic fiber of the universal deformation ring $R_{\bar{\rho}}$ of $\bar{\rho}$.

THEOREM 1.2. *Fix an algebraic irreducible representation W of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$. Let $f \in R_{\bar{\rho}}$ such that f vanishes on all classical points $z \in Y_{\bar{\rho}}$ corresponding to irreducible automorphic representations Π with $\Pi_\infty = W$. Then f vanishes in $\Gamma(Y_{\bar{\rho}}, \mathcal{O}_Y)$.*

We prove Theorem 1.1 by extending Theorem 1.2 to the space of trianguline representations $X(\bar{\rho}_{w_0})$, using a map $f : Y_{\bar{\rho}} \to X(\bar{\rho}_{w_0})$ constructed in Theorem 3.5 below. Here $\bar{\rho}_{w_0}$ is the restriction of $\bar{\rho}$ to the decomposition group at some place w_0 of E dividing p. The second step in the proof of Theorem 1.1 then is to globalize the situation following [GK14] and [EG14].

Namely, one realizes a given residual representation $\bar{r} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ as the restriction to the decomposition group at w_0 of a $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$-representation arising from an automorphic representation of $G(\mathbb{A}_F)$. Using Theorem 1.2, we are then able to prove that an element of $R_{\bar{\rho}_{w_0}}$ vanishing at all crystabelline points of Hodge–Tate weight $\mathbf{k}$ has to vanish on all those irreducible components of $X(\bar{\rho}_{w_0})$ that contain (the image of) an eigenvariety.

The final step is to use the Zariski density of the image of the space of trianguline representations in the deformation space, which is the main result of [Na10] and [Ch13]. However, we have to refine this density statement replacing this space of trianguline representations by the union of its irreducible components containing automorphic points of finite slope, i.e. we prove the following theorem (see the body of the paper for a more precise definition of $X(\bar{\rho}, W_\infty)^{\mathrm{aut}}$ and $Y(W_\infty, S, e)_{\bar{\rho}}$).

THEOREM 1.3. *Let $X(\bar{\rho}, W_\infty)^{\mathrm{aut}}$ denote the union of those components of the space of trianguline representations that contain the image of some eigenvariety $Y(W_\infty, S, e)_{\bar{\rho}}$ of non-specified level S away from p. Then $X(\bar{\rho}, W_\infty)^{\mathrm{aut}}$ has Zariski-dense image in the rigid analytic generic fiber of the universal deformation ring of $\bar{\rho}_{w_0}$.*

In order to prove this theorem, we have to prove that this union of components contains sufficiently many crystalline points which are non-critical and whose all refinements are non-critical *and* stay in this particular union of irreducible components. We then reduce this existence to the proof of the fact that such *generic* crystalline points form a Zariski-open subset of the *scheme* parametrizing crystalline representations of fixed Hodge–Tate weights together with a density statement of automorphic points in a union of irreducible components of this space. The first of these two facts is proved using the existence of an universal Breuil-Kisin module on such a space and the second using the theory of Taylor–Wiles–Kisin systems.

Notation

We fix the following notation. Let $\bar{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$, $|\cdot|$ and val the norm and valuation on $\bar{\mathbb{Q}}_p$ extending the p-adic norm and valuation of $\mathbb{Q}_p$. Let $K \subset \bar{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ and let K_0 denote the maximal unramified subextension of $\mathbb{Q}_p$ in K. We fix a compatible system $\epsilon_n \in \bar{\mathbb{Q}}_p$ of p^nth roots of unity. Let $K_n = K(\epsilon_n) \subset \bar{\mathbb{Q}}_p$, $K_\infty = \bigcup_n K_n$ and K_0' the maximal unramified subextension of $\mathbb{Q}_p$ in K_∞. We will write $\mathcal{G}_L = \mathrm{Gal}(\bar{\mathbb{Q}}_p/L)$ for any subfield $L \subset \bar{\mathbb{Q}}_p$. Finally, we write $\Gamma = \Gamma_K = \mathrm{Gal}(K_\infty/K)$. We define the Hodge–Tate weights of a de Rham representation as the opposite of the gaps of the filtration on the covariant de Rham functor, so that the Hodge–Tate weight of the cyclotomic character is $+1$.

We choose a uniformizer $\varpi \in \mathcal{O}_K$ and normalize the reciprocity isomorphism $\mathrm{rec}_K : K^\times \to W_K^{\mathrm{ab}}$ of local class field theory such that ϖ is mapped to a geometric Frobenius automorphism. Here W_K^{ab} is the abelization of the Weil group $W_K \subset \mathcal{G}_K$ and the reciprocity map allows us to identify $\mathcal{O}_K^\times$ with a subgroup of $\mathcal{G}_K^{\mathrm{ab}}$, the maximal abelian quotient of $\mathcal{G}_K$. Further, we write $\varepsilon : \mathcal{G}_K \to Z_p^\times$ for the cyclotomic character.

Given a crystalline (respectively, semi-stable) representation $\rho : \mathcal{G}_K \to \mathrm{GL}_d(\bar{\mathbb{Q}}_p)$, we write $D_{\mathrm{cris}}(\rho)$ (respectively, $D_{\mathrm{st}}(\rho)$) for the filtered φ-module (respectively, (φ, N)-module) associated to ρ by Fontaine (cf. [Fo94a]). Further, we write $\mathrm{WD}(D_{\mathrm{cris}}(\rho))$ and $\mathrm{WD}(D_{\mathrm{st}}(\rho))$ for the Weil–Deligne representations associated respectively to $D_{\mathrm{cris}}(\rho)$ and $D_{\mathrm{st}}(\rho)$ by the recipe of Fontaine in [Fo94b]. A similar notation is used for potentially crystalline (respectively, potentially semi-stable representations).

If R is a complete local noetherian $\mathbb{Z}_p$-algebra with finite residue field (i.e. a quotient of $\mathbb{Z}_p[\![T_1, \ldots, T_m]\!]$ for some m), we write $(\mathrm{Spf}\, R)^{\mathrm{rig}}$ for the rigid analytic fiber of the formal spectrum

of R and often refer to it as the generic fiber of $\operatorname{Spf} R$ in the sense of Berthelot. Moreover, we will always use the expression 'rigid analytic spaces' for rigid analytic spaces locally of finite type over $\mathbb{Q}_p$.

If F is a number field, we will denote by $\mathbb{A}_F$ (respectively, $\mathbb{A}_{F,f}$ or $\mathbb{A}_{F,f}^p$) its ring of adeles (respectively, of finite adeles or of finite adeles outside of the places dividing p).

2. The space of trianguline representations

Let X be a rigid analytic space and recall the definition of the sheaf of relative Robba rings $\mathcal{R}_X = \mathcal{R}_{X,K}$ and $\mathcal{R}_X^+ = \mathcal{R}_{X,K}^+$ for K. If the base field K is understood we will omit the subscript K from the notation. This is the sheaf of functions that converge on the product of X with some boundary part of the open unit disc over K_0'; see [He12a, 2.2] or [KPX14, Definition 2.2.3], for example.[1] If $X = \operatorname{Sp} L$ for a finite extension L of $\mathbb{Q}_p$ we will write $\mathcal{R}_L = \mathcal{R}_{L,K}$ for (the global sections of) this sheaf. This sheaf of rings is endowed with a continuous $\mathcal{O}_X$-linear ring homomorphism $\varphi : \mathcal{R}_X \to \mathcal{R}_X$ and a continuous $\mathcal{O}_X$-linear action of the group Γ. Recall that a (φ, Γ)-module over a rigid space X consists of an $\mathcal{R}_X$-module D that is locally on X finite free over $\mathcal{R}_X$ together with a φ-linear isomorphism $\Phi : D \to D$ and a semi-linear Γ-action commuting with Φ.

Let us write $\mathbb{U}_L$ for the open unit disc over a p-adic field L and $\mathbb{U}_{r,L} \subset \mathbb{U}_L$ for the admissible open subspace of points of absolute value less than or equal to r for some $r \in p^{\mathbb{Q}} \cap [0, 1)$. Given such an r, we write $\mathcal{R}_X^r$ for the sheaf

$$X \supset U \longmapsto \Gamma(U \times \mathbb{U}_{r,K_0}, \mathcal{O}_{U \times \mathbb{U}_{r,K_0}})$$

and we write $\mathcal{R}_X^+$ for the sheaf $\mathcal{R}_X^0$ of functions converging on the product $X \times \mathbb{U}_{K_0}$.

Given a family of $\mathcal{G}_K$-representations $\mathcal{V}$ over a rigid space X, the work of Berger and Colmez [BC08] and Kedlaya and Liu [KL10] associates to $\mathcal{V}$ a (φ, Γ)-module $\mathbf{D}_{\mathrm{rig}}^\dagger(\mathcal{V})$ over $\mathcal{R}_X$.

Given a (φ, Γ)-module D over X, we write $H_{\varphi,\Gamma}^*(D)$ for the cohomology of the complex

$$C_{\varphi,\Gamma}^\bullet(D) = [D^\Delta \xrightarrow{\varphi - \mathrm{id}, \gamma - \mathrm{id}} D^\Delta \oplus D^\Delta \xrightarrow{(\mathrm{id} - \gamma) \oplus (\varphi - \mathrm{id})} D^\Delta],$$

where $\Delta \subset \Gamma$ is the p-torsion subgroup of Γ and $\gamma \in \Gamma/\Delta$ is a topological generator. It is known that the cohomology sheaves $H_{\varphi,\Gamma}^i(D)$ are coherent $\mathcal{O}_X$-modules for $i = 0, 1, 2$; see [KPX14, Theorem 4.4.5].

2.1 The parameters

In this section, we recall the construction of the space (φ, Γ)-modules of rank 1 over $\mathcal{R}$ essentially following [Co08]. This is first step toward a construction of the *trianguline space*.

Let $\mathcal{W} = \operatorname{Hom}_{\mathrm{cont}}(\mathcal{O}_K^\times, \mathbb{G}_m(-))$ be the *weight space* of K. This functor on the category of rigid analytic spaces is representable by the generic fiber of $\operatorname{Spf} \mathbb{Z}_p[\![\mathcal{O}_K^\times]\!]$. Further, let $\mathcal{T} = \operatorname{Hom}_{\mathrm{cont}}(K^\times, \mathbb{G}_m(-))$. There is a natural projection $\mathcal{T} \to \mathcal{W}$ given by restriction to $\mathcal{O}_K^\times$. The choice of the uniformizer ϖ gives rise to a section of this projection and identifies $\mathcal{T}$ with $\mathbb{G}_m \times \mathcal{W}$ via $\delta \mapsto (\delta(\varpi), \delta|_{\mathcal{O}_K^\times})$. It follows that $\mathcal{T}$ is representable by a rigid analytic space.

We recall how the (φ, Γ)-modules of rank 1 over a rigid space X are classified by $\mathcal{T}(X)$; see [KPX14, Theorem 6.2.14] (and also [Na09, 1.4] for the case $X = \operatorname{Sp} L$ in the context of B-pairs).

Let X be a rigid space over $\mathbb{Q}_p$ and let D be a rank-1 family of K-filtered φ-modules over X. Recall that this is a coherent $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0$-module that is locally on X free of rank 1 together

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[1] The sheaf $\mathcal{R}_X$ is denoted by $\mathscr{B}_{X,\mathrm{rig}}^\dagger$ in [He12a].

with an id $\otimes \varphi$-linear automorphism $\Phi : D \to D$ and a filtration $\mathrm{Fil}^\bullet$ on $D_K = D \otimes_{K_0} K$ by $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K$ submodules that are locally on X direct summands as $\mathcal{O}_X$-modules.

Assume that X is affinoid and defined over the normalization K^{norm} of K inside $\bar{\mathbb{Q}}_p$, and assume that D is free. Then such a K-filtered φ-module may be described as follows. There exist a uniquely determined $a \in \Gamma(X, \mathcal{O}_X^\times)$ and uniquely determined $k_\sigma \in \mathbb{Z}$ for each embedding $\sigma : K \hookrightarrow K^{\mathrm{norm}}$ such that $D \cong D(a; (k_\sigma)_\sigma)$ where $\Phi^{[K_0:\mathbb{Q}_p]}$ acts on $D(a; (k_\sigma)_\sigma)$ via multiplication with $a \otimes \mathrm{id} \in \Gamma(X, \mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0)^\times$ and

$$(\mathrm{gr}^i D_K) \otimes_{\mathcal{O}_X \otimes_{\mathbb{Q}_p} K, \mathrm{id} \otimes \sigma} \mathcal{O}_X \cong \begin{cases} 0 & i \neq -k_\sigma \\ \mathcal{O}_X & i = -k_\sigma \end{cases} \tag{2.1}$$

for all embeddings $\sigma : K \hookrightarrow \bar{\mathbb{Q}}_p$.

Given $k_\sigma \in \mathbb{Z}$ for each embedding $\sigma : K \hookrightarrow \bar{\mathbb{Q}}_p$ we consider the following special K-filtered φ-module $D((k_\sigma)_\sigma)$ over $L = K^{\mathrm{norm}}$ whose filtration is given by (2.1) and which has a basis on which $\Phi^{[K_0:\mathbb{Q}_p]}$ acts via multiplication with $\prod_\sigma \sigma(\varpi)^{k_\sigma}$.

Let X be an affinoid space defined over K^{norm} and let D be a K-filtered φ-module over X. Associated to D there is a (φ, Γ)-module $\mathcal{R}_X(D)$ of rank 1 as follows. We write

$$D = D(a; (0)_\sigma) \otimes_{K_0} D((k_\sigma)_\sigma)$$

for some $k_\sigma \in \mathbb{Z}$ and $a \in \Gamma(X, \mathcal{O}_X^\times)$ and define

$$\mathcal{R}_X(D(a; (0)_\sigma)) = D(a; (0)_\sigma) \otimes_{\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0} \mathcal{R}_X,$$

where φ acts diagonally and Γ acts trivially on the first factor.

Given $\sigma : K \hookrightarrow K^{\mathrm{norm}}$, we write $t_\sigma \in \mathcal{R}_{K^{\mathrm{norm}}}^+$ for a period of the character $\sigma \circ \chi_{\mathrm{LT}}$ from $\mathcal{G}_K$ into $\mathcal{O}_{K^{\mathrm{norm}}}^\times$ as in [KPX14, Notation 6.2.7]. Then $\prod_\sigma t_\sigma$ is equal to the usual period of the cyclotomic character $t = \log([(1, \epsilon_1, \epsilon_2, \dots)]) \in \mathcal{R}_{\mathbb{Q}_p}^+ \subset \mathcal{R}_{K^{\mathrm{norm}}}^+$, up to multiplication by an invertible element of $\mathcal{R}_{K^{\mathrm{norm}}}^+$.

Using this notation, we write

$$\mathcal{R}_X(D((k_\sigma)_\sigma)) = \prod_\sigma t_\sigma^{k_\sigma} \mathcal{R}_X \subset \mathcal{R}_X\left[\frac{1}{t}\right]$$

with action of φ and Γ inherited from $\mathcal{R}_X[1/t]$. Finally, we set

$$\mathcal{R}_X(D) = \mathcal{R}_X(D(a; (0)_\sigma)) \otimes_{\mathcal{R}_X} \mathcal{R}_X(D((k_\sigma)_\sigma)).$$

More generally, let $\delta : K^\times \to \Gamma(X, \mathcal{O}_X^\times)$ be a continuous character. Then there is a (φ, Γ)-module $\mathcal{R}_X(\delta)$ of rank 1 associated to δ as follows; cf. [KPX14, Construction 6.1.4]. Write $\delta = \delta_1 \delta_2$ with $\delta_1|_{\mathcal{O}_K^\times} = 1$ and such that δ_2 extends to a character of $\mathcal{G}_K$. Then we set

$$\mathcal{R}_X(\delta) = \mathcal{R}_X(D(\delta_1(\varpi), (0)_\sigma)) \otimes_{\mathcal{R}_X} \mathbf{D}_{\mathrm{rig}}^\dagger(\delta_2).$$

We write $\delta(D)$ for the character of $K^\times$ such that $\mathcal{R}_X(\delta(D)) = \mathcal{R}_X(D)$.

Further, we can check that, given $k_\sigma \in \mathbb{Z}$, the character $\delta(D((k_\sigma)_\sigma))$ associated with the K-filtered φ-module $D((k_\sigma)_\sigma)$ is given by $\delta((k_\sigma)_\sigma)$, i.e. by the character $z \mapsto \prod_\sigma \sigma(z)^{k_\sigma}$. We write $\delta_{\mathcal{W}}((k_\sigma)_\sigma)$ for the restriction of $\delta((k_\sigma)_\sigma)$ to $\mathcal{O}_K^\times$. Finally, we have

$$\varepsilon \circ \mathrm{rec}_K = \delta(1, \dots, 1)|\delta(1, \dots, 1)|.$$

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Lemma 2.1. *Let $\delta \in \mathcal{T}(L)$ for a local field $L \supset K^{\mathrm{norm}}$. Then*

$$H^0_{\varphi,\Gamma}(\mathcal{R}_L(\delta)) \neq 0 \iff \delta = \delta((-k_\sigma)_\sigma) \quad \text{for some } (k_\sigma)_\sigma \in \prod_{\sigma: K \hookrightarrow L} \mathbb{Z}_{\geqslant 0},$$

$$H^2_{\varphi,\Gamma}(\mathcal{R}_L(\delta)) \neq 0 \iff \delta = \varepsilon \cdot \delta((k_\sigma)_\sigma) \quad \text{for some } (k_\sigma)_\sigma \in \prod_{\sigma: K \hookrightarrow L} \mathbb{Z}_{\geqslant 0}.$$

In particular, $H^1_{\varphi,\Gamma}(\mathcal{R}_L(\delta))$ has L-dimension $[K : \mathbb{Q}_p]$ if and only if

$$\delta \notin \left\{ \delta((-k_\sigma)_\sigma), \varepsilon \cdot \delta((k_\sigma)_\sigma) \mid (k_\sigma) \in \prod_\sigma \mathbb{Z}_{\geqslant 0} \right\}.$$

Proof. This is [KPX14, Proposition 6.2.8]; cf. also [Na09, Proposition 2.14]. □

Notation 2.2. (i) Let us write $\mathcal{T}_{\mathrm{reg}} \subset \mathcal{T}$ for the set of *regular characters*, i.e the characters

$$\delta \notin \left\{ \delta((-k_\sigma)_\sigma), \varepsilon \cdot \delta((k_\sigma)_\sigma) \mid (k_\sigma) \in \prod_\sigma \mathbb{Z}_{\geqslant 0} \right\}.$$

(ii) Let $d > 0$ be an integer. We define the set of *regular*[2] parameters $\mathcal{T}^d_{\mathrm{reg}} \subset \mathcal{T}^d$ to be the set of $(\delta_1, \ldots, \delta_d) \in \mathcal{T}^d$ such that $\delta_i/\delta_j \in \mathcal{T}_{\mathrm{reg}}$ for $i \neq j$. Note that by construction $\mathcal{T}^d_{\mathrm{reg}} \neq (\mathcal{T}_{\mathrm{reg}})^d$.

(iii) A weight $\delta \in \mathcal{W}(\bar{\mathbb{Q}}_p)$ is *algebraic of weight* $(k_\sigma)_\sigma$ if $\delta = \delta_{\mathcal{W}}((k_\sigma)_\sigma)$.

(iv) We say that $\delta \in \mathcal{W}(\bar{\mathbb{Q}}_p)$ is *locally algebraic of weight* $(k_\sigma)_\sigma$ if $\delta \otimes \delta_{\mathcal{W}}((-k_\sigma)_\sigma)$ becomes trivial after restricting to some open subgroup of $\mathcal{O}_K^\times$.

(v) An element $\mathbf{k} = (k_{\sigma,i})_\sigma \in \prod_\sigma \mathbb{Z}^d$ is called *strongly dominant* if $k_{\sigma,1} > k_{\sigma,2} > \cdots > k_{\sigma,d}$ for all σ.

(vi) Let $\mathbf{k} = (k_{\sigma,i})_\sigma \in \prod_\sigma \mathbb{Z}^d$. We say that $(\delta_1, \ldots, \delta_d) \in \mathcal{W}^d(\bar{\mathbb{Q}}_p)$ is *algebraic of weight* $\mathbf{k}$ if δ_i is algebraic of weight $(k_{\sigma,i})_\sigma$. An element $\delta = (\delta_1, \ldots, \delta_d) \in \mathcal{W}^d(\bar{\mathbb{Q}}_p)$ is called *locally algebraic of weight* $\mathbf{k}$ if δ_i is locally algebraic of weight $(k_{\sigma,i})_\sigma$. The set of weights that are locally algebraic of weight $\mathbf{k}$ is denoted by $\mathcal{W}^d_{\mathbf{k},\mathrm{la}} \subset \mathcal{W}^d(\bar{\mathbb{Q}}_p)$.

2.2 The space of trianguline (φ, Γ)-modules

We extend the construction of the space of trianguline (φ, Γ)-modules with regular parameters given in [Ch13] to our context. This extension relies on results of Kedlaya, Pottharst and Xiao [KPX14]. There are similar results (concerning triangulations in families) due to Liu [Liu15].

Let d be a positive integer and consider the functor $\mathcal{S}^\square_d$ that assigns to a rigid space X the isomorphism classes of quadruples $(D, \mathrm{Fil}_\bullet(D), \delta, \nu)$, where D is a (φ, Γ)-module over $\mathcal{R}_X$ and $\mathrm{Fil}_\bullet(D)$ is a filtration of D by sub-$\mathcal{R}_X$-modules that are stable under the action of φ and Γ and that are locally on X direct summands as $\mathcal{R}_X$-modules. Further, $\delta \in \mathcal{T}^d_{\mathrm{reg}}(X)$ and $\nu = (\nu_1, \ldots, \nu_d)$ is a collection of trivializations

$$\nu_i : \mathrm{Fil}_{i+1}(D)/\mathrm{Fil}_i(D) \xrightarrow{\cong} \mathcal{R}_X(\delta_i).$$

Similarly, we consider a variant of this functor parametrizing non-split extensions (cf. [He12b]), that is, the functor $\mathcal{S}^{\mathrm{ns}}_d$ that assigns to X the set of isomorphism classes of quadruples $(D, \mathrm{Fil}_\bullet(D), \delta, \nu_d)$ where D and $\mathrm{Fil}_\bullet(D)$ are as above and $\delta \in \mathcal{T}^d_{\mathrm{reg}}$ such that locally on X there exist short exact sequences

$$0 \longrightarrow \mathrm{Fil}_i(D) \longrightarrow \mathrm{Fil}_{i+1}(D) \longrightarrow \mathcal{R}_X(\delta_i) \longrightarrow 0$$

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[2] Note that this definition is a bit more restrictive that the definition of [Ch13].

that are non-split at every geometric point $x \in X$ as a sequence of (φ, Γ)-modules. Finally, ν_d is a trivialization

$$\nu_d : \mathrm{Fil}_{d+1}(D)/\mathrm{Fil}_d(D) \stackrel{\cong}{\longrightarrow} \mathcal{R}_X(\delta_d).$$

PROPOSITION 2.3. *Let* $\delta = (\delta_1, \ldots, \delta_d) \in (\mathcal{T}_{\mathrm{reg}})^d(X)$ *for some rigid space* X *and let* D *be a successive extension of the* $\mathcal{R}_X(\delta_i)$. *Then* $H^i_{\varphi, \Gamma}(D)$ *is a locally free* $\mathcal{O}_X$-*module of rank*

$$\begin{cases} 0 & \text{if } i = 0, 2 \\ d[K : \mathbb{Q}_p] & \text{if } i = 1 \end{cases}$$

and the canonical morphisms

$$H^i_{\varphi, \Gamma}(D) \otimes k(x) \longrightarrow H^i_{\varphi, \Gamma}(D \otimes k(x))$$

are isomorphisms for all $x \in X$.

Proof. By the base change formula [KPX14, Theorem 4.4.3] the second claim is a direct consequence of the first. Let us assume that the claim is true for $d = 1$. By way of induction we consider the short exact sequence

$$0 \longrightarrow \mathcal{R}_X(\delta_1) \longrightarrow D \longrightarrow D' \longrightarrow 0 \tag{2.2}$$

and we may assume that $H^1_{\varphi, \Gamma}(\mathcal{R}_X(\delta_1))$ and $H^1_{\varphi, \Gamma}(D')$ are locally free of rank $[K : \mathbb{Q}_p]$ and $(d-1)[K : \mathbb{Q}_p]$ respectively, and

$$H^0_{\varphi, \Gamma}(\mathcal{R}_X(\delta_1)) = H^2_{\varphi, \Gamma}(\mathcal{R}_X(\delta_1)) = H^0_{\varphi, \Gamma}(D') = H^2_{\varphi, \Gamma}(D') = 0.$$

Then the claim follows from the long exact cohomology sequence of (2.2).

It remains to prove the claim for $d = 1$. Using the base change formula [KPX14, Theorem 4.4.3(2)], we are reduced to considering the universal case $X = \mathcal{T}_{\mathrm{reg}}$ and $D = \mathcal{R}(\delta)$, where δ is the universal character on $\mathcal{T}_{\mathrm{reg}}$. Let $x \in X$. The base change formula induces a spectral sequence

$$E_2^{j, -i} = \mathrm{Tor}_i^X(H^j_{\varphi, \Gamma}(D), k(x)) \Rightarrow H^{j-i}_{\varphi, \Gamma}(D \otimes k(x)).$$

The fact that $H^j_{\varphi, \Gamma}(D) = 0$ for $j > 2$, together with the spectral sequence, implies that

$$H^2_{\varphi, \Gamma}(D) \otimes k(x) \longrightarrow H^2_{\varphi, \Gamma}(D \otimes k(x))$$

is an isomorphism. By Lemma 2.1 the target of this isomorphism vanishes and hence so does $H^2_{\varphi, \Gamma}(D) \otimes k(x)$. But as $H^2_{\varphi, \Gamma}(D)$ is a coherent sheaf by [KPX14, Theorem 4.4.2] (and again using the base change formula in the flat case), it follows that $H^2_{\varphi, \Gamma}(D) = 0$, as we have shown that all its fibers vanish.

As $H^2_{\varphi, \Gamma}(D) = 0$, it follows that $\mathrm{Tor}_1^X(H^2_{\varphi, \Gamma}(D), k(x)) = 0$ for all $x \in X$ and hence the above spectral sequence implies that

$$H^1_{\varphi, \Gamma}(D) \otimes k(x) \longrightarrow H^1_{\varphi, \Gamma}(D \otimes k(x))$$

is an isomorphism for all $x \in X$. By Lemma 2.1 we conclude that $H^1_{\varphi, \Gamma}(D) \otimes k(x)$ has dimension $[K : \mathbb{Q}_p]$ for all $x \in X$. Being a coherent sheaf of constant rank on a reduced space, it has to be locally free automatically.

Finally, we deduce that $E_2^{j, -i} = 0$ for $i, j > 0$ and any fixed $x \in X$. It follows that

$$H^0_{\varphi, \Gamma}(D) \otimes k(x) \longrightarrow H^0_{\varphi, \Gamma}(D \otimes k(x))$$

is an isomorphism for all $x \in X$. As the right-hand side vanishes for all $x \in X$ by Lemma 2.1 we again conclude that $H^0_{\varphi, \Gamma}(D) = 0$. $\qquad\square$

Theorem 2.4. (i) *The functors* $\mathcal{S}_d^{\square}$ *and* $\mathcal{S}_d^{\mathrm{ns}}$ *are representable by rigid spaces.*

(ii) *The map* $\mathcal{S}_d^{\square} \to \mathcal{T}_{\mathrm{reg}}^d$ *is smooth of relative dimension* $(d(d-1)/2)[K:\mathbb{Q}_p]$.

(iii) *The map* $\mathcal{S}_d^{\mathrm{ns}} \to \mathcal{T}_{\mathrm{reg}}^d$ *is smooth and proper and*

$$\dim \mathcal{S}_d^{\mathrm{ns}} = 1 + [K:\mathbb{Q}_p]\left(\frac{d(d+1)}{2}\right).$$

Proof. The proof is the same as the proof of [Ch13, Theorem 3.3] or [He12b, Proposition 2.3]. For the convenience of the reader we give a short sketch. The case $d = 1$ is settled by $\mathcal{S}_1^{\square} = \mathcal{S}_1^{\mathrm{ns}} = \mathcal{T}$. Now assume that $\mathcal{S}_{d-1}^{\square}$ and $\mathcal{S}_{d-1}^{\mathrm{ns}}$ are constructed with universal objects $\mathcal{D}_{d-1}^{\square}$ and $\mathcal{D}_{d-1}^{\mathrm{ns}}$, respectively. Let $U \subset \mathcal{T} \times \mathcal{S}_{d-1}^{\square}$ and $V \subset \mathcal{S}_{d-1}^{\mathrm{ns}} \times \mathcal{T}$ respectively be the preimage of $\mathcal{T}_{\mathrm{reg}}^d \subset \mathcal{T} \times \mathcal{T}_{\mathrm{reg}}^{d-1}$ under the canonical projection. Then Proposition 2.3 implies that

$$\mathscr{E}xt_{\mathcal{R}_U}^1(\mathcal{R}_U(\delta_1), \mathcal{D}_{d-1}^{\square}) = H_{\varphi,\Gamma}^1(\mathcal{D}_{d-1}^{\square}(\delta_1^{-1}))$$

and

$$\mathscr{E}xt_{\mathcal{R}_V}^1(\mathcal{R}_V(\delta_1), \mathcal{D}_{d-1}^{\mathrm{ns}}) = H_{\varphi,\Gamma}^1(\mathcal{D}_{d-1}^{\mathrm{ns}}(\delta_1^{-1}))$$

respectively are vector bundles of rank $(d-1)[K:\mathbb{Q}_p]$. As the Tate duality is a perfect pairing [KPX14, Theorem 4.4.5] we find that also

$$\mathcal{M}_U = \mathscr{E}xt_{\mathcal{R}_U}^1(\mathcal{D}_{d-1}^{\square}, \mathcal{R}_U(\delta_1))$$

and

$$\mathcal{M}_V = \mathscr{E}xt_{\mathcal{R}_V}^1(\mathcal{D}_{d-1}^{\mathrm{ns}}, \mathcal{R}_V(\delta_1))$$

respectively are vector bundles of rank $(d-1)[K:\mathbb{Q}_p]$. Now $\mathcal{S}_d^{\square} = \underline{\mathrm{Spec}}_U(\mathrm{Sym}^\bullet \mathcal{M}_U^\vee)$ is the geometric vector bundle over U associated to $\mathcal{M}_U$ while $\mathcal{S}_d^{\mathrm{ns}} = \mathbb{P}_V(\mathcal{M}_V^\vee)$ is the projective bundle associated to $\mathcal{M}_V$. Here $\underline{\mathrm{Spec}}$ is the relative spectrum in the sense of [Con06, 2.2] and, given a vector bundle $\mathcal{E}$, the projective bundle $\mathbb{P}(\mathcal{E}) = \underline{\mathrm{Proj}}(\mathrm{Sym}^\bullet \mathcal{E})$ is the relative Proj in the sense of [Con06, 2.3].

The universal object $\mathcal{D}_d^{\square}$ then is the universal extension

$$0 \longrightarrow \mathcal{R}(\delta_1) \longrightarrow \mathcal{D}_d^{\square} \longrightarrow \mathcal{D}_{d-1}^{\square} \longrightarrow 0$$

over $\mathcal{S}_d^{\square}$. In the non-split context consider the geometric vector bundle $\tilde{\mathcal{S}}_d^{\mathrm{ns}} = \underline{\mathrm{Spec}}_V(\mathrm{Sym}^\bullet \mathcal{M}_V^\vee)$ over V associated to $\mathcal{M}_V$. Then there is a universal extension

$$0 \longrightarrow \mathcal{R}(\delta_1) \longrightarrow \tilde{\mathcal{D}}_d^{\mathrm{ns}} \longrightarrow \mathcal{D}_{d-1}^{\mathrm{ns}} \longrightarrow 0$$

over $\tilde{\mathcal{S}}_d^{\mathrm{ns}}$. Consider the open subspace $\tilde{\mathcal{S}}_d^{\mathrm{ns}} \backslash V \subset \tilde{\mathcal{S}}_d^{\mathrm{ns}}$ where the image of the zero section $0 : V \hookrightarrow \tilde{\mathcal{S}}_d^{\mathrm{ns}}$ is removed. This space carries a natural action of $\mathbb{G}_m$ and this action lifts to an action on the restriction of $\tilde{\mathcal{D}}_d^{\mathrm{ns}}$ to $\tilde{\mathcal{S}}_d^{\mathrm{ns}} \backslash V$ by acting on $\mathcal{R}(\delta_1)$. Hence $\tilde{\mathcal{D}}_d^{\mathrm{ns}}$ descends to a (φ, Γ)-module $\mathcal{D}_d^{\mathrm{ns}}$ over $\mathbb{P}(\mathcal{M}_V^\vee) = (\tilde{\mathcal{S}}_d^{\mathrm{ns}} \backslash V)/\mathbb{G}_m$.

The computation of the dimension follows from the construction as well as the fact that $\mathcal{S}_d^{\square}$ is smooth over $\mathcal{T}_{\mathrm{reg}}^d$ and $\mathcal{S}_d^{\mathrm{ns}}$ is smooth and proper over $\mathcal{T}_{\mathrm{reg}}^d$. $\square$

Let $r \in p^{\mathbb{Q}} \cap [0, 1)$ and consider the ring $\mathcal{R}^r = \mathcal{R}_{\mathbb{Q}_p}^r$. If $n \gg 0$, then there is a morphism $\mathcal{R}^r \to K_n[\![t]\!]$ where the ring $K_n[\![t]\!]$ is viewed as the complete local ring at the point of $\mathbb{U}_{r,K_0'}$

corresponding to (the $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K_0')$-orbit of) $1 - \epsilon_n$. If D_r is a (φ, Γ)-module defined over $\mathcal{R}_L^r$ for some p-adic field L and some $r \in p^{\mathbb{Q}} \cap [0, 1)$ and if $D = D_r \otimes_{\mathcal{R}_L^r} \mathcal{R}_L$, then we define

$$D_{\mathrm{dR}}(D) = (K_\infty \otimes_{K_n} K_n((t)) \otimes_{\mathcal{R}_{\mathbb{Q}_p}^r} D_r)^\Gamma,$$

$$\mathrm{Fil}^i D_{\mathrm{dR}}(D) = (K_\infty \otimes_{K_n} t^i K_n[\![t]\!] \otimes_{\mathcal{R}_{\mathbb{Q}_p}^r} D_r)^\Gamma.$$

If L contains K^{norm}, then $D_{\mathrm{dR}}(D)$ splits up into a product $D_{\mathrm{dR}}(D) = \prod_\sigma D_{\mathrm{dR},\sigma}(D)$ and $\mathrm{Fil}^i D_{\mathrm{dR}}(D) = \prod_\sigma \mathrm{Fil}_\sigma^i D_{\mathrm{dR}}(D)$ splits up into filtrations $\mathrm{Fil}_\sigma^i D_{\mathrm{dR}}(D)$ of the $D_{\mathrm{dR},\sigma}(D)$.

As in [BC09, Definition 2.2.10], we can extend the notions of being crystalline or de Rham to (φ, Γ)-modules.

DEFINITION 2.5. Let L be a finite extension of $\mathbb{Q}_p$ and let D be a (φ, Γ)-module of rank d over $\mathcal{R}_L = \mathcal{R}_{L,K}$. Assume that $D = D_r \otimes_{\mathcal{R}_L^r} \mathcal{R}_L$ for some (φ, Γ)-module D_r defined over $\mathcal{R}_L^r$ and some $r < 1$.

(i) The (φ, Γ)-module D is called *de Rham* if $D_{\mathrm{dR}}(D)$ is a free $L \otimes_{\mathbb{Q}_p} K$-module of rank d.

(ii) The module D is called *crystalline* if $D_{\mathrm{cris}}(D) = D[1/t]^\Gamma$ is free of rank d over $L \otimes_{\mathbb{Q}_p} K_0$.

(iii) The module D is called *crystabelline* if $D \otimes_{\mathcal{R}_{L,K}} \mathcal{R}_{L,K'}$ is crystalline for some abelian extension K' of K.

The following proposition is the generalization of [BC09, Proposition 2.3.4][3] to our context and its proof is essentially the same as in the case $K = \mathbb{Q}_p$.

PROPOSITION 2.6. *Let L be a finite extension of $\mathbb{Q}_p$ containing K^{norm} and let D be a (φ, Γ)-module of rank d over $\mathcal{R}_L$ that is a successive extension of rank-1 objects $\mathcal{R}_L(\delta_i)$. Assume that $(\delta_1|_{\mathcal{O}_K^\times}, \ldots, \delta_d|_{\mathcal{O}_K^\times})$ is locally algebraic of weight $\mathbf{k} = (k_{\sigma,i})$ for some strongly dominant weight $\mathbf{k}$. Then D is de Rham with labeled Hodge–Tate weights $\mathbf{k}$.*

Proof. Write $R = \bigcup_n (L \otimes_{\mathbb{Q}_p} K_n[\![t]\!])$ for the moment. We proceed by induction on d. The case $d = 1$ easily follows from the fact that we may twist by characters δ such that $\delta|_{\mathcal{O}_K^\times} = 1$ and the fact that the claim is true for characters of $\mathcal{G}_K^{\mathrm{ab}} = \hat{\mathbb{Z}} \times \mathcal{O}_K^\times$ by the definition of locally algebraic weights.

For simplicity we only treat the case $p \neq 2$. In this case the group Γ is pro-cyclic. In the case $p = 2$ one concludes similarly after taking invariants under the 2-power torsion subgroup Δ of Γ.

Let $\gamma \in \Gamma$ be a topological generator and let $\Gamma_0 = \langle \gamma \rangle \subset \Gamma$. We will prove by induction on $1 \leqslant j \leqslant d$ that $(\prod_\sigma t^{k_{\sigma,j}} R \otimes_{\mathcal{R}_{\mathbb{Q}_p}^r} \mathrm{Fil}_j(D)_r)^{\Gamma_0} \neq 0$ for big enough r. Suppose we have the result for $j \leqslant d - 1$. One deduces from the short exact sequence

$$0 \to \prod_\sigma \mathrm{Fil}_\sigma^{-k_{\sigma,d}} D_{\mathrm{dR}}(\mathrm{Fil}_{d-1}(D)) \to \prod_\sigma \mathrm{Fil}_\sigma^{-k_\sigma} D_{\mathrm{dR}}(D) \to \prod_\sigma \mathrm{Fil}_\sigma^{-k_\sigma} \mathcal{R}_L(\delta_d)$$

$$\to H^1\left(\Gamma_0, \mathrm{Fil}_{d-1}(D)_r \otimes_{\mathcal{R}_{\mathbb{Q}_p}^r} \left(\prod_\sigma t_\sigma^{-k_{\sigma,d}}\right) R\right)$$

that it suffices to show that

$$H^1\left(\Gamma_0, \mathrm{Fil}_{d-1}(D)_r \otimes_{\mathcal{R}_{\mathbb{Q}_p}^r} \left(\prod_\sigma t_\sigma^{-k_{\sigma,d}}\right) R\right) = 0.$$

---

[3] Note that Bellaiche and Chenevier use a different sign convention.

To do so we are reduced to computing the first cohomology of $(\prod_\sigma t_\sigma^{-k_\sigma})R \otimes_L R(\delta_i)$ for $i \leqslant d-1$. However, this cohomology vanishes, as $\prod_\sigma t_\sigma^{-k_{\sigma,d}} R \otimes \delta_j \simeq \prod_\sigma t_\sigma^{-k_{\sigma,d}+k_{\sigma,j}} R \otimes R(\delta)$ with δ a finite order character, and $-k_{\sigma,d}+k_{\sigma,j} > 0$ for all σ and hence

$$H^1\left(\Gamma_0, \left(\prod_\sigma t_\sigma^{i_\sigma}\right)R\right) = \left(\prod_\sigma t_\sigma^{i_\sigma}\right)R \Big/ (\gamma-1)\left(\prod_\sigma t_\sigma^{i_\sigma}\right)R = 0$$

if $i_\sigma > 0$ for all embeddings σ. It follows that D has to be de Rham. $\qquad\square$

Let $\omega_d^\square : \mathcal{S}_d^\square \to \mathcal{W}^d$ and $\omega_d : \mathcal{S}_d^{\mathrm{ns}} \to \mathcal{W}^d$ denote the projection to the weight space.

COROLLARY 2.7. (i) *Let $w \in \mathcal{W}_{\mathrm{alg}}^d$ be a strongly dominant algebraic weight. Then there is a non-empty Zariski-open subset $Z_{\mathrm{cris}}(w) \subset \omega_d^{-1}(w)$ such that all points of $Z_{\mathrm{cris}}(w)$ are crystalline (φ,Γ)-modules.*

(ii) *Let $\mathbf{k} \in \prod_\sigma \mathbb{Z}^d$ be strongly dominant and let $w \in \mathcal{W}_{\mathbf{k},\mathrm{la}}^d$ be a locally algebraic weight. Then there is a non-empty Zariski-open subset $Z_{\mathrm{pcris}}(w) \subset \omega_d^{-1}(w)$ such that all points of $Z_{\mathrm{pcris}}(w)$ are crystabelline.*

Proof. The proof is identical to that of [Ch13, Theorem 3.14].

(i) As $w = (w_1,\ldots,w_d) \in \mathcal{W}^d$ is algebraic we may write $\mathcal{R}(\delta_i) = \mathcal{R}(D(\delta_i))$ for any character $\delta_i \in \mathcal{T}$ restricting to w_i on $\mathcal{O}_K^\times$. We write $D(\delta_i) = D(a_i, (k_\sigma)_\sigma)$ with $a_i = \delta_i(\sigma)\prod_\sigma \sigma(\varpi)^{-k_{\sigma,i}}$ and let

$$Z_{\mathrm{cris}}(w) = \left\{(D, \mathrm{Fil}_\bullet(D), \delta, \nu_d) \in \omega_d^{-1}(w) \;\middle|\; \frac{a_i}{a_j} \neq p^{\pm[K_0:\mathbb{Q}_p]} \text{ for } i < j\right\}.$$

Let D be a (φ,Γ)-module associated to some point in $Z_{\mathrm{cris}}(w)$. Then D is de Rham by Proposition 2.6 above and hence potentially semi-stable. As w is algebraic, D is a successive extension of crystalline (φ,Γ)-modules, hence it has to be semi-stable and we have to assure that the monodromy acts trivially. However, the monodromy operator maps the Φ^f-eigenspace with eigenvalue λ to the Φ^f-eigenspace with eigenvalue $p^f\lambda$, where $f = [K_0 : \mathbb{Q}_p]$. As the possible eigenvalues of Φ^f are given by the a_i the monodromy has to be trivial.

(ii) Let $w^{\mathrm{sm}} = w \cdot \delta(-\mathbf{k}) = (w_1,\ldots,w_n)$ and let K' be the abelian extension of K corresponding to $\bigcap_{1\leqslant i\leqslant n}\ker w_i \subset \mathcal{O}_K^\times \hookrightarrow \mathcal{G}_K^{\mathrm{ab}}$. Then the same argument as above yields a Zariski-open subset $Z_{\mathrm{pcris}}(w) \subset \omega_d^{-1}(w)$ whose points are (φ,Γ)-modules that become crystalline over K'. $\qquad\square$

Remark 2.8. In the case $d = 2$ the second claim of the corollary above applies, for example, to the weight $\mathbf{k} = ((0,1)_\sigma)$, i.e. to potentially Barsotti–Tate representations. If $d > 2$, a corresponding statement for potentially Barsotti–Tate representations can no longer hold true. There are no strongly dominant weights for potentially Barsotti–Tate representations in this case and the dimension of the flag variety parametrizing the Hodge filtrations for weights that are not strongly dominant will be strictly smaller than the dimension of the space of extensions of (φ,Γ)-modules.

LEMMA 2.9. *Let $L \subset \bar{\mathbb{Q}}_p$ be a finite extension of the Galois closure K^{norm} of K inside $\bar{\mathbb{Q}}_p$ and let V be a crystalline representation of $\mathcal{G}_K$ on a d-dimensional L-vector space with labeled Hodge–Tate weights $\mathbf{k} = (k_{\sigma,i})$ such that $\mathbf{k}$ is strongly dominant. Let $D = D_{\mathrm{cris}}(V)$ and assume*

that the $[K_0 : \mathbb{Q}_p]$th power of the crystalline Frobenius Φ_{cris} on $\mathrm{WD}(D) = D \otimes_{L \otimes_{\mathbb{Q}_p} K_0} \bar{\mathbb{Q}}_p$ is semi-simple. Let $\lambda_1, \ldots, \lambda_d$ be an ordering of its eigenvalues and assume that for all σ one has

$$\frac{[K : \mathbb{Q}_p]}{[K_0 : \mathbb{Q}_p]} \mathrm{val}(\lambda_1) < -k_{\sigma,2} - \sum_{\sigma' \neq \sigma} k_{\sigma',1},$$

$$\frac{[K : \mathbb{Q}_p]}{[K_0 : \mathbb{Q}_p]} \mathrm{val}(\lambda_1 \ldots \lambda_i) < -k_{\sigma,i+1} - \sum_{\sigma' \neq \sigma} k_{\sigma',i} - \sum_{\sigma'} \sum_{j=1}^{i-1} k_{\sigma',j}. \tag{2.3}$$

Then there is a triangulation $0 = D_0 \subset D_1 \subset \cdots \subset D_d = D_{\mathrm{rig}}^\dagger(V)$ such that $D_i/D_{i-1} \cong D(\delta_i)$ with $\delta_i : K^\times \to L^\times$ given by

$$\delta_i|_{\mathcal{O}_K^\times} : z \longmapsto \prod_\sigma \sigma(z)^{k_{\sigma,i}},$$

$$\delta_i(\varpi) = \lambda_i \prod_\sigma \sigma(\varpi)^{k_{\sigma,i}}.$$

Proof. Let $D_i \subset D_{\mathrm{rig}}^\dagger(V)$ be the filtration induced by a filtration $0 = D_0' \subset D_1' \subset \cdots \subset D_d' = D = D_{\mathrm{cris}}(V)$ by Φ_{cris}-stable subspaces such that the restriction of $\Phi_{\mathrm{cris}}^{[K_0:\mathbb{Q}_p]}$ to $\mathrm{WD}(D_i)$ has eigenvalues $\lambda_1, \ldots, \lambda_i$. Then D_i is stable under φ and Γ and we need to compute the graded pieces. However, the graded pieces are as claimed, if the filtration $D_\bullet'$ is in general position with all the Hodge filtrations $\mathrm{Fil}_\sigma^\bullet$, which is to say that

$$(D_i' \otimes_{K_0 \otimes L, \sigma \otimes \mathrm{id}} \bar{\mathbb{Q}}_p) \oplus (\mathrm{Fil}^{-k_{\sigma,i+1}} D_K \otimes_{K \otimes L, \sigma \otimes \mathrm{id}} \bar{\mathbb{Q}}_p) = D \otimes_{K_0 \otimes L, \sigma \otimes \mathrm{id}} \bar{\mathbb{Q}}_p = \mathrm{WD}(D).$$

One can easily see that this is assured by weak admissibility and condition (2.3). □

2.3 Construction of Galois representations

Let $\bar{\rho} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ be an absolutely irreducible continuous representation, where $\mathbb{F}$ is a finite field of characteristic p. Write $R_{\bar{\rho}}$ for the universal deformation ring of $\bar{\rho}$ and $\mathfrak{X}_{\bar{\rho}}$ for the generic fiber of $\mathrm{Spf}\, R_{\bar{\rho}}$ in the sense of Berthelot.

Recall that a pseudo-character $T : G \to R$ of a group G with values in a ring R is a map satisfying several axioms; see [BC09, 1.2.1] and the references cited therein, for example. If, moreover, G is a topological group and R is a topological ring, the pseudo-character T is called *continuous* if the map T is continuous (as a map of topological spaces).

Let X be a rigid space and let $T : \mathcal{G}_K \to \Gamma(X, \mathcal{O}_X)$ be a continuous pseudo-character of dimension d. We say that T has *residual type* $\bar{\rho}$ if for all $x \in X$ the semi-simple representation $\rho_x : \mathcal{G}_K \to \mathrm{GL}_d(\mathcal{O}_{\bar{\mathbb{Q}}_p})$ with $\mathrm{tr}\, \rho_x = (T \otimes k(x)) \otimes_{k(x)} \bar{\mathbb{Q}}_p$ (which is uniquely determined up to conjugation) reduces to (the isomorphism class of) $\bar{\rho}$ modulo the maximal ideal of $\mathcal{O}_{\bar{\mathbb{Q}}_p}$.

Then the rigid space $\mathfrak{X}_{\bar{\rho}}$ represents the functor that assigns to a rigid space X the pseudo-characters $T : \mathcal{G}_K \to \Gamma(X, \mathcal{O}_X)$ of dimension d and residual type $\bar{\rho}$.

By [He12a, Theorem 5.2] there exists a natural rigid space $\mathcal{S}_d^{\mathrm{ns,adm}}$ which is étale over $\mathcal{S}_d^{\mathrm{ns}}$ and a vector bundle $\mathcal{V}$ on $\mathcal{S}_d^{\mathrm{ns,adm}}$ together with a continuous representation $\rho : \mathcal{G}_K \to \mathrm{GL}(\mathcal{V})$ such that $\mathbf{D}_{\mathrm{rig}}^\dagger(\mathcal{V})$ is the restriction of the universal trianguline (φ, Γ)-module. In the setup of adic spaces (cf. [Hu96]) the spaces $\mathcal{S}_d^{\mathrm{ns,adm}}$ is an open subspace of $\mathcal{S}_d^{\mathrm{ns}}$. In what follows we will embed the category of rigid spaces into the category of adic spaces as in [Hu96, 1.1.11].

Let us write $\mathcal{S}(\bar{\rho}) \subset \mathcal{S}_d^{\mathrm{ns,adm}}$ for the open and closed subspace where the pseudo-character $\mathrm{tr}\,\rho$ has residual type $\bar{\rho}$. Then we obtain a canonical map

$$\pi_{\bar{\rho}} : \mathcal{S}(\bar{\rho}) \longrightarrow \mathfrak{X}_{\bar{\rho}} \times \mathcal{T}_{\mathrm{reg}}^d.$$

It is clear, reasoning along the lines of [He12b, Proposition 3.10] for the case $K = \mathbb{Q}_p$, that this map is injective at the level of rigid analytic points; see Lemma 2.12 below.

DEFINITION 2.10. Let $X(\bar{\rho})$ be the Zariski closure of $\mathrm{Im}(\pi_{\bar{\rho}}) \subset \mathfrak{X}_{\bar{\rho}} \times \mathcal{T}_{\mathrm{reg}}^d$ in $\mathfrak{X}_{\bar{\rho}} \times \mathcal{T}^d$. This is a Zariski-closed subset of a rigid analytic space and we view it itself as an rigid analytic space with the induced reduced structure. The space $X(\bar{\rho})$ is called the *trianguline space* in the following.

THEOREM 2.11. *The space $X(\bar{\rho})$ is equidimensional of dimension*

$$\dim X(\bar{\rho}) = 1 + [K : \mathbb{Q}_p]\frac{d(d+1)}{2}.$$

If $x = (\rho, \delta) \in X(\bar{\rho}) \subset \mathfrak{X}_{\bar{\rho}} \times \mathcal{T}^d$ is a rigid analytic point, then the Galois representation ρ is trianguline. Moreover, the morphism $\pi_{\bar{\rho}} : \mathcal{S}(\bar{\rho}) \to X(\bar{\rho})$ is an isomorphism onto a Zariski-open and dense subspace of $X(\bar{\rho})$.

The image of $\pi_{\bar{\rho}}$ is then a Zariski-open and dense subset $X(\bar{\rho})^{\mathrm{reg}} \subset X(\bar{\rho})$, which we call the *regular trianguline space*.

For the proof of Theorem 2.11, we need some preliminary lemmas.

LEMMA 2.12. *Let $(\rho, \delta_1, \ldots, \delta_d)$ be a rigid point of $\mathfrak{X}_{\bar{\rho}} \times \mathcal{T}_{\mathrm{reg}}^d$. Then the (φ, Γ)-module $D_{\mathrm{rig}}^{\dagger}(\rho)$ has at most one triangulation of parameter $(\delta_1, \ldots, \delta_d)$. In particular, the map $\pi_{\bar{\rho}}$ is injective on rigid points.*

Proof. We are reduced to proving that if D is a trianguline (φ, Γ)-module having a triangulation whose successive subquotients are isomorphic to $\mathcal{R}_{L,K}(\delta_i)$, then this triangulation is unique. We can prove this statement by induction, the case where $d = 1$ being clear. Assume the claim proved for $d - 1 \geqslant 1$ and consider D a (φ, Γ)-module of parameter $(\delta_1, \ldots, \delta_d) \in \mathcal{T}_{\mathrm{reg}}^d$. Using the induction hypothesis, it is sufficient to prove that $\mathrm{Hom}_{(\varphi, \Gamma)}(\mathcal{R}_{L,K}(\delta_1), D)$ has L-dimension 1. This is a consequence of the fact that $\mathrm{End}_{(\varphi, \Gamma)}(\mathcal{R}_{L,K}(\delta_1))$ has L-dimension 1 and that $\mathrm{Hom}_{(\varphi, \Gamma)}(\mathcal{R}_{L,K}(\delta_1), \mathcal{R}_{L,K}(\delta_i)) = 0$ for $i \geqslant 2$, the last equality being a consequence of $(\delta_1, \ldots, \delta_d) \in \mathcal{T}_{\mathrm{reg}}^d$ and Lemma 2.1. $\qquad\square$

LEMMA 2.13. *Let X be an adic space of finite type over $\mathbb{Q}_p$ and let $U \subset X$ be an open subset. Suppose that U is constructible for the Zariski topology, i.e. U is a finite union of subsets that are locally closed for the Zariski topology. Then $U \subset X$ is Zariski-open.*

Proof. The claim is local on X and hence we may assume $X = \mathrm{Spa}(A, A^+)$ with (A, A^+) an affinoid Tate algebra topologically of finite type over $\mathbb{Q}_p$. Given $x \in X$, we write $\mathrm{supp}\, x$ for the support of the valuation defined by x. This support is a prime ideal and the corresponding map $f : X \to \mathrm{Spec}\, A$ is continuous and surjective (for surjectivity it is enough to consider the case of $\mathbb{Q}_p\langle T_1, \ldots, T_n\rangle$, where it is easily seen to be surjective). Moreover, a subset $U \subset X$ is Zariski-open (respectively, constructible for the Zariski topology) if $V = f^{-1}(\mathcal{V})$ for some $\mathcal{V} \subset \mathrm{Spec}\, A$ open (respectively, constructible).

Write $V = f^{-1}(\mathcal{V})$ for some constructible subset $\mathcal{V} \subset \mathrm{Spec}\, A$. It is enough to show that $\mathcal{V}$ is stable under generalization. Let $\mathfrak{p} \rightsquigarrow \mathfrak{q} = \mathrm{supp}\, x$ be a specialization[4] with $x \in U$. As U is open

---

[4] Note that specializations in $\mathrm{Spec}\, A$ are different from specializations in $\mathrm{Spa}(A, A^+)$. In fact the morphism $\mathrm{Spa}(A, A^+) \to \mathrm{Spec}\, A$ identifies points x and y such that x is a specialization of y in the sense of adic spaces.

there exists a neighborhood U' of x contained in $U = f^{-1}(\mathcal{U})$. However, then U' contains a point y with $\operatorname{supp} y = \mathfrak{p}$ and hence $\mathfrak{p} \in \mathcal{U}$: this comes down to showing that, if $\operatorname{Spa}(B, B^+) \hookrightarrow \operatorname{Spa}(A, A^+)$ is the inclusion of an affinoid subdomain with $\operatorname{supp} x \in \operatorname{Spa}(B, B^+)$, then $\operatorname{Spec} B$ contains all generalizations of $\mathfrak{p} = \operatorname{supp} x$ in $\operatorname{Spec} A$. However, this follows from the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\
\uparrow & & \uparrow \\
\operatorname{Spec} \hat{B}_\mathfrak{p} & \overset{=}{\longrightarrow} & \operatorname{Spec} \hat{A}_\mathfrak{p}
\end{array}
$$

and the fact that the image of $\operatorname{Spec} \hat{A}_\mathfrak{p} \to \operatorname{Spec} A$ contains all generalizations of the prime ideal $\mathfrak{p}$. □

LEMMA 2.14. *Let A be a Tate algebra over $\mathbb{Q}_p$ and let $f : X \to \operatorname{Spa}(A, A^+)$ be a projective morphism. Let $U \subset X$ be a Zariski-open subset. Then $f(U) \subset \operatorname{Spa}(A, A^+)$ is constructible for the Zariski topology.*

Proof. Let $g : \mathcal{X} \to \operatorname{Spec} A$ be a scheme of finite type over $\operatorname{Spec} A$. Then there is an analytification $g^{\mathrm{an}} : \mathcal{X}^{\mathrm{an}} \to \operatorname{Spa}(A, A^+)$ which can be written as a fiber product

$$
\begin{array}{ccc}
\mathcal{X}^{\mathrm{an}} & \longrightarrow & \mathcal{X} \\
g^{\mathrm{an}} \downarrow & & \downarrow g \\
\operatorname{Spa}(A, A^+) & \longrightarrow & \operatorname{Spec} A
\end{array}
$$

as in [Hu94, Proposition 3.8]. Here the morphism $\operatorname{Spa}(A, A^+) \to \operatorname{Spec} A$ maps a valuation to its support. By [Kö74, §3] there is also a functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ from the category $\operatorname{Coh}_\mathcal{X}$ of coherent sheaves on $\mathcal{X}$ to the category $\operatorname{Coh}_{\mathcal{X}^{\mathrm{an}}}$ of coherent sheaves on $\mathcal{X}^{\mathrm{an}}$. By [Kö74, §§4, 5] this functor is an equivalence of categories when g is projective.

Applying this equivalence to the sheaf of ideals of an embedding of X into some projective space over $\operatorname{Spa}(A, A^+)$, we find a projective morphism $g : \mathcal{X} \to \operatorname{Spec} A$ such that $f = g^{\mathrm{an}}$. Moreover, there exists a closed subscheme $\mathcal{Z} \subset \mathcal{X}$ such that $U = \mathcal{X}^{\mathrm{an}} \backslash \mathcal{Z}^{\mathrm{an}}$. Let us write $\mathcal{U} = \mathcal{X} \backslash \mathcal{Z}$; then we have $U = \mathcal{U}^{\mathrm{an}}$.

As the morphism g is a morphism of finite type between noetherian schemes the theorem of Chevalley (cf. [EGAIV, Theorem 1.8.4]) implies that there is a decomposition $\operatorname{Spec} A = \bigcup \mathcal{V}_i$ with $\mathcal{V}_i \subset \operatorname{Spec} A$ locally closed such that $g_i(\mathcal{U}_i) \subset \mathcal{V}_i$ is Zariski-open, where we write

$$
g_i = g|_{g^{-1}(\mathcal{V}_i)} : g^{-1}(\mathcal{V}_i) \longrightarrow \mathcal{V}_i
$$

and $\mathcal{U}_i = \mathcal{U} \cap g^{-1}(\mathcal{V}_i)$. As $\bigcup f(\mathcal{U}_i)^{\mathrm{an}}$ then is constructible for the Zariski topology it remains to show that $g_i^{\mathrm{an}}(\mathcal{U}_i^{\mathrm{an}}) = (g_i(\mathcal{U}_i))^{\mathrm{an}}$. However, this is easily verified. □

LEMMA 2.15. *Let $X = \operatorname{Spa}(A, A^+)$ be an affinoid adic space of finite type over $\mathbb{Q}_p$ and let $U \subset X$ be constructible for the Zariski topology such that U contains all rigid analytic points of X. Then $U = X$.*

Proof. As U is constructible for the Zariski topology we have $U = q^{-1}(\mathcal{U})$ for some constructible set $\mathcal{U} \subset \operatorname{Spec} A$, where $q : \operatorname{Spa}(A, A^+) \to \operatorname{Spec} A$ denotes the morphism mapping a valuation to its support. It follows that $\mathcal{U}$ contains all maximal ideals of A and hence $\mathcal{U} = \operatorname{Spec} A$ since A is a Jacobson ring. □

Proof of Theorem 2.11. Let us write $M \subset X(\bar{\rho})$ for the set of rigid analytic points in the image of $\pi_{\bar{\rho}}$. Further, we write ρ^{un} for the pullback of the universal $\mathcal{G}_K$-representation on $\mathfrak{X}_{\bar{\rho}}$ to $X(\bar{\rho})$. By Lemma 2.12, all points of M are strictly trianguline in the sense of [KPX14, Definition 6.3.1]. Then by [KPX14, Corollary 6.3.10] there exist a morphism

$$f : \tilde{X}(\bar{\rho}) \longrightarrow X(\bar{\rho})$$

and a filtration $\mathcal{F}_i$ of $\tilde{D} = \mathbf{D}^{\dagger}_{\mathrm{rig}}(f^* \rho^{\mathrm{un}})$ by submodules that are stable under the action of φ and Γ such that there are short exact sequences

$$0 \longrightarrow \mathcal{F}_i/\mathcal{F}_{i-1} \longrightarrow \mathcal{R}_{\tilde{X}(\bar{\rho})}(\delta_i) \otimes \mathcal{L}_i \longrightarrow \mathcal{M}_i \longrightarrow 0,$$

where the $\mathcal{L}_i$ are line bundles on $\tilde{X}(\bar{\rho})$ and $\mathcal{M}_i$ is supported on a Zariski-closed subset Z_i of $\tilde{X}(\bar{\rho})$ not containing any irreducible component of $\tilde{X}(\bar{\rho})$. Moreover, f is projective and birational (in fact it is the composition of the normalization of $X(\bar{\rho})$ with some blow-ups) and

$$f^{-1}(M) \subset \tilde{X}(\bar{\rho}) \Big\backslash \bigcup_i Z_i.$$

Let $U \subset \tilde{X}(\bar{\rho})$ be a Zariski-open and dense subset such that $U \subset \tilde{X}(\bar{\rho}) \backslash \bigcup_i Z_i$ and such that f induces an isomorphism of U onto its image in $X(\bar{\rho})$ (which we will again denote by U). For $x \in U$ we find $\mathcal{M}_i \otimes \kappa(x) = 0$ and hence $\mathcal{F}_i|_U$ is locally on U a direct factor of the $\mathcal{R}_U$-module $\tilde{D}$. Hence we find short exact sequences

$$0 \longrightarrow \mathcal{F}_{i-1} \otimes k(x) \longrightarrow \mathcal{F}_i \otimes k(x) \longrightarrow \mathcal{R}(\delta_i) \otimes k(x) \longrightarrow 0. \qquad (2.4)$$

Shrinking U if necessary, we may assume that all the extensions (2.4) for $x \in U$ are non-split and that $U \subset \mathfrak{X}_{\bar{\rho}} \times \mathcal{T}^d_{\mathrm{reg}}$. Then the filtration $\mathcal{F}_{\bullet}|_U$ defines a section

$$s : U \longrightarrow \mathcal{S}(\bar{\rho}) \qquad (2.5)$$

to $\pi_{\bar{\rho}}^{-1}(U) \longrightarrow U$. However, as $\pi_{\bar{\rho}}$ is separated this section is a closed immersion. On the other hand, $\pi_{\bar{\rho}}$ is injective on rigid analytic points and hence $\pi_{\bar{\rho}}^{-1}(U) \cong U$, as $\pi_{\bar{\rho}}^{-1}(U)$ is smooth and, in particular, reduced. It follows that U is equidimensional of the claimed dimension and hence so is $X(\bar{\rho})$ as U is Zariski-open and dense in $X(\bar{\rho})$. Moreover, [KPX14, Theorem 6.3.13] implies that all the representations $\rho^{\mathrm{un}} \otimes k(x)$ for rigid analytic points $x \in X(\bar{\rho})$ are trianguline.

We let

$$V \subset \Big(\tilde{X}(\bar{\rho}) \Big\backslash \bigcup_i Z_i \Big) \cap f^{-1}(\mathfrak{X}_{\bar{\rho}} \times \mathcal{T}^d_{\mathrm{reg}}) \qquad (2.6)$$

denote the Zariski-open subset of $\tilde{X}(\bar{\rho})$, where all the extensions (2.4) are non-split. Then the rigid analytic points in $f(V)$ are precisely the points of M, as both sets precisely consist of the non-split triauguline representations with parameters in $\mathcal{T}^d_{\mathrm{reg}}$. Moreover, for the same reasons as before, the morphism $U \to \mathcal{S}(\bar{\rho})$ from (2.5) extends to a morphism $s : V \to \mathcal{S}(\bar{\rho})$. In particular, we find $f(V) \subset \operatorname{Im} \pi_{\bar{\rho}}$.
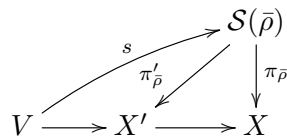
We will show in Proposition 2.17 below that the morphism $\pi_{\bar{\rho}} : \mathcal{S}(\bar{\rho}) \to X(\bar{\rho})$ is étale at rigid analytic points $y \in \mathcal{S}(\bar{\rho})$. Let us remark that the proof of that proposition does not use what follows. Let us finish the proof of the theorem assuming this claim. It then follows from [Hu96, Proposition 1.7.11] that $\pi_{\bar{\rho}}$ is étale as a morphism of adic space and hence $\operatorname{Im} \pi_{\bar{\rho}}$ is open by [Hu96, Proposition 1.7.8].

For $\mathrm{Spa}(A, A^+) \subset X(\bar{\rho})$ open affinoid, Lemma 2.14 implies that $f(V) \cap \mathrm{Spa}(A, A^+)$ is constructible. On the other hand, $f(V) = \pi_{\bar{\rho}}(s(V)) \subset \mathrm{Im}\,\pi_{\bar{\rho}}$ and hence restricting to some affinoid neighborhood of $x \in f(V)$ contained in $\mathrm{Im}\,\pi_{\bar{\rho}}$ Lemma 2.15 implies that $f(V)$ is open. It follows from Lemma 2.13 that $f(V) \subset X(\bar{\rho})$ is Zariski-open. We define $X(\bar{\rho})^{\mathrm{reg}}$ to be the Zariski-open subset $f(V)$.

It is left to show that the morphism $\pi_{\bar{\rho}} : \mathcal{S}(\bar{\rho}) \to X(\bar{\rho})^{\mathrm{reg}}$ is an isomorphism (still assuming that we already know that it is étale). As remarked above, the morphism $\pi_{\bar{\rho}}$ is injective on rigid analytic points and, in particular, (locally) quasi-finite as a morphism of rigid analytic spaces. Then [Hu96, Proposition 1.5.7] implies that $\pi_{\bar{\rho}}$ is (locally) quasi-finite as a morphism of adic spaces. Further, $\pi_{\bar{\rho}}$ is partially proper, as $\mathcal{S}(\bar{\rho})$ is partially proper (cf. [He12b, Proposition 3.6]) and $X(\bar{\rho})$ is separated. It follows from [Hu96, Proposition 1.5.6] that $\pi_{\bar{\rho}}$ is locally on $\mathcal{S}(\bar{\rho})$ and $X(\bar{\rho})$ of the form $g : Z \to T$ with g finite. Moreover, g then has to be finite étale and hence it is an isomorphism if it has degree 1. However, we have constructed above a section $s : U \to \mathcal{S}(\bar{\rho})$ to $\pi_{\bar{\rho}}$ on some Zariski-open and dense subset $U \subset X(\bar{\rho})$. As g is injective it has to be an isomorphism. $\qquad\square$

LEMMA 2.16. *Let $x \in \mathrm{Im}\,\pi_{\bar{\rho}}$ be a rigid point, Then the complete local ring $\hat{\mathcal{O}}_{X(\bar{\rho}),x}$ is a domain.*

Proof. Let $X = X(\bar{\rho})$. By [vdPSch95, Corollary 5], the ring $\mathcal{O}_{X,x}$ is a henselian local ring. It is then a consequence of [Ra70, Corollary 1] that $\mathcal{O}_{X,x}$ is a domain if and only if $\hat{\mathcal{O}}_{X,x}$ is a domain. To prove that $\mathcal{O}_{X,x}$ is a domain, it is sufficient to prove that, in the normalization $X' \to X$, the fiber over x is just one point. As the space $\mathcal{S}(\bar{\rho})$ is smooth, it is normal, and then the map $\pi_{\bar{\rho}}$ has a canonical factorization $\pi'_{\bar{\rho}}$ through X'. Let x' a point of X' over x, and $\tilde{x}$ a point of the canonical blow-up $\tilde{X} = \tilde{X}(\bar{\rho})$ over x'. Writing again M for the set of rigid analytic points in the image of $\pi_{\bar{\rho}}$, the construction of $f : \tilde{X} \to X$ implies that $f^{-1}(M) \subset V$, where V is the Zariski-open subset defined in (2.6). In particular, this means $\tilde{x} \in V$. As constructed above, we have a morphism $s : V \to \mathcal{S}(\bar{\rho})$. To summarize, we have the following diagram.

As the outer triangle is commutative, by the universal property of the normalization, we must have $g = \pi'_{\bar{\rho}} \circ s$, proving that $x' = g(\tilde{x}) = \pi'_{\bar{\rho}}(s(\tilde{x}))$ is in the image of $\pi'_{\bar{\rho}}$. As the map $\pi_{\bar{\rho}}$ is injective, the map $\pi'_{\bar{\rho}}$ must be injective too. We have proved that the entire fiber of $X' \to X$ over x is contained in the image of $\pi'_{\bar{\rho}}$. Hence we can conclude that there is only one point of X' over x. $\qquad\square$

PROPOSITION 2.17. *The morphism $\pi_{\bar{\rho}} : \mathcal{S}(\bar{\rho}) \to X(\bar{\rho})$ is étale at rigid analytic points.*

Proof. Let $x = (\rho, \delta) \in \mathrm{Im}\,\pi_{\bar{\rho}} \subset \mathfrak{X}_{\bar{\rho}} \times \mathcal{T}_{\mathrm{reg}}^d$ and write $R_\rho = \hat{\mathcal{O}}_{\mathfrak{X}_{\bar{\rho}},r}$ for the complete local ring at ρ. This is the complete local noetherian ring pro-representing the deformation functor of ρ. Further, the choice of $y \in \mathcal{S}(\bar{\rho})$ mapping to x defines a triangulation $\mathrm{Fil}_\bullet$ on $\mathbf{D}_{\mathrm{rig}}^\dagger(\rho)$. Then the functor of (trianguline) deformations of $(\mathbf{D}_{\mathrm{rig}}^\dagger(\rho), \mathrm{Fil}_\bullet)$ is pro-representable by $\hat{\mathcal{O}}_{\mathcal{S}(\bar{\rho}),y}$. On the other hand, by [BC09, Proposition 2.3.6] this functor is a subfunctor[5] of the deformation functor

---

[5] Strictly speaking, [BC09] treat the case $K = \mathbb{Q}_p$. However, the same argument applies in the general case.

pro-represented by R_ρ. It follows that the canonical morphism $R_\rho \to \hat{\mathcal{O}}_{\mathcal{S}(\bar{\rho}),y}$ is surjective. On the other hand, this morphism factors through $\hat{\mathcal{O}}_{X(\bar{\rho}),x}$ and hence the canonical morphism

$$\hat{\mathcal{O}}_{X(\bar{\rho}),x} \longrightarrow \hat{\mathcal{O}}_{\mathcal{S}(\bar{\rho}),y}$$

is surjective. Further, both rings are reduced (the left one by excellence) and have the same dimension. Now it follows from Lemma 2.16 that $\hat{\mathcal{O}}_{X(\bar{\rho}),x}$ is a domain and hence the morphism from $\hat{\mathcal{O}}_{X(\bar{\rho}),x} \to \hat{\mathcal{O}}_{\mathcal{S}(\bar{\rho}),y}$ has to be an isomorphism. It follows from [Hu96, Definition 1.7.10] that the morphism $\mathcal{S}(\bar{\rho}) \to X(\bar{\rho})$ is étale at y. □

We will refer to the space $X(\bar{\rho})^{\mathrm{reg}}$ constructed in Theorem 2.11 as the *regular part of the trianguline space*. The following lemma is a direct consequence of the construction of $\mathcal{S}^{\mathrm{ns}}(\bar{\rho})$ as an open subspace of a successive extension of vector bundles over $\mathcal{T}^d_{\mathrm{reg}}$.

LEMMA 2.18. *Let $x \in X(\bar{\rho})^{\mathrm{reg}}$ be a rigid analytic point. Then there exists a neighborhood of x in $X(\bar{\rho})^{\mathrm{reg}}$ that is isomorphic to the product of a neighborhood of $\omega_d(x) \in \mathcal{W}^d$ with the closed unit disc of dimension $(1 + d(d-1)/2)[K : \mathbb{Q}_p]$.*

Similar to the space $\mathcal{S}^{\mathrm{ns}}(\bar{\rho})$, we can define a subspace $\mathcal{S}^\square(\bar{\rho}) \subset \mathcal{S}^\square_d$ consisting of those trianguline (φ, Γ)-modules in $\mathcal{S}^\square$ that come from a $\mathcal{G}_K$-representation whose associated pseudo-character has residual type $\bar{\rho}$. As in the discussion above, we have a map $\pi^\square_{\bar{\rho}} : \mathcal{S}^\square(\bar{\rho}) \to \mathfrak{X}_{\bar{\rho}} \times \mathcal{T}^d$.

LEMMA 2.19. *The map $\pi^\square_{\bar{\rho}} : \mathcal{S}^\square(\bar{\rho}) \to \mathfrak{X}_{\bar{\rho}} \times \mathcal{T}^d$ factors over $X(\bar{\rho})$.*

Proof. As $X(\bar{\rho})$ is closed and $\mathcal{S}^\square(\bar{\rho})$ is reduced it is enough to show that a dense subset of $\mathcal{S}^\square(\bar{\rho})$ maps to $X(\bar{\rho})$. However, the set $\tilde{\mathcal{S}}^\square_d$ of all points $x \in \mathcal{S}^\square_d$, where all the extensions

$$0 \longrightarrow \mathrm{Fil}_i(D) \otimes k(x) \longrightarrow \mathrm{Fil}_{i+1}(D) \otimes k(x) \longrightarrow \mathcal{R}_{k(x)}(\delta_i) \longrightarrow 0$$

are non-split, is Zariski-open and dense. It follows that $\tilde{\mathcal{S}}^\square_d$ meets every component of $\mathcal{S}^\square(\rho)$ and in fact the intersection $\mathcal{S}^\square(\bar{\rho}) \cap \tilde{\mathcal{S}}^\square_d$ is Zariski-open and dense in $\mathcal{S}^\square(\bar{\rho})$. We now conclude by remarking that there is a canonical map $\tilde{\mathcal{S}}^\square_d \to \mathcal{S}^{\mathrm{ns}}_d$ (which is in fact a $\mathbb{G}^{d-1}_m$-torsor) that induces a map $q : \tilde{\mathcal{S}}^\square_d \cap \mathcal{S}^\square(\bar{\rho}) \to \mathcal{S}^{\mathrm{ns}}(\bar{\rho})$ such that the map $\pi^\square_{\bar{\rho}}$ factors through q. □

3. Application of eigenvarieties

In this section, we recall some facts about eigenvarieties attached to definite unitary groups and prove a density statement about them which will be used in the proof of the main theorem.

3.1 The eigenvarieties

The eigenvarieties that we will use are studied in Chenevier's paper [Ch09]. The first result that we need is the analogue of the results in [He12b], where the corresponding eigenvarieties were studied in [BC09]. We recall the setup of Chenevier's paper.

Notation 3.1. (i) We choose a totally real field F such that $[F : \mathbb{Q}]$ is even and let E be a CM quadratic extension of F. We write c for the complex conjugation of E over F and assume that there is a place v_0 of F dividing p such that $v_0 = w_0 w_0^c$ splits in E and such that $F_{v_0} = E_{w_0} \cong K$. We fix such an isomorphism and view the uniformizer ϖ of K as an uniformizer of F_{v_0}.

(ii) We fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and embeddings $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $I_\infty = \mathrm{Hom}(F, \mathbb{C}) = \mathrm{Hom}(F, \mathbb{R})$ denote the set of infinite places of F. Given a place v of F dividing p, the set $I(v) = \mathrm{Hom}(F_v, \bar{\mathbb{Q}}_p)$ is identified with a subset $I_\infty(v) \subset I_\infty$ via our choice of embeddings ι_∞ and ι_p.

(iii) Let $d \geqslant 1$ be an integer and let us write G for the unique unitary group in d variables defined over F which splits over E, is quasi-split at all finite places and compact at all infinite places. The existence of such a group can be deduced from the considerations of [Cl91, § 2].

(iv) As v_0 splits in E, there exists an isomorphism $G(F_{v_0}) \cong \mathrm{GL}_d(K)$ that we fix for the following. We write S_p for the set of places v of F dividing p and $S'_p = S_p \backslash \{v_0\}$.[6]

(v) Let T denote the diagonal torus in $\mathrm{GL}_d(K)$ and denote by T^0 its maximal compact subgroup. Further, we fix the Borel $B \subset \mathrm{GL}_d(K)$ of upper triangular matrices in order to have a notion of dominant weights. Let $L \subset \bar{\mathbb{Q}}_p$ be a subfield containing $\sigma(F_{v_0})$ for all $\sigma \in I(v_0)$. We define the weight space for the automorphic representations to be

$$\mathcal{W}^{\mathrm{aut}} = \mathrm{Hom}_{\mathrm{cont}}(T^0, \mathbb{G}_{m,L}(-)),$$

as a rigid space over L. In particular, we have a canonical identification $\mathcal{W}^{\mathrm{aut}} \cong \mathcal{W}_L^d$.

(vi) Fix a finite set S of finite places of F containing S_p and all places such that $G(F_v)$ ramifies and fix a compact open subgroup $H = \prod_v H_v \subset G(\mathbb{A}_{F,f})$ such that H_v is maximal hyperspecial for all $v \notin S$ and such that H_{v_0} is $\mathrm{GL}_d(\mathcal{O}_K)$. Write $S' = S \backslash \{v_0\}$. We define $H' = \prod_v H'_v$ such that $H'_v = H_v$ is $v \neq v_0$ and H'_{v_0} is the Iwahori subgroup I of $\mathrm{GL}_d(\mathcal{O}_K)$ of matrices whose reduction modulo ϖ is upper triangular. Further, let $\mathcal{H}^{\mathrm{un}} = \mathcal{O}_L[G(\mathbb{A}_{F,f}^S)/\!\!/H^S]$ denote the spherical Hecke algebra outside of S. Furthermore, we require that H is small enough, i.e. for $g \in G(\mathbb{A}_{F,f})$,

$$G(F) \cap gHg^{-1} = 1. \tag{3.1}$$

(vii) For each place $v \in S'$ we fix an idempotent element e_v in the Hecke algebra $\mathcal{O}_L[G(F_v)/\!\!/H_v]$ and write $e = (\bigotimes_{v \in S'} e_v) \otimes 1_{\mathcal{H}^{\mathrm{un}}}$ for the resulting idempotent element of the Hecke algebra $\mathcal{O}_L[G(\mathbb{A}_{F,f}^{v_0})/\!\!/H^{v_0}]$.[7]

(viii) For $1 \leqslant i \leqslant d$, let $t_i = \mathrm{diag}(1, \ldots, 1, \varpi, 1 \ldots, 1) \in T$, where the uniformizer is the ith diagonal entry. Let $T^- \subset T$ denote the set of $\mathrm{diag}(x_1, \ldots, x_d) \in T$ such that $\mathrm{val}(x_1) \geqslant \cdots \geqslant \mathrm{val}(x_d)$. We regard $\mathbb{Z}[T/T^0]$ as a subring of the Iwahori–Hecke algebra of $G(F_{v_0})$ with coefficients in $\mathbb{Z}[1/p]$ by means of $t \mapsto \mathbb{1}_{H_{v_0} t H_{v_0}}$. This subalgebra is generated by the Hecke operators $\mathbb{1}_{H_{v_0} t H_{v_0}}$ for $t \in T^-$ and their inverses. Finally, let $\mathcal{H} = \mathcal{H}^{\mathrm{un}} \bigotimes_{\mathbb{Z}} \mathbb{Z}[T/T^0]$, which is a subalgebra of $L[G(\mathbb{A}_{F,f})/\!\!/H']$.

Let W_∞ be an irreducible algebraic representation of $\prod_{v \in S'_p, w \in I_\infty(v)} G(F_w)$ and let $\mathcal{A} = \mathcal{A}(W_\infty, S, e)$ denote the set of isomorphism classes of all irreducible automorphic representations Π of $G(\mathbb{A}_F)$ such that $\bigotimes_{v \in S'_p, w \in I_\infty(v)} \Pi_w$ is isomorphic to W_∞ and $e(\Pi_f)^{H'_{v_0}} \neq 0$. Further, define the set of classical points to be

$$\mathcal{Z} = \left\{ (\Pi, \chi) \;\middle|\; \begin{array}{c} \Pi \in \mathcal{A}, \chi : T/T^0 \to \bar{\mathbb{Q}}_p^\times \text{ continuous} \\ \text{such that } \Pi_{v_0} |\det|_{v_0}^{(1-d)/2} \text{ is a subobject of } \mathrm{Ind}_B^{\mathrm{GL}_d(K)} \chi \end{array} \right\} \tag{3.2}$$

where the parabolic induction is normalized.

---

[6] Let us remark that here S_p is not exactly the same as in [Ch09].

[7] We can view e_v (respectively, e) also as idempotent elements of the full Hecke algebra, i.e. the convolution algebra of all compactly supported smooth functions on $G(F_v)$ (respectively, $G(\mathbb{A}_{F,f}^{v_0})$) (which are bi-invariant under some compact open subgroup). In particular, we can apply these idempotents to any representation of $G(F_v)$ (respectively, $G(\mathbb{A}_{F,f}^{v_0})$).

Associated to these data there is an *eigenvariety*, i.e. a reduced rigid analytic space $Y(W_\infty, S, e)$ over L together with a morphism

$$\kappa : Y(W_\infty, S, e) \longrightarrow \mathcal{W}^{\mathrm{aut}}$$

and

$$\psi = \psi^{\mathrm{un}} \otimes \psi_{v_0} : \mathcal{H} \longrightarrow \Gamma(Y(W_\infty, S, e), \mathcal{O}_{Y(W_\infty, S, e)})$$

a morphism of algebras such that $Y(W_\infty, S, e)$ contains a set Z as a Zariski-dense accumulation[8] subset. These data are due to the property that there is a bijection between Z and $\mathcal{Z}$ sending a point $z \in Z$ on the pair $(\Pi_z, \chi_z) \in \mathcal{Z}$ according to the following rule.

The evaluation $\psi^{\mathrm{un}}(z) : \mathcal{H}^{\mathrm{un}} \to k(z)$ is the character of the spherical Hecke algebra associated to the representation Π_z^S. For $w \in I_\infty(v_0)$, let $\kappa_{\Pi_{z,w}}$ denote the algebraic character of T_{v_0} obtained from $\Pi_{z,w}$ following the rule of [Ch09, §1.4]. Then $\kappa(z) = \prod_{w \in I_\infty(v_0)} \kappa_{\Pi_{z,w}}$. Let $\kappa_\varpi(z)$ be the unique character $T/T^0 \to \bar{\mathbb{Q}}_p^\times$ such that $\kappa_\varpi(z)(t) = \kappa(z)(t)$ when t is a diagonal matrix whose entries are powers of ϖ. Finally, the component ψ_{v_0} of the morphism ψ is given by

$$\psi_{v_0}(z)|_{T_{v_0}^-} : \mathbb{1}_{H_{v_0} t H_{v_0}} \longmapsto \chi_z(t) \cdot \delta_{B_v}^{-1/2}(t) |\det(t)|^{(d-1)/2} \kappa_\varpi(z)(t),$$

where δ_{B_v} is the modulus character.

In what follows, we fix the data (W_∞, S, e) and write simply Y for $Y(W_\infty, S, e)$.

In [Ch09, §2], Chenevier constructs these eigenvarieties using a space of overconvergent p-adic automorphic forms. More precisely, if V is an open affinoid of $\mathcal{W}^{\mathrm{aut}}$, one defines a certain $r_V \geqslant 1$, and constructs for each $r \geqslant r_V$ an $\mathcal{O}(V)$-Banach space denoted $e\mathcal{S}(V, r)$ with a continuous action of $\mathcal{H}$ such that the operator U_{v_0} corresponding to $\mathrm{diag}(\varpi^{d-1}, \varpi^{d-2}, \dots, 1) \in \mathbb{Z}[T/T^0] \subset \mathcal{H}$ acts as a compact operator (see also (3.5) below for a definition of U_{v_0}). We say that a character of $\mathcal{H}$ is U_{v_0}-finite if the image of U_{v_0} is non-zero. Then we have the following interpretation of points of Y, which is a consequence of Buzzard's construction of eigenvarieties [Bu07, §5].

PROPOSITION 3.2. *Let* $t \in \mathcal{W}(\bar{\mathbb{Q}}_p)$. *Then there is a natural bijection between* $\bar{\mathbb{Q}}_p$-*points of* Y *mapping to* t *and the* $\bar{\mathbb{Q}}_p$-*valued* U_{v_0}-*finite system of eigenvalues of* $\mathcal{H}$ *on* $\varinjlim_{V,r} e\mathcal{S}(V, r) \otimes_{\mathcal{O}(V)} k(t)$.

3.2 The map to the trianguline space

In the above section we have recalled the construction of an eigenvariety $Y \to \mathcal{W}^{\mathrm{aut}}$. Now assume that the extension E/F is unramified at finite places and $S \setminus \{v_0\}$ contains only places which split in E. As above, we write $Z \subset Y$ for the set of classical points (3.2). Let $(\Pi, \chi) \in Z$ and let $\pi = \bigotimes'_v \mathrm{BC}(\Pi_v)$ be the representation of $\mathrm{GL}_d(\mathbb{A}_E)$ defined by local base change for GL_d. Then by [Ch09, Theorems 3.2 and 3.3] there are Galois representations $\rho_\Pi : \mathcal{G}_E \to \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$ attached to the automorphic representations $\Pi \in Z$, unramified outside a finite set of places and such that the semi-simplification of the Weil–Deligne representation attached to $\rho_\Pi|_{\mathcal{G}_v}$ equals the Langlands parameter of $\pi_v |\cdot|^{(1-d)/2}$, where $\mathcal{G}_v \subset \mathcal{G}_E$ is the decomposition group at v for v not dividing p.

Let $\mathbb{B} \subset \mathbb{G} = \mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GL}_d$ denote the Weil restriction of the Borel subgroup of upper triangular matrices and let $\mathbb{T} \subset \mathbb{B}$ denote the Weil restriction of the diagonal torus. Using the canonical isomorphism $\mathbb{G}_{\bar{\mathbb{Q}}_p} \cong \prod_\sigma \mathrm{GL}_{d, \bar{\mathbb{Q}}_p}$, an algebraic weight $\mathbf{n}$ of $(\mathbb{G}_{\bar{\mathbb{Q}}_p}, \mathbb{T}_{\bar{\mathbb{Q}}_p})$ that is dominant with respect to $\mathbb{B}_{\bar{\mathbb{Q}}_p}$ can be identified with a tuple $(n_{\sigma,1}, \dots, n_{\sigma,d})_{\sigma \in I(v_0)} \in \prod_{\sigma \in I(v_0)} \mathbb{Z}^d$ such that

---

[8] Recall that a subset $A \subset Y$ of a rigid space accumulates at a point $x \in Y$ if $A \cap U$ is Zariski-dense in U for every connected affinoid neighborhood U of x in Y.

$n_{\sigma,1} \geqslant \cdots \geqslant n_{\sigma,d}$ for all σ. Note that this algebraic weight is already canonically defined over the reflex field $E_{\mathbf{n}}$ of the weight $\mathbf{n}$, i.e. over the subfield of $\bar{\mathbb{Q}}_p$ defined by

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/E_{\mathbf{n}}) = \{\psi \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \mid n_{\sigma,i} = n_{\psi \circ \sigma,i} \text{ for all embeddings } \sigma\}$$

and, in particular, over our fixed field L. Hence $\mathbf{n}$ defines an L-valued point of $\mathcal{W}^d$.

Let $z = (\Pi, \chi) \in Z$ and for $\sigma \in I(v_0)$ let $n_{\sigma,1} \geqslant \cdots \geqslant n_{\sigma,d}$ denote the highest weight of $\Pi_{v(\sigma)}$, where $v(\sigma) = \iota_\infty \iota_p^{-1} \sigma \in I_\infty(v_0)$. We say that z is regular (with respect to v_0) if $n_{\sigma,1} > \cdots > n_{\sigma,d}$ for all $\sigma \in I(v_0)$ and if

$$\frac{\lambda_i}{\lambda_j} \notin \{1, p^{\pm[K_0:\mathbb{Q}_p]}\},$$

where we set

$$\lambda_i = \chi(t_i).$$

We further say that $z = (\Pi, \chi)$ is uncritical if in addition condition (2.3) holds with λ_i as above and $k_{\sigma,i} = n_{\sigma,i} - (i-1)$. We write $Z^{\mathrm{reg}} \subset Z$ for the set of regular points and $Z^{\mathrm{un}} \subset Z$ for the set of uncritical regular points.

LEMMA 3.3. *The subsets Z^{reg} and Z^{un} are Zariski-dense in the eigenvariety Y and accumulate at all classical points $z \in Z$.*

Proof. The proof is the same as the usual proof of density of classical points. Let us denote by $Y_0 \subset \mathcal{W}^d \times \mathbb{G}_m$ the Fredholm hypersurface cut out by the Fredholm determinant of $U_{v_0} = \mathrm{diag}(\varpi^{d-1}, \ldots, \varpi, 1)$. Let $z \in Z \subset Y$ be a classical point and let $U \subset Y$ be a connected affinoid neighborhood. After shrinking U, we may assume that there is an affinoid open subset $V \subset \mathcal{W}^d$ such that $U \to V$ is finite and torsion free. As U is quasi-compact, there exist $C_1, \ldots, C_d$ such that

$$C_i \geqslant \mathrm{val}_x(\psi_{v_0}(x)(t_1 \cdots t_i))$$

for all $x \in U$, where val_x is the valuation on $k(x)$ normalized by $\mathrm{val}_x(p) = 1$. Let us write $A \subset V$ for the set of dominant algebraic weights $n_{\sigma,1} \geqslant \cdots \geqslant n_{\sigma,d}$ such that $C_i < n_{\sigma,i} - n_{\sigma,i+1} + 1$ for all i and $\sigma : K \hookrightarrow \bar{\mathbb{Q}}_p$.

Then one can easily see that A accumulates at the point $\kappa(z)$. It follows from [Ch09, Theorem 1.6(vi)] that the points $x \in U$ such that $\kappa(x) \in A$ are classical. Moreover, using the relation

$$\frac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]} \mathrm{val}_x(\lambda_i(x)) + \sum_\sigma k_{\sigma,i} = \frac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]} \mathrm{val}_x(\psi_{v_0}(x)(t_i)),$$

we obtain

$$\frac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]} \mathrm{val}_x(\lambda_1(x)) \leqslant C_1 - \sum_{\sigma'} k_{\sigma',1},$$

$$\frac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]} \mathrm{val}_x(\lambda_1(x) \cdots \lambda_i(x)) \leqslant C_i - \sum_{\sigma'} \sum_{j=1}^{i} k_{\sigma',j} \quad \text{for all } i, \tag{3.3}$$

which implies that these points lie in Z^{un}. The claim now follows from this as the map $U \to V$ is finite and torsion free. $\qquad \square$

Let us fix an identification of the decomposition group $\mathcal{G}_{w_0}$ of $\mathcal{G}_E$ at w_0 with the local Galois group $\mathcal{G}_K$.

Proposition 3.4. *Let $\Pi = \Pi_z$ for some $z \in Z^{\mathrm{reg}}$. For an infinite place $v \in I$ let $n_{v,1} \geqslant \cdots \geqslant n_{v,d}$ denote the highest weight of Π_v. Then the representation $\rho_\Pi|_{\mathcal{G}_K}$ is crystalline with Hodge–Tate weights*[9]

$$k_{\sigma,i} = n_{v(\sigma),i} - (i-1), \tag{3.4}$$

where $v(\sigma) = \iota_\infty \iota_p^{-1} \sigma$. Moreover, the Frobenius $\Phi_{\mathrm{cris},\Pi}$, i.e. the $[K_0 : \mathbb{Q}_p]$th power of the crystalline Frobenius on

$$\mathrm{WD}(\rho_\Pi|_{\mathcal{G}_K}) = D_{\mathrm{cris}}(\rho_\Pi|_{\mathcal{G}_K}) \otimes_{K_0 \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p} \bar{\mathbb{Q}}_p,$$

is semi-simple, and its eigenvalues are distinct and given by $\lambda_i = \chi_z(t_i)$.

Proof. It follows from [Ch09, Theorem 3.2] that the representation is semi-stable with Hodge–Tate weights and Frobenius eigenvalues as described above. The condition

$$\frac{\lambda_i}{\lambda_j} \neq p^{\pm[K_0:\mathbb{Q}_p]}$$

assures that the monodromy operator has to vanish and hence the representation is crystalline. Further, the condition $\lambda_i/\lambda_j \neq 1$ assures that the Frobenius has distinct eigenvalues and is *a priori* semi-simple. $\square$

By [Ch09, Corollary 3.9] there is a pseudo-character $T_Y : \mathcal{G}_{E,S} \to \Gamma(Y, \mathcal{O}_Y)$ such that for all $\Pi \in Z^{\mathrm{reg}}$ one has $T \otimes k(\Pi) = \mathrm{tr}\,\rho_\Pi$. Let us write $\mathcal{G}_{E,S}$ for the Galois group of the maximal extension E_S inside $\bar{\mathbb{Q}}$ that is unramified outside S and fix a continuous residual representation $\bar{\rho} : \mathcal{G}_{E,S} \to \mathrm{GL}_d(\mathbb{F})$ with values in a finite extension $\mathbb{F}$ of $\mathbb{F}_p$ such that the restriction $\bar{\rho}_{w_0} = \bar{\rho}|_{\mathcal{G}_{w_0}}$ is absolutely irreducible. We write $R_{\bar{\rho},S}$ (respectively, $R_{\bar{\rho}_{w_0}}$) for the universal deformation rings of $\bar{\rho}$ (respectively, $\bar{\rho}_v$) and let $\mathfrak{X}_{\bar{\rho},S}$ (respectively, $\mathfrak{X}_{\bar{\rho}_v}$) denote their rigid analytic generic fibers. As we assume $\bar{\rho}_{w_0}$ (and hence also $\bar{\rho}$) to be absolutely irreducible, [Ch14, Theorems A and B] implies that the universal deformation rings $R_{\bar{\rho}_{w_0}}$ and $R_{\bar{\rho},S}$ agree with the universal deformation rings of the corresponding pseudo-characters $\mathrm{tr}\,\bar{\rho}_{w_0}$ (respectively, $\mathrm{tr}\,\bar{\rho}$).

Let $Y_{\bar{\rho}} \subset Y$ denote the open and closed subset where the pseudo-character T_Y has residual type $\bar{\rho}$. Then the restriction to the decomposition group $\mathcal{G}_K \cong \mathcal{G}_{w_0} \subset \mathcal{G}_{E,S}$ at w_0 induces a map $f_{\bar{\rho}} : Y_{\bar{\rho}} \to \mathfrak{X}_{\bar{\rho}_{w_0}}$. Let $N_{K/\mathbb{Q}_p} : K^\times \to \mathbb{Q}_p^\times$ denote the norm map of K. We define $g_i : Y \to \mathbb{G}_m$ by

$$z \longmapsto \psi_{v_0}(t_{v_0,i}) \cdot (|N_{K/\mathbb{Q}_p}(\varpi)| N_{K/\mathbb{Q}_p}(\varpi))^{1-i}.$$

Further, we define a morphism

$$\omega_Y = (\omega_{Y,i})_i : Y \longrightarrow \mathcal{W}^d$$

by setting $\omega_{Y,i} = \kappa_i \cdot \delta_{\mathcal{W}}((1-i, \dots, 1-i))$.

Theorem 3.5. *The map*

$$f = (f_{\bar{\rho}}, \omega_Y, (g_i)_i) : Y_{\bar{\rho}} \longrightarrow \mathfrak{X}_{\bar{\rho}_{w_0}} \times \mathcal{W} \times \mathbb{G}_m^d = \mathfrak{X}_{\bar{\rho}_{w_0}} \times \mathcal{T}^d$$

factors over the trianguline space $X(\bar{\rho}_{w_0}) \subset \mathfrak{X}_{\bar{\rho}_{w_0}} \times \mathcal{T}^d$ and fits into the following commutative diagram.

$$
\begin{array}{ccc}
Y_{\bar{\rho}} & \xrightarrow{\ f\ } & X(\bar{\rho}_{w_0}) \\
& \searrow_{\omega_Y} & \downarrow_{\omega_d} \\
& & \mathcal{W}^d
\end{array}
$$

---

[9] Again note that we use a different sign convention as [Ch09].

Proof. The subset $X(\bar{\rho}_{w_0}) \subset \mathfrak{X}_{\bar{\rho}_{w_0}} \times \mathcal{T}^d$ is Zariski-closed, hence it suffices to check that $f(z) \in X(\bar{\rho}_{w_0})$ for all $z = (\Pi_z, \chi_z) \in Z^{\mathrm{un}} \cap Y(\bar{\rho})$, as this subset is Zariski-dense by Lemma 3.3 and as $Y_{\bar{\rho}}$ is reduced. By Lemma 2.19 this amounts to saying that for $z \in Z^{\mathrm{un}}$ the representation $\rho_{\Pi_z}|_{D_{w_0}}$ is trianguline with graded pieces $\mathcal{R}(\delta_i)$, where $\delta_i : K^\times \to \bar{\mathbb{Q}}_p^\times$ is the character

$$\delta_i|_{\mathcal{O}_K^\times} : z \longmapsto \prod_\sigma \sigma(z)^{n_{v(\sigma),i}+1-i},$$

$$\delta_i(\varpi) = \psi_{v_0}(z)(t_{v_0,i})(|N_{K/\mathbb{Q}_p}(\varpi)|N_{K/\mathbb{Q}_p}(\varpi))^{1-i}.$$

where, as above, $v(\sigma) = \iota_\infty \iota_p^{-1} \sigma$ and where we write $(n_{v(\sigma),i})$ for the highest weight of $\Pi_{z,v(\sigma)}$. By our choice of Z^{un} this follows from Lemma 2.9 and Proposition 3.4. $\qquad\square$

3.3 A density result for the space of p-adic automorphic forms

We now introduce the Banach space of p-adic automorphic forms of tame level H^{v_0} and prove that an element of $R_{\bar{\rho},S}$ vanishing on this space, vanishes on the eigenvariety $Y(W_\infty, S, e)_{\bar{\rho}}$ too.

Recall that we have fixed a finite extension L of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizer ϖ_L. If $H = \prod_v H_v \subset G(\mathbb{A}_{F,f})$ is a compact open subgroup such that $H_v \subset \mathrm{GL}_d(F_v)$ for $v|p$, we can define, for W_0 a finite $\mathcal{O}$-module with a continuous action of $\mathrm{GL}_d(\mathcal{O}_F \otimes \mathbb{Z}_p)$, the space of automorphic forms of level H and weight W_0 by

$$S_{W_0}(H, \mathcal{O}) = \{f : G(F)\backslash G(\mathbb{A}_F^\infty) \to W_0 \mid f(gh) = h^{-1}f(g) \text{ for all } h \in H\}.$$

If $H^{v_0} = \prod_{v \neq v_0} H^v$, we can define

$$S_{W_0}(H^{v_0}, \mathcal{O}) = \varinjlim_{H_{v_0} \subset G(\mathcal{O}_{F_{v_0}})} S_{W_0}(H^{v_0} H_{v_0}, \mathcal{O}),$$

where the limit is taken over all compact open subgroups of $G(\mathcal{O}_{F_{v_0}})$. This spaces carries an action of $G(F_{v_0})$ which is induced by right translation on functions.

Let $\hat{S}_{W_0}(H^{v_0}, \mathcal{O})$ be the ϖ_L-adic completion of $S_{W_0}(H^{v_0}, \mathcal{O})$. When W_0 is the trivial representation, we omit it from the notation.

If $\mathbf{n}$ is a dominant algebraic weight, we write $\mathbb{W}_\mathbf{n}$ for the irreducible representation of $\mathbb{G}_{\bar{\mathbb{Q}}_p} = (\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d)_{\bar{\mathbb{Q}}_p}$ of highest weight $\mathbf{n}$ relative to our choice of Borel subgroup. Note that this representation is already canonically defined over the reflex field $E_\mathbf{n}$ of the weight $\mathbf{n}$ and, in particular, over our fixed field L, because we assumed that L contains all the Galois conjugates $\sigma(K)$ of K inside $\bar{\mathbb{Q}}_p$. Finally, we write $W_\mathbf{n}$ for the representation of $\mathrm{GL}_d(K)$ or $\mathrm{GL}_d(\mathcal{O}_K)$ given by composing the embedding

$$\mathrm{GL}_d(K) \longrightarrow \prod_\sigma \mathrm{GL}_d(L) = (\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d)(L)$$
$$x \longmapsto (\sigma(x))_\sigma.$$

with the evaluation of $\mathbb{W}_\mathbf{n}$ on L-valued points (and similarly for its restriction to $\mathrm{GL}_d(\mathcal{O}_K)$).

Recall that W_∞ is the representation of $\prod_{v \in S_p'} \prod_{w \in I_\infty(v)} G(F_v)$ fixed in § 3.1. Using ι_p and ι_∞ and choosing L big enough, we can view W_∞ as a representation of $\prod_{v \in S_p'} G(F_v)$ and put an L-structure on it. Let us write $\hat{S}_\mathbf{n}(H^{v_0}, L) = \hat{S}_{W^0}(H^{v_0}, \mathcal{O}) \otimes_\mathcal{O} L$, where W^0 is a stable $\mathcal{O}_L$-lattice of the representation $W_\mathbf{n} \otimes_{\mathcal{O}_L} W_\infty$ of $G(\mathcal{O}_F \otimes \mathbb{Z}_p) = \mathrm{GL}_d(\mathcal{O}_K) \times \prod_{v \in S_p'} G(\mathcal{O}_{F_v})$.

If H is a compact open subgroup of $G(\mathbb{A}_{F,f})$ we write $\mathcal{H}(H)$ for the image of $\mathcal{H}^{\mathrm{un}}$ in $\mathrm{End}(\hat{S}_\mathbf{k}(H, L))$.

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We now fix H as in §3.1, and assume that all places $v|p$ are split in E and H_v is maximal at theses places.[10] Recall that we fixed a Galois representation $\bar{\rho}$ which is *automorphic* of level H, i.e. there exists $z \in Z$ such that $\bar{\rho}$ is isomorphic to the reduction mod ϖ_L of ρ_{Π_z}. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{H}^{\mathrm{un}}$ such that for $v \notin S$, the conjugacy class of $\bar{\rho}(\mathrm{Frob}_v)$ coincides via the Satake correspondence with the morphism $\mathcal{H}(G(F_v), H_v) = \mathcal{O}_L[G(F_v)/\!\!/H_v] \to \mathcal{H}(H)/\mathfrak{m} \simeq k_L$.

Given a compact open subgroup $H_{v_0} \subset G(\mathcal{O}_{F_{v_0}})$, we write $\mathcal{H}_{\mathfrak{m}}(H_{v_0})$ for the image of $\mathcal{H}_{\mathfrak{m}}^{\mathrm{un}}$ in $\mathrm{End}(S(H^{v_0}H_{v_0}, \mathcal{O})_{\mathfrak{m}})$. It follows from [Tho12, Proposition 6.7] that there is a unique continuous map

$$\theta : R_{\bar{\rho}} \longrightarrow \mathcal{H}_{\mathfrak{m}}(H_{v_0})$$

with the following property. Given a morphism $\psi : \mathcal{H}_{\mathfrak{m}}(H_{v_0}) \to \bar{\mathbb{Q}}_p$ of $\mathcal{O}_L$-algebras, the deformations ρ of $\bar{\rho}$ corresponding to $\psi \circ \theta$ are such that for $v \notin S$, the conjugacy class of $\rho(\mathrm{Frob}_v)$ coincides via the Satake correspondence with the morphism

$$\mathcal{H}(G(F_v), H_v) \longrightarrow \mathcal{H}_{\mathfrak{m}}(H_{v_0}) \xrightarrow{\psi} \bar{\mathbb{Q}}_p.$$

By unicity, these maps glue into a map

$$\theta : R_{\bar{\rho}} \longrightarrow \varprojlim_{H_{v_0}} \mathcal{H}_{\mathfrak{m}}(H_{v_0})$$

giving a continuous action of $R_{\bar{\rho}}$ on $\hat{S}(H^{v_0}, \mathcal{O})_{\mathfrak{m}}$.

We can now fix an idempotent e as in §3.1 such that $e\hat{S}(H^{v_0}, L)_{\mathfrak{m}} \neq 0$. We will prove that if an element $t \in R_{\bar{\rho}}$ vanishes on $e\hat{S}(H^{v_0}, L)_{\mathfrak{m}}$, then it vanishes on $Y_{\bar{\rho}} = Y(W_\infty, S, e)_{\bar{\rho}}$ too.

LEMMA 3.6. *If* $\mathbf{n}$ *is a dominant algebraic weight, there exists a* $\mathrm{GL}_d(\mathcal{O}_K) \times \mathcal{H}^{\mathrm{un}}$-*equivariant homeomorphism*

$$\hat{S}_{\mathbf{n}}(H^{v_0}, L)_{\mathfrak{m}} \simeq W_{\mathbf{n}} \otimes_L \hat{S}(H^{v_0}, L)_{\mathfrak{m}},$$

supposing that $\mathcal{H}^{\mathrm{un}}$ *acts trivially on* $W_{\mathbf{n}}$.

Proof. It is sufficient to prove this before the localization in $\mathfrak{m}$, by $\mathcal{H}^{\mathrm{un}}$-equivariance. Then we can use the following list of $\mathrm{GL}_d(\mathcal{O}_K) \times \mathcal{H}^{\mathrm{un}}$-equivariant isomorphisms

$$
\begin{aligned}
W_{\mathbf{n}}^0 \otimes_{\mathcal{O}_L} \hat{S}(H^{v_0}, \mathcal{O}_L) &= \varprojlim_n (W_{\mathbf{n}}^0/\varpi_L^n) \otimes_{\mathcal{O}_L} S(H^{v_0}, \mathcal{O}_L/\varpi_L^n) \\
&= \varprojlim_n (W_{\mathbf{n}}^0/\varpi_L^n) \otimes_{\mathcal{O}_L} (\varinjlim_{H_{v_0}} S(H_{v_0}H^{v_0}, \mathcal{O}_L/\varpi_L^n)) \\
&= \varprojlim_n \varinjlim_{H_{v_0}} (S_{W_{\mathbf{n}}^0/\varpi_L^n}(H_{v_0}H^{v_0}, \mathcal{O}_L/\varpi_L^n)) \\
&= \hat{S}_{\mathbf{n}}(H^{v_0}, \mathcal{O}_L). \qquad \square
\end{aligned}
$$

PROPOSITION 3.7. *The* $\mathrm{GL}_d(\mathcal{O}_K)$-*representation* $\hat{S}(H^{v_0}, \mathcal{O})_{\mathfrak{m}}$ *is isomorphic to a direct factor of* $C(G(\mathcal{O}_K), L)^r$ *for some* $r \geqslant 0$, *where* $C(G(\mathcal{O}_K), L)$ *denotes the space of continuous* L-*valued functions on* $G(\mathcal{O}_K)$.

---

[10] This restriction is only to allow us to apply the idempotent e at the spaces $S_W(H, L)$.

Proof. Using Lemma 3.6, it is sufficient to prove this when $\mathbf{n} = \mathbf{0}$. In this case we remark that the Banach space $\hat{S}(H^{v_0}, L) = \hat{S}_{\mathbf{0}}(H^{v_0}, L)$ is the Banach space of continuous functions

$$G(F)\backslash G(\mathbb{A}_{F,f})/H^{v_0} \to W_\infty.$$

Let $g_1, \ldots, g_{r'} \in G(\mathbb{A}_{F,f})$ be a set of representatives of $G(F)\backslash G(\mathbb{A}_{F,f})/H$. We have $G(F) \cap g_i H g_i^{-1} = \{1\}$ for each i, proving that $G(F)\backslash G(\mathbb{A}_{F,f})/H^{v_0}$ is isomorphic to $\mathrm{GL}_d(\mathcal{O}_K)^{r'}$. This proves that $\hat{S}(H^{v_0}, L)$ is $\mathrm{GL}_d(\mathcal{O}_K)$-equivariantly isomorphic to $C(\mathrm{GL}_d(\mathcal{O}_K), L)^r$ with $r = r' \dim W_\infty$. Using the fact that $\varprojlim_{H_{v_0}} \mathcal{H}(H_{v_0}, \mathcal{O}_L)$ and its action on $\hat{S}(H^{v_0}, L)$ commutes to $\mathrm{GL}_d(\mathcal{O}_K)$, we can conclude that $\hat{S}(H^{v_0}, L)_{\mathfrak{m}}$ is isomorphic to a direct factor of $C(\mathrm{GL}_d(\mathcal{O}_K), L)^r$. $\qquad\square$

By Lemma 3.6, there is an $\mathcal{H}^{\mathrm{un}}$-equivariant isomorphism

$$S_{\mathbf{n}}(H, L)_{\mathfrak{m}} \simeq \mathrm{Hom}_{\mathrm{GL}_d(\mathcal{O}_K)}(W_{\mathbf{n}}^*, \hat{S}(H^{v_0}, L)_{\mathfrak{m}}).$$

This implies that if $t \in R_{\bar{\rho}}$ vanishes on $\hat{S}(H^{v_0}, L)_{\bar{\rho}}$ it will vanish at each point of $Z \subset Y_{\bar{\rho}}$. These points being Zariski-dense in Y, the function t vanishes on $Y_{\bar{\rho}}$.

3.4 A density result for the eigenvariety

We now fix $\mathbf{k}$ a strongly dominant weight. We say that a closed point $y \in Y_{\bar{\rho}}$ is crystabelline of Hodge–Tate weights $\mathbf{k}$ if its image in $\mathfrak{X}(\bar{\rho})$ is crystalline on an abelian extension of K and its Hodge–Tate weights are given by $\mathbf{k}$. The purpose of this section is to prove that if $t \in R_{\bar{\rho}}$ vanishes on the subset of points of $Y_{\bar{\rho}}$ which are crystabelline of Hodge–Tate weights $\mathbf{k}$, then t vanishes on $Y_{\bar{\rho}}$.

Recall that we have fixed a Borel subgroup $B \subset \mathrm{GL}_d(K)$ and let us write $N \subset B$ for its unipotent radical. Further, we write $N_0 = N \cap \mathrm{GL}_d(\mathcal{O}_K)$.

Recall that, given a representation Π of $G(F_{v_0})$, the operator U_{v_0} is defined to be

$$U_{v_0} v = \sum_{n \in N_0/z_{v_0} N_0 z_{v_0}^{-1}} nzv \qquad (3.5)$$

on Π^{N_0} with $z_{v_0} = \mathrm{diag}(\varpi^{d-1}, \varpi^{d-2}, \ldots, 1)$.

DEFINITION 3.8. Let Π be an irreducible smooth representation of $G(F_{v_0})$. We say that Π has finite slope if the operator U_{v_0} has a non-zero eigenvalue on the space Π^{N_0}. If Π is an irreducible automorphic representation of $G(\mathbb{A}_F)$, we say that Π has finite slope if Π_{v_0} has finite slope as a smooth representation of $G(F_{v_0})$.

The following result tells us that finite slope automorphic representations of $G(\mathbb{A}_F)$ give rise to closed points of $Y_{\bar{\rho}}$.

PROPOSITION 3.9. *Let Π be an irreducible automorphic representation of $G(\mathbb{A}_F)$ of finite slope whose isomorphism class lies in $\mathcal{A}(W_\infty, S, e)$. Then there exists a point $z \in Y(W_\infty, S, e)$ such that $\psi_z|_{\mathcal{H}^{\mathrm{un}}} = \psi_\Pi|_{\mathcal{H}^{\mathrm{un}}}$. Moreover, if $\bigotimes_{w \in I_\infty(v_0)} \Pi_w$ is isomorphic to $W_{\mathbf{n}}$, then the Galois-representation attached to z becomes semi-stable of weight $\mathbf{k} = (k_{\sigma,i})$, when restricted to the Galois group of an abelian extension of K, where $k_{\sigma,i} = n_{\sigma,i} - (i-1)$. If moreover, for $\eta_i = \omega_{Y,i}(z)\delta_{\mathcal{W}}((k_{\sigma,i}))^{-1}$, we have $\eta_i \neq \eta_j$ for $i \neq j$ then this Galois representation is potentially crystalline of weight $\mathbf{k}$.*

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Proof. By assumption, there exists $f \in e\hat{S}(H^{v_0}, L)^{N_0}$ which is an eigenvector of $\mathcal{H} \times L[T^0]$ such that the character of $\mathcal{H}$ is ψ_Π and the eigenvalue of U_{v_0} is non-zero. Let χ be the character of T^0 giving the action of T^0 on f. By [Loe11, Proposition 3.10.1], we have

$$e\hat{S}(H^{v_0}, L)^{N_0}[\chi] \simeq \varinjlim_r e\mathcal{S}(\chi, r).$$

By Proposition 3.2, there exists a point z of $Y(W_\infty, S, e)$ such that $\psi_z|_{\mathcal{H}^{\mathrm{un}}} = \psi_\Pi|_{\mathcal{H}^{\mathrm{un}}}$. The claim about semi-stability after an abelian extension follows easily using the map to the trianguline space and the fact that the fibers over strongly dominant locally algebraic characters have this property. By Theorem 3.5, the character $(\eta_i)_i$ gives the action of the inertia on the Weil–Deligne module of this Galois representation, which is non-monodromic under the last assumption of the proposition. $\qquad\square$

Let $\mathbf{k}$ and $\mathbf{n}$ be as in the proposition. This proposition shows us that if we want to prove that an element $t \in \mathcal{H}^{\mathrm{un}}$ vanishing on all crystabelline points of type $\mathbf{k}$ of $Y(W_\infty, S, e)_{\bar{\rho}}$ is zero, it is sufficient to prove the following: an element $t \in \mathcal{H}^{\mathrm{un}}$ vanishing on $eS_{\mathbf{n}}(H^{v_0}, L)_{\mathfrak{m}}[\Pi_f]$ for all irreducible automorphic representations Π of finite slope such that $e\Pi_f \neq 0$, satisfies $t = 0$ on $e\hat{S}(H^{v_0}, L)_{\mathfrak{m}}$.

To produce sufficiently many automorphic finite slope representations we can use the following result. We now write I_n for the level n Iwahori subgroup of $\mathrm{GL}_d(\mathcal{O}_K)$, i.e. the group of elements of $\mathrm{GL}_d(\mathcal{O}_K)$ such that the entries below the diagonal are divisible by ϖ^n, and $B_0 = B \cap I_n$ and $N_0 = N \cap B_0$. Recall that the level of a smooth character $\chi : \mathcal{O}_K^\times \to \mathbb{C}^\times$ is the least integer n such that $1 + \varpi^{n+1}\mathcal{O}_K$ is contained in $\ker(\chi)$.

PROPOSITION 3.10. *Let $\chi = \bigotimes_{i=1}^d \chi_i$ be a smooth character of T^0 such that for $1 \leqslant i \leqslant n-1$, the level of χ_i is strictly bigger than the level of χ_{i+1}. Then there exists an open subgroup $I(\chi)$ such that $I(\chi) = (I(\chi) \cap \bar{N})T^0(I(\chi) \cap N)$, $I(\chi) \cap B = B^0$ and, if we write χ for the composite $I(\chi) \to T^0 \to \mathbb{C}^\times$, then the pair $(I(\chi), \chi)$ is a type for the inertial conjugacy class of (T, χ), more precisely, if π is a smooth irreducible representation of $\mathrm{GL}_d(K)$, then*

$$\mathrm{Hom}_{I(\chi)}(\chi, \pi) \neq 0 \iff \pi \cong \mathrm{Ind}_B^{\mathrm{GL}_d(K)}(\eta)$$

with η a character of T such that $\eta|_T = \chi$. Moreover, in this case, π has finite slope.

Proof. Let n_i be the level of χ_i and define $I(\chi)$ as the subgroup of I of matrices $(a_{i,j})_{1 \leqslant i,j \leqslant d}$ such that $\varpi^{n_j}|a_{i,j}$ for $j < i$. It is immediate to check that χ can be extended to a character of $I(\chi)$. It is enough to prove that $(I(\chi), \chi)$ is a type for the $\mathrm{GL}_d(K)$-inertial equivalence class of (T, χ). In order to do so, we use the characterization of part 2 of the introduction of [BK99]. Its only non-trivial condition is (iii). We follow closely the arguments of [BK99] where the situation is much more general. Let z be the element of T whose diagonal entries are $(\varpi^{n-1}, \varpi^{n-2}, \dots, \varpi, 1)$ and f_z the element of $\mathcal{H}(G, \chi)$ with support $I(\chi)zI(\chi)$ such that $f_z(z) = 1$. We only have to prove that f_z is an invertible element of $\mathcal{H}(G, \chi)$. Let $f_{z^{-1}}$ be the element of support $I(\chi)z^{-1}I(\chi)$ such that $f_{z^{-1}}(z^{-1}) = 1$. We want to prove that $g = f_{z^{-1}} * f_z$ has support in $I(\chi)$. The support of g is contained in $I(\chi)z^{-1}I(\chi)zI(\chi)$. Now remark that

$$I(\chi)zI(\chi) = \coprod_{u \in I(\chi)/(I(\chi) \cap zI(\chi)z^{-1})} uzI(\chi)$$

and that each class of $I(\chi)$ modulo $I(\chi) \cap zI(\chi)z^{-1}$ contains an element of $N \cap I(\chi) = N_0$, so that $I(\chi)z^{-1}I(\chi)zI(\chi) = I(\chi)z^{-1}N_0zI(\chi) \subset I(\chi)NI(\chi)$. By [BH06, Proposition 11.1.2.], it is then sufficient to check that if an element $u \in N$ intertwines the character χ, then $u \in N_0$. We can restrict ourselves to the case $d = 2$. Let $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. Suppose that $x \notin \mathcal{O}_K$ and choose $n = n_1 - v(x)$ with n_i the level of χ_i. If u intertwines χ, an easy computation shows that we must have $\chi_1(a + x\varpi^n c)\chi_2(d - x\varpi^n c) = \chi_1(a)\chi_2(d)$ for each $(a,d,c) \in \mathcal{O}_K^\times \times \mathcal{O}_K^\times \times \mathcal{O}_K$. As $n_1 \geqslant n_2 + 1$, we have $\chi_2(d - x\varpi^n c) = \chi_2(d)$, so that we have $\chi_1(1 + x\varpi^n c) = 1$ for all $c \in \mathcal{O}_K$, which contradicts the fact that $n_1 = n + v(x)$ is the level of χ_1. $\qquad\square$

Let $\tilde{\mathcal{T}}^0$ be the set of smooth characters $T^0 \to \mathbb{C}_p$ of the form $\chi_1 \otimes \cdots \otimes \chi_d$ such that the level of χ_i is strictly bigger than the level of χ_{i+1} for $1 \leqslant i \leqslant d - 1$.

PROPOSITION 3.11. *Let B be the Banach space of continuous function $\mathcal{O}_K^\times \to \mathbb{C}_p$ and, for $n \in \mathbb{N}$, let $B_{\geqslant n}$ denote the subspace generated by the characters $\mathcal{O}_K^\times \to \mathbb{C}_p^\times$ of finite level bigger than n. Then $B_{\geqslant n}$ is dense in B.*

Proof. As the space of smooth functions from $\mathcal{O}_K^\times \to \mathbb{C}_p$ is dense in B and a basis of this space is given by the set of all characters of finite level, the closure of $B_{\geqslant n}$ in B is a subspace of finite codimension. If it is strictly included in B, there exists a continuous map $\lambda : B \to \mathbb{C}_p$ which is U-equivariant for some open subgroup $U \subset \mathcal{O}_K^\times$ acting trivially on $\mathbb{C}_p$. Then λ gives rise to a non-trivial Haar measure on U which cannot exist.

COROLLARY 3.12. *Let $C(T^0, \mathbb{C}_p)$ denote the space of continuous $\mathbb{C}_p$-valued functions on T^0. Then the subspace of $C(T^0, \mathbb{C}_p)$ generated by the elements of $\tilde{\mathcal{T}}^0$ is dense.*

PROPOSITION 3.13. *Let $C(N_0 \backslash \mathrm{GL}_d(\mathcal{O}_K), \mathbb{C}_p)$ denote the space of $\mathbb{C}_p$-valued continuous functions on $\mathrm{GL}_d(\mathcal{O}_K)$ which are left invariant under N_0. Then the subspace*

$$\sum_{\chi \in \mathcal{T}_d} \mathrm{Ind}_{I(\chi)}^{\mathrm{GL}_d(\mathcal{O}_K)}(\chi) \subset C(N_0 \backslash \mathrm{GL}_d(\mathcal{O}_K), \mathbb{C}_p)$$

is dense.[11]

Proof. If $\chi \in \tilde{\mathcal{T}}^0$, the character χ of T^0 uniquely extends to a character χ of $I(\chi)$ which is trivial on $I_n \cap N$ and $I_n \cap \bar{N}$. Let us call such a function a *character function* for the moment. More generally, for $g \in \mathrm{GL}_d(\mathcal{O}_K)$, the function $\chi(\cdot g)$ of support $I(\chi)g^{-1}$ is named a *right translated character function*. For $\chi \in \tilde{\mathcal{T}}^0$, the space $\mathrm{Ind}_{I(\chi)}^{\mathrm{GL}_d(\mathcal{O}_K)}(\chi)$ is exactly the subspace of $C(N_0 \backslash \mathrm{GL}_d(\mathcal{O}_K, \mathbb{C}_p))$ generated by the right translated character functions. Let $f : \mathrm{GL}_d(\mathcal{O}_K) \to \mathbb{C}_p$ be a continuous function, invariant on the left under N_0. We have to prove that we can approximate f by right translated character functions. Let $g_1, \ldots, g_r$ be a system of representatives of the quotient $I_1 \backslash \mathrm{GL}_d(\mathcal{O}_K)$. Let $f_i = f(\cdot g_i^{-1})|_{I_1}$, so that $f = \sum_{i=1}^r f_i(\cdot g_i)$. Now fix $1 \leqslant i \leqslant r$ and $\epsilon > 0$. As I_1 is compact, we can find $n \geqslant 1$, such that for $h \in I_n \cap \bar{N}$, we have $||f_i - f_i(\cdot h)|| < \epsilon$. Let $h_1, \ldots, h_s \in I_1$ be a system of representatives of $(I_n \cap \bar{N}) \backslash (I_1 \cap \bar{N})$, which is also a system of representatives of $I_n \backslash I_1$, and define $f_{i,j} = f_i(\cdot h_j^{-1})|_{I_n}$. Let $f'_{i,j}$ be

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[11] V. Paškūnas informed us that he has more general versions of this result in his forthcoming work with M. Emerton.

the function on I_n defined by $f'_{i,j}(nt\bar{N}) = f_{i,j}(t)$ for $(n, t, \bar{N}) \in (N \cap I_n) \times T^0 \times (\bar{N} \cap I_n)$. As $(\bar{N} \cap I_n)$ is a normal subgroup of $(\bar{N} \cap I_1)$, we have $||f_{i,j} - f'_{i,j}|| < \epsilon$. Using Corollary 3.12, for each $(i, j) \in [1, r] \times [1, s]$, we can find elements $\tilde{f}_{i,j} \in \mathcal{T}_d$, such that $||f'_{i,j}|_{T^0} - \tilde{f}_{i,j}|| < \epsilon$. Now we can write each $\tilde{f}_{i,j}$ as $\sum_k a_{i,j,k}\chi_{i,j,k}$ with $I(\chi_{i,j,k}) \subset I_n$. We extend each $\chi_{i,j,k}$ to $I(\chi_{i,j,k})$ as previously described. As $I(\chi_{i,j,k}) \subset I_n$, we can write $\tilde{f}_{i,j}$ as a finite sum of right translated character functions. If $\tilde{f} = \sum_{i,j} \tilde{f}_{i,j}(\cdot\, h_j g_i)$, we have $||f - \tilde{f}|| < \epsilon$, and $\tilde{f}$ is a finite sum of right translated character functions. □

We now can prove the density statement about eigenvarieties.

PROPOSITION 3.14. *Let $t \in R_{\bar{\rho}}$ such that t is zero on each $eS_{\mathbf{n}}(H^{v_0}I(\chi), \mathbb{C}_p)_{\mathfrak{m}}[\chi]$ such that $\chi \in \tilde{\mathcal{T}}^0$. Then $t = 0$ on $e\hat{S}(H^{v_0}, L)_{\mathfrak{m}}^{N_0}$.*

Proof. We know that $\sum_\chi \mathrm{Ind}_{I(\chi)}^{\mathrm{GL}_n(\mathcal{O}_K)}(\chi)$ is dense in $C(N_0\backslash \mathrm{GL}_n(\mathcal{O}_K), \mathbb{C}_p)$ and that $\hat{S}_{\mathbf{n}}(H^{v_0}, L)_{\mathfrak{m}}$ is isomorphic to a direct summand of $C(\mathrm{GL}_d(\mathcal{O}_K), L)^r$ for some r. It follows that the space $S_1 = \hat{S}_{\mathbf{n}}(H^{v_0}, L)_{\mathfrak{m}} \hat{\otimes}_L \mathbb{C}_p$ is isomorphic to a direct summand of $C(\mathrm{GL}_d(\mathcal{O}_K), \mathbb{C}_p)^r$. We can write $C(\mathrm{GL}_d(\mathcal{O}_K), \mathbb{C}_p)^r = S_1 \oplus S_2$.

As the functor $F = \bigoplus_\chi \mathrm{Hom}_{I(\chi)}(\chi, -)$ commutes with finite direct sums, we know that $F(S_1) \oplus F(S_2)$ is dense in $[C(\mathrm{GL}_d(\mathcal{O}_K), \mathbb{C}_p)^r]^{N_0}$. As the functor of N_0-invariants commutes with direct sums, we conclude that $F(S_1) \subset S_1^{N_0}$ must be dense. By assumption, t vanishes on $F(S_1)$, hence on $S_1^{N_0}$, which contains $\hat{S}_{\mathbf{n}}(H^{v_0}, L)_{\mathfrak{m}}^{N_0}$. Finally, we conclude by remarking that

$$\hat{S}_{\mathbf{n}}(H^{v_0}, L)_{\mathfrak{m}}^{N_0} = W_{\mathbf{n}} \otimes_L \hat{S}(H^{v_0}, L)_{\mathfrak{m}}^{N_0}.$$

COROLLARY 3.15. *Let $f \in R_{\bar{\rho}}$ be a function vanishing on all points of $Y_{\bar{\rho}}$ which are crystabelline of Hodge–Tate weights $\mathbf{k}$. Then the image of f in $\Gamma(Y_{\bar{\rho}}, \mathcal{O}_Y)$ is zero.*

3.5 Conclusion

Let us summarize what we have proven so far using eigenvarieties. The following definition will be useful.

DEFINITION 3.16. Let X be a rigid space and R be a ring together with a ring homomorphism $\psi : R \to \Gamma(X, \mathcal{O}_X)$:

(i) a subset $Z \subset X$ is called *R-closed* if $Z = \{x \in X \mid \psi(f)(x) = 0 \text{ for all } f \in I\}$ for some ideal $I \subset R$;

(ii) a subset $U \subset X$ is called *R-open* if its complement is *R*-closed.

Further, we have an obvious notion of the *R*-closure of some subset $Z \subset X$ and a notion of *R*-density.

Let $\mathbf{k} = (k_{\sigma,i})_\sigma \in \prod_\sigma \mathbb{Z}^d$ be a strongly dominant algebraic weight and $X_{\mathbf{k}}(\bar{\rho}_{w_0})$ denote the $R_{\bar{\rho}_{v_0}}$-closure of the set of crystabelline points of $X(\bar{\rho}_{w_0})$ which have labeled Hodge–Tate weights $\mathbf{k}$. We have finally proved the following result.

THEOREM 3.17. *The image of $Y(W_\infty, S, e)_{\bar{\rho}}$ in $X(\bar{\rho}_{w_0})$ under the map defined by Theorem 3.5 is contained in $X_{\mathbf{k}}(\bar{\rho}_{w_0})$.*

4. The main theorem

Let us fix a continuous absolutely irreducible representation $\bar{r} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$. We need to embed our local situation into a global one. For this we use the results of the appendix of [EG14] and the following proposition.

PROPOSITION 4.1. *Let* $\mathbf{k}$ *be a set of labeled Hodge–Tate weights. Then* $\bar{r}$ *has a potentially diagonal lift of Hodge–Tate weights* $\mathbf{k}$.

Proof. Let K' be the unique unramified extension of degree d of K. Then there exists a character η of $\mathcal{G}_{K'}$ such that $\bar{r} \simeq \mathrm{Ind}_{\mathcal{G}_{K'}}^{\mathcal{G}_K} \eta$. If $\mathbf{k} = (k_{\sigma,i})$ is a strongly dominant weight, let θ be a character $\mathcal{G}_{K'} \to \bar{\mathbb{Q}}_p^\times$ such that the restriction of θ to the inertia group of K' corresponds, via local class field theory, to the character of $\mathcal{O}_{K'}^\times$ given by $x \mapsto \prod_{\sigma,i} \psi_{\sigma,i}(x)^{k_{\sigma,i}}$, $(\psi_{\sigma,i})_{1 \leqslant i \leqslant d}$ being the set of embeddings of K' in $\bar{\mathbb{Q}}_p$ whose restriction to K is σ. Let δ be a locally constant lift of $\eta\bar{\theta}^{-1}$, for example using the Teichmüller lift. Then $\mathrm{Ind}_{\mathcal{G}_{K'}}^{\mathcal{G}_K}(\theta\delta)$ is a lift of $\bar{r}$ which has Hodge–Tate weights $\mathbf{k}$ and whose restriction to $\mathcal{G}_{K'}$ is diagonal. $\square$

We also assume that p does not divide $2d$. Then [EG14, Corollary A.7] and Proposition 4.1 tell us that we can find F a totally real field, E a totally imaginary quadratic extension of F which is unramified at each finite place,[12] and a continuous irreducible representation $\bar{\rho} : \mathcal{G}_F \to \mathcal{G}_d(\bar{\mathbb{F}}_p)$ (see, for example, [EG14, § 5.1] for the definition of $\mathcal{G}_d$) such that:

- $4|[F : \mathbb{Q}]$;
- each place $v|p$ of F splits in E and $F_v \simeq K$;
- for each place $v|p$ of F, there is a place $\tilde{v}$ of E dividing p and such that $\bar{\rho}|_{\mathcal{G}_{F_{\tilde{v}}}} \simeq \bar{r}$;
- $\bar{\rho}$ is unramified outside of p;
- $\bar{\rho}^{-1}(\mathrm{GL}_d(\bar{\mathbb{F}}_p) \times \mathrm{GL}_1(\bar{\mathbb{F}}_p)) = \mathcal{G}_E$;
- $\bar{\rho}(\mathcal{G}_{E(\zeta_p)})$ is adequate (in the sense of [Tho12, § 2]);
- $\bar{\rho}$ is automorphic (we will now explain what this means).

Let v_1 be a place[13] of F which is prime to p and satisfies the same hypothesis as in [EG14, § 5.3]. We define the compact open subgroup $H_{\max} = \prod_v H_{\max,v} \subset G(\mathbb{A}_F)$ so that $H_{\max,v} \simeq \mathrm{GL}_d(\mathcal{O}_K)$ if $v|p$, $H_{\max,v} \subset G(F_v)$ is maximal hyperspecial if $v \nmid p$ and $v \neq v_1$, and $H_{\max,v_1}$ an open pro-ℓ-subgroup of $G(F_{v_1})$ for ℓ the residual characteristic of v_1. We say that $\bar{\rho}$ is automorphic if $S_W(H_{\max}, L)_{\bar{\rho}} \neq 0$ for some irreducible locally algebraic representation W of $G_p^0 = \prod_{v \in S_p} H_{\max,v}$.

Remark 4.2. Of course the group $H_{\max}$ is not a maximal compact subgroup. However, it will be the largest compact open subgroup that will appear in the following.

From now on we fix a place v_0 of F dividing p such that $F_{v_0} \simeq K$.

All the constructions we made in § 3 depend on the choice of a representation W_∞ of the group $\prod_{v \in S'_p, w \in I_\infty(v)} G(F_w)$. Even if that is not explicit in the notation, the space $\hat{S}(H_{\max}^{v_0}, L)$ actually depends on W_∞. That is why we need the following lemma.

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[12] This condition is not exactly in [EG14], but it can be obtained by a suitable solvable base change.

[13] The introduction of this auxiliary place is only needed so that the group H satisfies condition (3.1).

Lemma 4.3. *There exists an irreducible algebraic representation W_∞ of the group*

$$\prod_{v \in S_p'} \prod_{w \in I_\infty(v)} G(F_v)$$

such that $\hat{S}(H_{\max}^{v_0}, L)_{\bar\rho} \neq 0$.

Proof. By definition, we know that there exists an irreducible locally algebraic representation W of G_p^0 such that $S_W(H_{\max}, L)_{\bar\rho} \neq 0$. Let

$$\hat{S}(H_{\max}^p, L)_{\bar\rho} = \varprojlim_n \varinjlim_{H_p \subset G_p^0} S(H_{\max}^p H_p, \mathcal{O}_L/\varpi_L^n)_{\bar\rho}.$$

We see, as in §3.3, that $S_W(H_{\max}, L)_{\bar\rho} \simeq \mathrm{Hom}_{G_p^0}(W^*, \hat{S}(H_{\max}^p, L)_{\bar\rho})$ and that, as a G_p^0-representation, $S(H_{\max}^p, L)_{\bar\rho}$ is isomorphic to a non-zero direct factor of $(G_p^0)^r$ for some $r \geqslant 1$.

We can then find an irreducible algebraic representation W^{v_0} of $\prod_{v \in S_p'} \mathrm{GL}_d(\mathcal{O}_{F_v})$ such that $\mathrm{Hom}_{\prod_{v \in S_p'} \mathrm{GL}_d(\mathcal{O}_{F_v})}(W^{v_0}, \hat{S}(H_{\max}^p, L)_{\bar\rho}) \neq 0$ and choose for W_∞ the irreducible algebraic representation of $\prod_{v \in S_p'} \prod_{w \in I_\infty(v)} \mathrm{GL}_d(F_v)$ associated to W^{v_0} as explained in §3.3. Then we have by definition

$$\hat{S}(H_{\max}^{v_0}, L)_{\bar\rho} \simeq \mathrm{Hom}_{\prod_{v \in S_p'} \mathrm{GL}(\mathcal{O}_{F_v})}(W^{v_0}, \hat{S}(H_{\max}^p, L)_{\bar\rho}). \qquad \square$$

From now on, all pairs (S, H^{v_0}) to which we will apply the results of §3 (where H^{v_0} is a compact open subgroup of $G(\mathbb{A}_{F,f}^{v_0})$ and S a finite set of places of F splitting in E such that H_v is maximal hyperspecial for $v \notin S$) are assumed to satisfy $H^{v_0} \subset H_{\max}^{v_0}$ and $S \supset S_p \cup \{v_1\}$.

4.1 A result of density in crystalline deformation spaces

Fix $\mathbf{n}$ a dominant algebraic weight such that $\mathrm{Hom}_{\mathrm{GL}_d(\mathcal{O}_K)}(W_{\mathbf{n}}^*, \hat{S}(H^{v_0}, L)_{\bar\rho}) \neq 0$. We write $\mathbf{k}$ for the strongly dominant weight associated to $\mathbf{n}$ by the recipe of (3.4).

By Kisin's result [Ki08] there exists a reduced p-torsion free quotient $R_{\bar r, \mathbf{k}}^{\mathrm{cris}}$ of $R_{\bar r}$ such that a continuous homomorphism $\zeta : R_{\bar r} \to \bar{\mathbb{Q}}_p$ factors through $R_{\bar r, \mathbf{k}}^{\mathrm{cris}}$ if and only if $\zeta \circ r^{\mathrm{univ}}$ is crystalline of Hodge–Tate weight $\mathbf{k}$. Here r^{univ} is the universal deformation of $\bar r$ on $R_{\bar r}$. Moreover, the ring $R_{\bar r, \mathbf{k}}^{\mathrm{cris}}[1/p]$ is formally smooth. Let us denote by $\mathfrak{X}_{\bar r, \mathbf{k}}^{\mathrm{cris}}$ the generic fiber of the formal scheme $\mathrm{Spf}\, R_{\bar r, \mathbf{k}}^{\mathrm{cris}}$. In this section we will use patching techniques to prove that the $R_{\bar r}$-closure (or equivalently the $R_{\bar r, \mathbf{k}}^{\mathrm{cris}}$-closure) of automorphic points of $\mathfrak{X}_{\bar r, \mathbf{k}}^{\mathrm{cris}}$ is a union of connected components of $\mathfrak{X}_{\bar r, \mathbf{k}}^{\mathrm{cris}}$.

Now let H^p be a compact open subgroup of $G(\mathbb{A}_F^p)$ which is such that H_v^p is hyperspecial for every finite place v of F which is inert in E. We will say that a $\bar{\mathbb{Q}}_p$-point of $\mathrm{Spec}(R_{\bar r, \mathbf{k}}^{\mathrm{cris}})$ is H^p-automorphic if the corresponding deformation of $\bar r$ is isomorphic to the restriction to $\mathcal{G}_{E_{w_0}}$ of a deformation ρ of $\bar\rho$ associated to an automorphic form of G of type W_∞. More precisely, let $\mathfrak{p}$ be the kernel of the map $R_{\bar\rho}[1/p] \to \bar{\mathbb{Q}}_p$ associated to ρ. Then ρ is called H^p-automorphic if $S_{W_{\mathbf{n}}}(G_p^0 H^p, L)[\mathfrak{p}] \neq 0$, where we recall that $G_p^0 = \prod_{v \in S_p} \mathrm{GL}_d(\mathcal{O}_{F_v})$.

Further, an irreducible component of $\mathrm{Spec}(R_{\bar r, \mathbf{k}}^{\mathrm{cris}}[1/p])$ is called H^p-automorphic if it contains an H^p-automorphic point.

Finally, we say that a $\bar{\mathbb{Q}}_p$-point (respectively, a connected component) of $\mathrm{Spec}(R_{\bar r, \mathbf{k}}^{\mathrm{cris}}[1/p])$ is automorphic if it is H^p-automorphic for some H^p as above.

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Remark 4.4. Note that actually our definition of being automorphic depends on the representation W_∞ chosen as in Lemma 4.3. However, for the sake of simplicity we often will suppress W_∞ from the notation.

Our goal in this section is to use the usual patching construction to prove the following result.

THEOREM 4.5. *Let Z be an automorphic component of* $\mathrm{Spec}(R_{\bar r,\mathbf k}^{\mathrm{cris}}[1/p])$. *Then the set of automorphic points in Z is Zariski-dense.*

Let $\mathfrak X_{\bar r,\mathbf k}^{\mathrm{aut}}$ be the union the components of the rigid analytic generic fiber $\mathfrak X_{\bar r,\mathbf k}^{\mathrm{cris}}$ of $\mathrm{Spf}\, R_{\bar r,\mathbf k}^{\mathrm{cris}}$ that correspond to automorphic components of $\mathrm{Spec}(R_{\bar r,\mathbf k}^{\mathrm{cris}}[1/p])$.

COROLLARY 4.6. *The set automorphic points in $\mathfrak X_{\bar r,\mathbf k}^{\mathrm{aut}}$ is $R_{\bar r}$-dense in $\mathfrak X_{\bar r,\mathbf k}^{\mathrm{aut}}$.*

Given a rigid space $\mathfrak X$ over $\mathbb Q_p$, we write $|\mathfrak X|$ for the underlying point set of $\mathfrak X$. Similarly, we write $|X|$ for the set of closed points of a $\mathbb Q_p$-scheme X. Let R be a complete local noetherian $\mathbb Z_p$-algebra with finite residue field and let $\mathfrak X$ denote the generic fiber of R in the sense of Berthelot. Further, let $X = \mathrm{Spec}\, R[1/p]$. Then we have $|X| = |\mathfrak X|$, as, if we write

$$R = \mathbb Z_p[\![T_1, \ldots, T_n]\!]/(f_1, \ldots, f_m),$$

then both sets are identified with the $\mathrm{Gal}(\bar{\mathbb Q}_p/\mathbb Q_p)$-orbits in

$$\{x = (x_1, \ldots, x_n) \in \bar{\mathbb Q}_p \mid |x_i| < 1 \text{ and } f_j(x) = 0\};$$

cf. also [dJ95, Lemma 7.1.9]. Further, a subset $Z \subset |X| = |\mathfrak X|$ is dense in X if and only it is R-dense in $\mathfrak X$. This proves that the theorem implies the corollary.

Proof of Theorem 4.5. Let $H^p \subset G(\mathbb A_F^p)$ be a compact open subgroup, hyperspecial at all finite places of F which are inert in E, such that the component Z has H^p-automorphic points. Let S be the finite set of places v of F which are either dividing p or such that $H_v^{v_0}$ is not maximal. It is classical that, up to enlarging S and taking an open subgroup of H^p, we can assume that the group $H^p \prod_{v \in S} G_p^0$ satisfies the relation (3.1).

If $v \in S \backslash S_p$, fix $\tilde v$ a place of E dividing v and define $R_{\tilde v}^\square$ as the universal ring pro-representing the functor of lifts of $\bar\rho|_{G_{\tilde v}}$, and $R_{\tilde v}^{\square,\mathrm{rtf}}$ its biggest quotient which is reduced and p-torsion free. We have to take care of the fact that we fixed the weight $W_\infty = \bigotimes_{v \in S_p', w \in I_\infty(v)} W_w$. As a consequence, for $v \in S_p'$, let R_v' be the quotient $R_{\bar\rho|_{\mathcal G_{E_{\tilde v}}}, \mathbf k_{\mathbf v}}^{\mathrm{cris}}$ of $R_{\bar\rho|_{\mathcal G_{E_{\tilde v}}}}$ where $\mathbf k_v$ is the set of Hodge–Tate weights associated to the highest weight of the algebraic representation $\bigotimes_{w \in I_\infty(v)} W_w$. Let $R_{v_0}' = R_{\bar r,\mathbf k}^{\mathrm{cris}}$,

$$R^{\mathrm{loc}} = \widehat{\bigotimes_{v \in S_p}} R_v' \,\hat\otimes\, \widehat{\bigotimes_{v \in S \backslash S_p}} R_{\tilde v}^{\square,\mathrm{rtf}}$$

and $R_{\infty,g} = R^{\mathrm{loc}}[\![x_1, \ldots, x_g]\!]$. Then the patching construction (see [Tho12, §6]) gives us, for g big enough, a $R_{\infty,g}$-module M_∞ of finite type whose support in $\mathrm{Spec}\, R_{\infty,g}$ is a union of irreducible components, and $M_\infty[1/p]$ is a projective $R_\infty[1/p]$-module. Let $\bar R$ be the quotient of R^{loc} corresponding to one of these irreducible components. Then $\bar R \otimes_{R^{\mathrm{loc}}} M_\infty[1/p]$ is a faithful $\bar R[\![x_1, \ldots, x_g]\!][1/p]$-module. Moreover, M_∞ is constructed as an inverse limit of the modules M_n, where M_n is a quotient of a space of automorphic forms of weight $\mathbf k$, on which the action of $R_{\infty,g}$

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factors through a ring of Hecke operators. Let us specify that these spaces of automorphic forms have tame level $\tilde{H}^p$ where $\tilde{H}^p$ is an open subgroup of H^p still satisfying the condition that $\tilde{H}^p$ is hyperspecial at places of F which are inert in E.

[Ki09, Lemma 3.4.12] shows that the irreducible components of the scheme $\operatorname{Spec}(R^{\mathrm{loc}}[\![x_1, \ldots, x_g]\!])$ are of the form $\operatorname{Spec}(\bar{R}[\![x_1, \ldots, x_g]\!])$ where $\bar{R} = \bigotimes_{v \in S} \bar{R}'_v$ and $\bar{R}'_v$ is an irreducible component of R'_v if $v|p$ and of $R^{\square,\mathrm{rtf}}_{\bar{v}}$ if $v \in S \backslash S_p$. Now let $\bar{R}'_{v_0}$ be the quotient of $R^{\mathrm{cris},\square}_{\bar{r},\mathbf{k}}$ such that $Z = \operatorname{Spec}(\bar{R}'_{v_0}[1/p])$. By assumption, there exists an irreducible component $\bar{R}'[\![x_1, \ldots, x_g]\!]$ of $R^{\mathrm{loc}}[\![x_1, \ldots, x_g]\!]$ containing $\bar{R}'_{v_0}$ in the support of M_∞. If $t \in R^{\mathrm{cris},\square}_{\bar{r},\mathbf{k}}$ vanishes on each automorphic point, then t acts trivially on the spaces M_n, and so on M_∞. Now $\bar{R} \otimes_{R^{\mathrm{loc}}} M_\infty[1/p]$ being a faithful $\bar{R}[1/p]$-module, the image of t in $\bar{R}'_{v_0}[1/p]$ is zero, which implies that $t = 0$ in $\bar{R}'_{v_0}$ since $\bar{R}'_{v_0}$ is p-torsion free. □

4.2 Variation on the density of trianguline representations

In order to deduce density statements in the generic fiber of a local deformation ring from density statements in the space of trianguline representations one needs to show that the image of the space of trianguline representations is dense. This density result is included in the work of Chenevier [Ch13] and Nakamura [Na11]. In our case the situation will be a bit more restrictive: we can only make a statement about the components of the space of trianguline representations that are met by some eigenvariety. Hence we need to sharpen this density result a bit. In order to do so we need to compare the fibers of the space of trianguline representations over strongly dominant algebraic weights with Kisin's crystalline deformation rings.

Let $\mathbf{k} = (k_{\sigma,i})$ be a strongly dominant weight and let us continue to assume that $\bar{r}$ is absolutely irreducible. In order not to overload the notation let us write $A_{\mathbf{k}} = R^{\mathrm{cris}}_{\bar{r},\mathbf{k}}$ for the rest of this subsection. Further (for the rest of this subsection), we write

$$\mathfrak{Z}_{\mathbf{k}} = (\operatorname{Spf} A_{\mathbf{k}})^{\mathrm{rig}},$$
$$Z_{\mathbf{k}} = \operatorname{Spec}(A_{\mathbf{k}}[1/p]).$$

DEFINITION 4.7. We say that a crystalline representation $r_x : \mathcal{G}_K \to \mathrm{GL}_d(\kappa(x))$ corresponding to a rigid analytic point $x \in \mathfrak{Z}_{\mathbf{k}}$ or to a closed point $x \in Z_{\mathbf{k}}$ is *regular* if the eigenvalues $(\lambda_1, \ldots, \lambda_d)$ of the Frobenius on the Weil–Deligne representation $\mathrm{WD}(D_{\mathrm{cris}}(r_x))$ associated to r_x satisfy $\lambda_i/\lambda_j \notin \{1, p^{\pm[K_0:\mathbb{Q}_p]}\}$ for $i \neq j$. We further say that x (respectively, r_x) is *generic* if all $d!$ filtrations on $D_{\mathrm{cris}}(r_x)$ induced by an ordering of the Frobenius eigenvalues are in general position with respect to all Hodge–Tate filtrations (with respect to all embeddings $\psi : K \hookrightarrow L$).

Remark 4.8. Note that if r_x is crystalline generic, then all parameters $(\delta_1, \ldots, \delta_d)$ of all possible triangulations of its (φ, Γ)-module $\mathbf{D}^\dagger_{\mathrm{rig}}(r_x)$ have the property that $(\delta_1|_{\mathcal{O}_K^\times}, \ldots, \delta_d|_{\mathcal{O}_K^\times})$ is algebraic of strongly dominant weight $\mathbf{k}$. Moreover, given such a trianguline filtration Fil_i of $\mathbf{D}^\dagger_{\mathrm{rig}}(r_x)$, the extensions

$$0 \longrightarrow \mathrm{Fil}_i \longrightarrow \mathrm{Fil}_{i+1} \longrightarrow \mathcal{R}_{k(x)}(\delta_i) \longrightarrow 0$$

are non-split. This may be checked after taking the quotient by Fil_{i-1} and hence we are reduced to the two-dimensional case where one can easily check that the direct sum of two (φ, Γ)-modules $\mathcal{R}(\delta) \oplus \mathcal{R}(\delta')$ such that $\delta|_{\mathcal{O}_K^\times} = \delta_{\mathcal{W}}(a) \neq \delta_{\mathcal{W}}(a') = \delta'|_{\mathcal{O}_K^\times}$ is not generic: one of the two possible triangulations is not in general position with the Hodge filtration. It follows that a point $x \in \mathfrak{Z}_k \subset \mathfrak{X}_{\bar{r}}$ that is, moreover, generic is in the image of $X(\bar{r})^{\mathrm{reg}} \subset X(\bar{r})$ under the map $X(\bar{r}) \to \mathfrak{X}_{\bar{r}}$.

By [Ki08, Theorem 2.5.5] there is a (locally free) $A_{\mathbf{k}} \otimes_{\mathbb{Z}_p} K_0$-module on $Z_{\mathbf{k}}$ together with an id $\otimes\,\varphi$-linear automorphism $\Phi : D \to D$ such that

$$(D, \Phi) \otimes \kappa(x) \cong D_{\mathrm{cris}}(r_x)$$

for all closed points $x \in Z_{\mathbf{k}}$. It follows that there is a Zariski-open subset $Z_{\mathbf{k}}^{\mathrm{reg}} \subset Z_{\mathbf{k}}$ with closed points precisely given by the regular points. Indeed, let us write $(\mathrm{WD}(D), \Phi_{\mathrm{WD}})$ for the Weil–Deligne representation on $Z_{\mathbf{k}}$, i.e. the locally free $A_{\mathbf{k}}$-module

$$\mathrm{WD}(D) = D \otimes_{A_{\mathbf{k}} \otimes_{\mathbb{Q}_p} K_0, \mathrm{id} \otimes \psi_0} A_{\mathbf{k}}$$

for some fixed embedding $\psi_0 : K_0 \hookrightarrow L$, endowed with the $A_{\mathbf{k}}$-linear Frobenius $\Phi^{[K_0:\mathbb{Q}_p]} \otimes \mathrm{id}$. Then the isomorphism class of $(\mathrm{WD}(D), \Phi_{\mathrm{WD}})$ does not depend on the choice of ψ_0. Moreover, we can consider the morphism $Z_{\mathbf{k}} \to \mathbb{A}^d$ that is defined by the coefficients of the characteristic polynomial of Φ_{WD}. Then it is easy to see that the condition on the eigenvalues of the crystalline Frobenius defining the set of regular points implies that $Z_{\mathbf{k}}^{\mathrm{reg}}$ is the preimage of a Zariski-open subset of $\mathbb{A}^d$.

Passing to the associated adic spaces, we find that there is a Zariski-open subset $\mathfrak{Z}_{\mathbf{k}}^{\mathrm{reg}} \subset \mathfrak{Z}_{\mathbf{k}}$ with rigid analytic points precisely given by the regular points in $\mathfrak{Z}_{\mathbf{k}}$. In fact we will see below that this subset is Zariski-dense (and, in particular, non-empty).

Recall that we have a map $\omega : \mathcal{S}^{\mathrm{ns}}(\bar{r}) \to \mathcal{W}^d$ to the weight space. We further view the weight $\mathbf{k}$ as an element of $\mathcal{W}^d$.

PROPOSITION 4.9. *The map $\omega^{-1}(\mathbf{k}) \to \mathfrak{X}_{\bar{r}}$ induces a map $g_{\mathbf{k}} : \omega^{-1}(\mathbf{k}) \to \mathfrak{Z}_{\mathbf{k}}$ which is étale over $\mathfrak{Z}_{\mathbf{k}}^{\mathrm{reg}}$. Further, $g_{\mathbf{k}}^{-1}(\mathfrak{Z}_{\mathbf{k}}^{\mathrm{reg}})$ is open and dense in $\omega^{-1}(\mathbf{k})$.*

Proof. By Corollary 2.7 there is a Zariski-open subset of $\omega^{-1}(\mathbf{k})$ on which the Galois representations are pointwise crystalline of weight $\mathbf{k}$. Hence the map $\omega^{-1}(\mathbf{k}) \to \mathfrak{X}_{\bar{r}}$ generically factors over $\mathfrak{Z}_{\mathbf{k}}$. As $\mathfrak{Z}_{\mathbf{k}} \subset \mathfrak{X}_{\bar{r}}$ is Zariski-closed the first claim follows. Further, it is easy to see that the preimage of $\mathfrak{Z}_{\mathbf{k}}^{\mathrm{reg}}$ is open and dense: it is identified with the preimage of the set of characters $(\delta_1, \ldots, \delta_d)$ such that $\delta_i(p)/\delta_j(p) \notin \{\prod_\sigma \sigma(\varpi)^{k_{i,\sigma} - k_{j,\sigma}}\}$.

It remains to prove the claim on étaleness which, in the adic setup can be checked using the infinitesimal lifting criterion; cf. [Hu96, Definition 1.6.5]. Consider an affinoid algebra A with ideal $I \subset A$ satisfying $I^2 = 0$ and the diagram

$$
\begin{array}{ccc}
\mathrm{Sp}(A/I) & \longrightarrow & g_{\mathbf{k}}^{-1}(\mathfrak{Z}_{\mathbf{k}}^{\mathrm{reg}}) \\
\downarrow & & \downarrow \\
\mathrm{Sp}(A) & \longrightarrow & \mathfrak{Z}_{\mathbf{k}}^{\mathrm{reg}}.
\end{array}
$$

This diagram gives rise to a family of filtered φ-modules over A and, as above, we write $(\mathrm{WD}(D), \Phi_{\mathrm{WD}})$ for the associated family of Weil–Deligne representations on $\mathrm{Sp}(A)$. Further, the upper arrow in the diagram gives us a filtration

$$0 \subsetneq \overline{\mathrm{Fil}}^1 \subsetneq \cdots \subsetneq \overline{\mathrm{Fil}}^{d-1} \subset \overline{\mathrm{Fil}}^d = \overline{\mathrm{WD}(D)} = \mathrm{WD}(D)/I$$

by subspaces that are locally on $\mathrm{Sp}(A)$ direct summands and stable under the Frobenius $\Phi_{\overline{\mathrm{WD}}} = \Phi_{\mathrm{WD}} \mod I$ and we have to prove that this filtration uniquely lifts to a Φ_{WD}-stable filtration $\mathrm{Fil}^\bullet$ of $\mathrm{WD}(D)$ such that locally on $\mathrm{Sp}(A)$ the Fil^i are direct summands of $\mathrm{WD}(D)$. After localizing

on $\mathrm{Sp}(A)$ we may assume that D is free and that all the $\overline{\mathrm{Fil}}^i$ are direct summands of $\overline{\mathrm{WD}(D)}$. We show that we can lift $\overline{\mathrm{Fil}}^1$ uniquely to a Φ-stable direct summand of $\mathrm{WD}(D)$. The rest will follow by induction. Let us chose a basis $\bar{e}_1, \ldots, \bar{e}_d$ of $\overline{\mathrm{WD}(D)}$ such that $\overline{\mathrm{Fil}}^i$ is generated by $\bar{e}_1, \ldots, \bar{e}_i$ and take arbitrary lifts e_j of the $\bar{e}_j$ in $\mathrm{WD}(D)$. We write $A = (a_{ij})$ for the matrix of Φ_{WD} in this basis. Then the above implies that $a_{ij} \in I$ for $i > j$ and that $a_{ii} \neq a_{jj}$ for all $i \neq j$, as this is true modulo all maximal ideals of A by definition of $\mathfrak{Z}_{\mathbf{k}}^{\mathrm{reg}}$. We have to show that there exist uniquely determined $\lambda_2 \ldots, \lambda_d \in I$ and a $\mu \in A^\times$ such that

$$\Phi\left(e_1 + \sum_{j=2}^{d} \lambda_j e_j\right) = \mu\left(e_1 + \sum_{j=2}^{d} \lambda_j e_j\right).$$

However, this comes down to showing that

$$(A - \mu E)e_1 + \sum_{j=2}^{d} \lambda_j (A - \mu E)e_j = 0$$

has (up to scalar) a unique solution with $\lambda_i \in I$ which is an easy consequence of $a_{ij} \in I$ for $i > j$ and $a_{ii} \neq a_{jj}$ for $i \neq j$. $\qquad\square$

The following result is a slightly stronger version of [Ch13, Lemma 4.4]. Its proof parallels the proof of that lemma, but relies on a stronger statement about integral models on Kisin's crystalline deformation ring.

PROPOSITION 4.10. *There is a Zariski-open and dense subset $Z_{\mathbf{k}}^{\mathrm{gen}} \subset Z_{\mathbf{k}}$ whose closed points are precisely the generic crystalline representations. In particular, there is an $R_{\bar{r}}$-open subset $\mathfrak{Z}_{\mathbf{k}}^{\mathrm{gen}} \subset \mathfrak{Z}_{\mathbf{k}}$ whose rigid analytic points are precisely the generic crystalline representations.*

Proof. After twisting with some power of the cyclotomic character we may assume that all Hodge–Tate weights are non-negative. The ring $A_{\mathbf{k}}$ is a quotient of the universal deformation ring $R_{\bar{r}}$ and (with the notations of [Ki08, 1.6.4]) it is even a quotient of $R_{\bar{r}}^{\leqslant h}$ for some $h \gg 0$. By construction of the quotient $R_{\bar{r}}^{\leqslant h}$ there is a free $A_{\mathbf{k}} \hat{\otimes}_{\mathbb{Z}_p} W\llbracket u \rrbracket$-module $\mathfrak{M}$ together with an injection $\Phi : \mathfrak{M} \to \mathfrak{M}$ that is semi-linear with respect to the identity on $A_{\mathbf{k}}$, the Frobenius on $W = \mathcal{O}_{K_0}$ and $u \mapsto u^p$ and whose cokernel is killed by $E(u)^h$. Here $E(u)$ denotes the minimal polynomial of a chosen uniformizer of K over K_0.

By [Ki08, Theorem 2.2.5] this module has the property that for all closed points $x \in Z_{\mathbf{k}}$ with corresponding crystalline representation $r_x : \mathcal{G}_K \to \kappa(x)$ one has a canonical isomorphism

$$D_{\mathrm{cris}}(r_x) \cong (\mathfrak{M}/u\mathfrak{M}, \Phi \bmod u) \otimes_{A_{\mathbf{k}}} \kappa(x).$$

Let us write $\mathfrak{N} = \mathfrak{M}[1/p]$. By the construction in [Ki08, 2.1, 2.2, (2.5.3)] or [PR09, 5.a.1, (5.30)] we have an isomorphism

$$\mathfrak{N}/u\mathfrak{N} \otimes_W K \cong \Phi(\varphi^* \mathfrak{N})/E(u)\Phi(\varphi^* \mathfrak{N})$$

under which (for all closed points $x \in X_{\mathbf{k}}$) the Hodge filtration on $D_{\mathrm{cris}}(r_x) \otimes_{K_0} K$ is induced by the filtration on $\Phi(\varphi^* \mathfrak{N})/E(u)\Phi(\varphi^* \mathfrak{N})$ by the $E(u)^i \mathfrak{N}/E(u)\Phi(\varphi^* \mathfrak{N})$. As the condition for two possible filtrations to be in general position is an open condition it follows that there is a Zariski-open subset $Z_{\mathbf{k}}^{\mathrm{gen}} \subset Z_{\mathbf{k}}$ whose closed points are precisely the generic crystalline points.

It remains to show that every connected component of $Z_{\mathbf{k}}$ contains a generic point (recall that $Z_{\mathbf{k}}$ is formally smooth and hence every connected component is irreducible). However, this is proven exactly as in [Ch13, Proposition 4.3]. $\qquad\square$

4.3 Making use of eigenvarieties

We now return to the setup with eigenvarieties. The data of F, E, G, H, $\bar{\rho}$ are satisfying the properties of §3.1, so that we can consider the eigenvariety $Y_{\bar{\rho}} = Y(W_{\infty}, S, e)_{\bar{\rho}}$ with $S \supset S_p$, $S \backslash S_p$ being the set of places where H is not hyperspecial and e a suitable idempotent. Recall that if v is a place of F not dividing p and inert in E, then H_v is hyperspecial.

Recall that we have defined a Zariski-open subset $X(\bar{r})^{\mathrm{reg}} \subset X(\bar{r})$ of the trianguline space. Moreover, this subset is smooth.

DEFINITION 4.11. Write $X(\bar{\rho}, W_{\infty})^{\mathrm{aut}}$ for the union of irreducible components of $X(\bar{r})$ whose intersection with the regular part $X(\bar{r})^{\mathrm{reg}} \subset X(\bar{r})$ is met by any of the eigenvarieties $Y(W_{\infty}, S, e)_{\bar{\rho}}$ under the map defined in Theorem 3.5 for some (S, e) where S contains S_p and is such that each element of $S \backslash S_p$ splits in E.

Remark 4.12. The space $X(\bar{\rho}, W_{\infty})^{\mathrm{aut}}$ (or rather a variant of it which is defined in a fixed *patching situation*) is studied in more detail in [BHS14].

LEMMA 4.13. *The space* $X(\bar{\rho}, W_{\infty})^{\mathrm{aut}}$ *is non-empty.*

Proof. Let C be an automorphic component of the crystalline deformation space $\mathfrak{Z}_{\mathbf{k}}$ for some weight $\mathbf{k}$. Then Proposition 4.10 and Theorem 4.5 imply that there exists an automorphic point $r \in C$ which is generic. It follows that for some (S, e) with corresponding eigenvariety $Y_{\bar{\rho}} = Y(W_{\infty}, S, e)_{\bar{\rho}}$, there is a classical point $y \in Y_{\bar{\rho}}$ mapping to r under the canonical morphism

$$Y_{\bar{\rho}} \xrightarrow{\ f\ } X(\bar{r}) \longrightarrow \mathfrak{X}_{\bar{r}}.$$

We claim that $f(y) \in X(\bar{r})^{\mathrm{reg}}$ which implies the claim.

Let us write Π for the automorphic representation that gives rise to the point $y \in Y_{\bar{\rho}}$. Then the representation Π_{v_0} is unramified and there is a character χ of $T_{v_0}/T_{v_0}^0$ such that $\Pi_{v_0}|\det|^{(1-d)/2}$ appears in the parabolic induction $\mathrm{Ind}_B^{\mathrm{GL}_d(K)} \chi$ and such that $f(y) = (r, \delta_1, \ldots, \delta_d) \in X(\bar{r}) \subset \mathfrak{X}_{\bar{r}} \times \mathcal{T}^d$, where δ_i is an algebraic character of weight $(k_{\sigma,i})_{\sigma}$ with

$$\delta_i(\varpi) = \chi(t_i) \prod_{\sigma} \sigma(\varpi)^{k_{\sigma,i}}$$

and $\chi(t_i)$ is an eigenvalue of the crystalline Frobenius on $\mathrm{WD}(D_{\mathrm{cris}}(r))$. By assumption r is generic and hence $(\delta_1, \ldots, \delta_n) \in \mathcal{T}_{\mathrm{reg}}^d$. Moreover, the fact that r is generic implies that the ordering of the Frobenius eigenvalues $\chi(t_1), \ldots, \chi(t_d)$ corresponds to the choice of a Frobenius stable flag that is in general position with the Hodge filtration (as all Frobenius stable flags have this property for generic r). It follows that r has a triangulation with parameter $\delta_1, \ldots, \delta_d$ which is nowhere split; cf. Remark 4.8. It follows that $f(y) = (r, \delta_1, \ldots, \delta_d) \in X(\bar{r})^{\mathrm{reg}}$ as claimed. $\qquad \square$

THEOREM 4.14. *The image of* $X(\bar{\rho}, W_{\infty})^{\mathrm{aut}}$ *in* $\mathfrak{X}_{\bar{\rho}_{w_0}} = \mathfrak{X}_{\bar{r}}$ *is Zariski-dense in a union of irreducible components of* $\mathfrak{X}_{\bar{\rho}_{w_0}}$.

Proof. Let us write T for the Zariski closure of the image of $X(\bar{\rho}, W_{\infty})^{\mathrm{aut}}$ in $\mathfrak{X}_{\bar{\rho}_{w_0}}$ for the moment. We view T as an adic space equipped with its reduced structure and write T^{sm} for the locus of regular points, which is Zariski-open and dense in T. Following the proof of [Ch13, 4.5] and [Na11, Theorem 4.3], we are reduced to showing that there exists a generic crystalline point $r \in T^{\mathrm{sm}}$ with the following additional properties:

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(i) if $\lambda_1, \dots, \lambda_d$ are the (pairwise distinct) eigenvalues of the Frobenius on $\mathrm{WD}(D_{\mathrm{cris}}(r))$, then $\lambda_i \lambda_j^{-1} \notin p^{\mathbb{Z}}$ for all i, j;

(ii) the space $X(\bar{\rho}, W_\infty, e)^{\mathrm{aut}}$ contains all possible triangulations of the representation r.

In fact the proof of Lemma 4.13 constructs representations that satisfy (i) and (ii) but that do not necessarily map to T^{sm}. In order to take care of this additional property we have to redo the proof, changing the point and even its weight.

Let us write $\pi : X(\bar{\rho}, W_\infty)^{\mathrm{aut}} \to \mathfrak{X}_{\bar{r}}$ for the canonical projection that by definition factors over T.

Fix a connected component $\mathcal{C} \subset X(\bar{\rho}, W_\infty)^{\mathrm{aut}} \cap X(\bar{r})^{\mathrm{reg}}$ such that $\pi^{-1}(T^{\mathrm{sm}}) \cap \mathcal{C} \neq \emptyset$ and hence $\pi^{-1}(T^{\mathrm{sm}}) \cap \mathcal{C}$ is Zariski-open and dense in $\mathcal{C}$. Then, by definition of $X(\bar{\rho}, W_\infty)^{\mathrm{aut}}$, there is a pair (S, e) such that $Y_{\bar{\rho}} = Y(W_\infty, S, e)_{\bar{\rho}} \neq \emptyset$ and a point $y_0 \in Y_{\bar{\rho}}$ such that $f(y_0) \in \mathcal{C}$. Here $f : Y_{\bar{\rho}} \to X(\bar{\rho}, W_\infty)^{\mathrm{aut}}$ is the map defined by Theorem 3.5.

LEMMA 4.15. *There is a classical point $y_1 \in Y_{\bar{\rho}}$ such that its image r_{y_1} in $\mathfrak{X}_{\bar{r}}$ lies on an automorphic component C of $\mathfrak{Z}_{\mathbf{k}} \subset \mathfrak{X}_{\bar{r}}$ for some strongly dominant weight $\mathbf{k} = \omega_Y(y_1)$ such that $C \cap T^{\mathrm{sm}}$ is non-empty (and hence Zariski-open and dense in C).*

Indeed, it suffices to construct y_1 classical such that $r_{y_1} \in C$ and $C \cap T^{\mathrm{sm}}$ is non-empty: as y is associated with an automorphic representation the component $C \subset \mathfrak{Z}_{\mathbf{k}}$ is automatically automorphic (note that we did not claim that $\mathbf{k}$ is the weight of y_0).

Proof of Lemma 4.15. Let $U \subset X(\bar{r})^{\mathrm{reg}} \cap \mathcal{C}$ be a connected open neighborhood of the point $x_0 = f(y_0) \in X(\bar{r})^{\mathrm{reg}} \cap X(\bar{\rho}, W_\infty)^{\mathrm{aut}}$ such that $\omega_d|_U$ has connected fibers. Indeed, such a neighborhood exists by Lemma 2.18. As $\pi^{-1}(T^{\mathrm{sm}}) \cap \mathcal{C}$ is Zariski-open and dense in $\mathcal{C}$, we find that $\pi^{-1}(T^{\mathrm{sm}}) \cap U$ is non-empty and hence it is Zariski-open and dense in U.

Let $V \subset Y_{\bar{\rho}}$ be a quasi-compact neighborhood of y_0 such that $f(V) \subset U$. Then there are constants $M_1, \dots, M_d$ such that

$$M_i \geqslant \mathrm{val}_x(\psi_{v_0}(t_{v_0,1} \dots t_{v_0,i})(x)) + 1$$

for all $x \in V$, where val_x is the valuation on $k(x)$ normalized by $\mathrm{val}_x(p) = 1$. As the strongly dominant algebraic weights $\mathbf{k} \in \mathcal{W}^d$ such that $M_i < k_{\sigma,i} - k_{\sigma,i+1}$ accumulate at $\omega(y_0)$ there is a weight $\mathbf{k}$ such that

$$k_{\sigma,i} - k_{\sigma,i+1} > M_i,$$
$$\omega_d^{-1}(\mathbf{k}) \cap U \cap \pi^{-1}(T^{\mathrm{sm}}) \neq \emptyset \quad \text{and} \quad \omega_d^{-1}(\mathbf{k}) \cap U \cap f(Y_{\bar{\rho}}) \neq \emptyset.$$

The last assumption is possible by the fact that the map $\omega : Y_{\bar{\rho}} \to \mathcal{W}^d$ is locally (on $Y_{\bar{\rho}}$) finite with open image.

It follows (see, for example, [Ch09, Theorem 1.6 (vi)]) that a point $y \in V$ with $f(y) \in \omega^{-1}(\mathbf{k}) \cap U$ is classical (i.e. associated to an automorphic representation) and has the required property that $T^{\mathrm{sm}} \cap C \neq \emptyset$, where $C \subset \mathfrak{Z}_{\mathbf{k}}$ is the connected component containing $r_y = \pi(f(y))$: as $U \cap \omega_d^{-1}(\mathbf{k})$ is connected it maps to C and hence $\omega_d^{-1}(\mathbf{k}) \cap U \cap \pi^{-1}(T^{\mathrm{sm}}) \neq \emptyset$ implies $T^{\mathrm{sm}} \cap C \neq \emptyset$. $\qquad\square$

Let us fix the weight $\mathbf{k}$ and the automorphic component $C \subset \mathfrak{Z}_{\mathbf{k}}$ as in the preceding lemma.

LEMMA 4.16. *There exist a tuple (S', e') and (up to lowering the tame level H^{v_0}) a classical point $y' \in Y'_{\bar{\rho}} = Y(W_\infty, S', e')_{\bar{\rho}}$ such that the crystalline representation $r_{y'}$ given by the image of y' in $\mathfrak{X}_{\bar{r}}$ is generic and y' lies on $C \subset \mathfrak{Z}_{\mathbf{k}} \subset \mathfrak{X}_{\bar{r}}$.*

Proof of Lemma 4.16. By Corollary 4.6 the automorphic points are $R_{\bar{r}}$-dense in C and by Proposition 4.10 the generic locus is $R_{\bar{r}}$-open in C. Hence there exist a pair (S', e') and a point $y' \in Y'_{\bar{\rho}} = Y(W_\infty, S', e')_{\bar{\rho}}$ with the desired properties (at least after a lowering of the tame level H^{v_0}). $\square$

We write $f' : Y'_{\bar{\rho}} \to X(\bar{r})$ for the map defined by Theorem 3.5 corresponding to the eigenvariety $Y'_{\bar{\rho}}$. Now the triangulations of $r_{y'}$ are in bijection with the orderings of the Frobenius eigenvalues as the representation is regular and hence there are exactly $d!$ such triangulations.

LEMMA 4.17. *All these possible triangulations of $r_{y'}$ lie in $f'(Y'_{\bar{\rho}})$.*

Proof of Lemma 4.17. Let us write Π for the automorphic representation associated to y'. As Π_{v_0} is unramified, there are $d!$ distinct characters χ_i of $T_{v_0}/T_{v_0}^0$ such that $\Pi_{v_0}|\det|^{(1-d)/2}$ appears in the parabolic induction $\mathrm{Ind}_B^{\mathrm{GL}_d(K)} \chi_i$ and as the representation is generic the Hodge filtrations are in general position with the triangulation given by the ordering of the Frobenius eigenvalues. It follows, as in the proof of Lemma 4.13, that the parameters of the triangulation are prescribed by the character χ_i. $\square$

Let us finish the proof of the theorem. The preceding lemma shows that there are points $z_1, \ldots, z_{d!} \in X(\bar{\rho}, W_\infty)^{\mathrm{aut}}$ with $z_i = f'(\Pi, \chi_i)$ mapping to $r_{y'} \in \mathfrak{X}_{\bar{r}}$, and $z_1, \ldots, z_{d!}$ are precisely the $d!$ possible triangulations of the crystalline representation $r_{y'}$. Note that the assumption that $r_{y'}$ is generic implies that the trianguline filtrations have to be nowhere split and hence $z_i \in X(\bar{\rho}, W_\infty)^{\mathrm{aut}} \cap X(\bar{r})^{\mathrm{reg}}$. However, we still cannot conclude as we do not know whether $r_{y'}$ lies in the smooth locus T^{sm}.

Using Theorem 2.11, we may identify open subsets of $X(\bar{r})^{\mathrm{reg}}$ with open subsets of $\mathcal{S}^{\mathrm{ns}}(\bar{r})$. Now there exist connected open neighborhoods

$$U_i \subset X^{\mathrm{reg}}(\bar{r}) \cap X(\bar{\rho}, W_\infty)^{\mathrm{aut}} \cap \omega^{-1}(\mathbf{k})$$

of the z_i such that $U_i \cap U_j = \emptyset$. As the map $g_{\mathbf{k}} : \omega^{-1}(\mathbf{k}) \to \mathfrak{Z}_{\mathbf{k}}$ from Proposition 4.9 is étale at all the z_i it follows from [Hu96, Proposition 1.7.8] that $g_{\mathbf{k}}$ is open in some neighborhood of the z_i. Hence it follows that (after eventually shrinking the U_i) the image of the U_i in $\mathfrak{Z}_{\mathbf{k}}^{\mathrm{reg}} \cap C$ is open, and after shrinking the U_i even further we may assume that $U_z = g_{\mathbf{k}}(U_i) = g_{\mathbf{k}}(U_j)$ for all i, j. It follows that all the crystalline representations $r \in U_z$ have the property that $X(\bar{\rho}, W_\infty)^{\mathrm{aut}} \cap X^{\mathrm{reg}}(\bar{r})$ contains all their possible triangulations.

Finally, the subset of all generic points $r \in C$ such that the Frobenius eigenvalues satisfy condition (i) from above is a Zariski-open and dense subset of $C' \subset C$. Hence $U_z \cap C' \cap T^{\mathrm{sm}} \neq \emptyset$ and an element of this intersection is a point lying in T^{sm} and satisfying conditions (i) and (ii) from above. $\square$

4.4 End of the proof

Let us retain the notation from the preceding subsection. Fix a strongly dominant weight $\mathbf{k} = (k_{\sigma,i})_\sigma \in \prod_\sigma \mathbb{Z}^d$ (not necessarily as in the preceding section) and recall the subset $\mathcal{W}_{\mathbf{k},\mathrm{la}}^d \subset \mathcal{W}^d$. Recall further that we wrote $X_{\mathbf{k}}(\bar{r}) \subset X(\bar{r})$ for the $R_{\bar{r}}$-closure of all crystabelline points of Hodge–Tate weight $\mathbf{k}$.

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We prove the following theorem which will imply the desired result on the density of potentially crystalline representations of fixed weight.

THEOREM 4.18. *We have an inclusion $X(\bar{\rho}, W_\infty)^{\mathrm{aut}} \subset X_{\mathbf{k}}(\bar{r})$.*

Proof. It follows from Corollary 3.15 or Theorem 3.17 that

$$f(Y_{\bar{\rho}}) \subset X_{\mathbf{k}}(\bar{r})$$

for every eigenvariety $Y_{\bar{\rho}} = Y(W_\infty, S, e)_{\bar{\rho}}$.

Let (S, e) and $Y_{\bar{\rho}} = Y(W_\infty, S, e)_{\bar{\rho}}$ be chosen so that we can find a point $y \in Y_{\bar{\rho}}$ such that $x = f(y) \in X(\bar{r})^{\mathrm{reg}}$ (cf. the proof of Lemma 4.13). By definition $\omega(x) = \omega_Y(y)$ is a strongly dominant algebraic weight $\mathbf{k}'$. Let us write $X'_{\mathbf{k}}$ for the intersection of $\omega^{-1}(\mathbf{k}')$ with $X(\bar{\rho}, W_\infty)^{\mathrm{aut}} \cap X(\bar{r})^{\mathrm{reg}}$. Let C denote the connected component of $X_{\mathbf{k}'}$ containing x and let $C^{\mathrm{cris}} \subset C$ be the Zariski-open (and dense) subset of crystalline points.

LEMMA 4.19. *The component C is contained in $X_{\mathbf{k}}(\bar{r})$.*

Proof of Lemma 4.19. By construction C maps under the projection to $\mathfrak{X}_{\bar{r}}$ to a connected component C' of $\mathfrak{X}_{\bar{r}, \mathbf{k}'}^{\mathrm{cris}} = \mathfrak{Z}_{\mathbf{k}'}$. We write R' for the quotient of $R_{\bar{r}, \mathbf{k}'}^{\mathrm{cris}}$ corresponding to C'. Further, the component C' is an automorphic component, as by assumption the representation defined by x extends to an automorphic Galois representation.

Let $t \in R_{\bar{r}}$ be an element vanishing on $X_{\mathbf{k}}(\bar{r})$ and consider its image, still denoted by t, in R'. Let $z' \in C'$ be an automorphic point corresponding to a crystalline $\mathcal{G}_K$-representation $r_{z'}$. By definition, we can find an irreducible cuspidal automorphic representation Π of an unitary group G, of some level $\tilde{H} \subset H$ as in §3.1 such that $(\rho_\Pi)_{w_0} \simeq r_{z'}$. Then there exist a triple (W_∞, S', e') and a character χ of T/T^0 such that $(\Pi, \chi) \in \mathcal{Z}$. Here $\mathcal{Z}$ is the set of classical points of the eigenvariety $Y(W_\infty, S', e')$. Now Theorem 3.17 implies that the image of $(\Pi, \chi) \in Y(W_\infty, S', e')_{\bar{\rho}_\Pi}$ in $X(\bar{r})$ is in fact contained in $X_{\mathbf{k}}(\bar{r})$, or equivalently that $t(z') = 0$. From Corollary 4.6 we can deduce that the image t in R' is zero. As the image of C under $\pi : X(\bar{r}) \to \mathfrak{X}_{\bar{r}}$ is contained in C', we have shown that any $t \in R_{\bar{r}}$ vanishing on $X_{\mathbf{k}}(\bar{r})$ also vanishes on C. This implies that $C \subset X_{\mathbf{k}}(\bar{r})$. □

We can now conclude. Fix $\mathcal{C}$ a component of $X(\bar{\rho}, W_\infty)^{\mathrm{aut}}$ and $y_0 \in Y_{\bar{\rho}}$ such that $f(y_0) = x_0 \in X(\bar{\rho}, W_\infty)^{\mathrm{aut}} \cap X(\bar{r})^{\mathrm{reg}}$. Let $\mathbf{k}'_0 = \omega_Y(y)$ and pick $\tilde{x}_0 \in \mathcal{S}^{\mathrm{ns}}(\bar{r})$ such that $\pi_{\bar{r}}(\tilde{x}_0) = x_0$. By Lemma 2.18 there is a quasi-compact connected neighborhood U of x inside $X^{\mathrm{reg}}(\bar{r})$ such that U is isomorphic to a product of an open subset $U_1 \subset \mathcal{W}^d$ with a rigid space U_2 which we may choose to be connected. After shrinking U_1, we may also assume that $f(Y_{\bar{\rho}}) \cap U$ surjects onto U_1. As U is quasi-compact, there exist $M_1, \ldots, M_d$ such that

$$M_i \geqslant \mathrm{val}_x(\psi_{v_0}(t_{v_0,1} \ldots t_{v_0,i})(x)) + 1$$

for all $x \in U$, where val_x is the valuation on $k(x)$ normalized by $\mathrm{val}_x(p) = 1$. Let us write $Z_1 \subset U_1$ for the set of strongly dominant algebraic weights $k_{\sigma,1} > \cdots > k_{\sigma,d}$ such that $M_i < k_{\sigma,i} - k_{\sigma,i+1}$ for all i and $\sigma : K \hookrightarrow \bar{\mathbb{Q}}_p$. Then Z_1 is Zariski-dense in U_1 and hence $\omega^{-1}(Z_1) \cap U$ is Zariski-dense in U.

A point $y \in f^{-1}(U)$ mapping to $\mathbf{k}' \in Z_1$ is classical by the choice of M_i and [Ch09, Theorem 1.6(vi)]. Applying Lemma 4.19 with the point y and the component of $X_{\mathbf{k}'}$ containing $\omega^{-1}(\{\mathbf{k}'\}) \cap U \simeq U_2$, we find that $\omega^{-1}(\{\mathbf{k}'\}) \cap U \subset X_{\mathbf{k}}(\bar{r})$. This implies that $X_{\mathbf{k}}(\bar{r})$ contains $\omega^{-1}(Z_1) \cap U$ which is Zariski-dense in U and hence $U \subset X_{\mathbf{k}}(\bar{r})$. This implies that $X_{\mathbf{k}}(\bar{r})$ contains $\mathcal{C}$. □

THEOREM 4.20. *Let $p \nmid 2d$ and let K be a finite extension of $\mathbb{Q}_p$. Let $\bar{r} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ be an absolutely irreducible continuous representation and let $R_{\bar{r}}$ be its universal deformation ring. Assume that $\bar{r} \not\cong \bar{r}(1)$. Let $\mathbf{k} = (k_{i,\sigma}) \in \prod_{\sigma : K \hookrightarrow \bar{\mathbb{Q}}_p} \mathbb{Z}^d$ be a strongly dominant weight. Then the representations that are crystabelline of labeled Hodge–Tate weight $\mathbf{k}$ are Zariski-dense in $\mathrm{Spec}\ R_{\bar{r}}[1/p]$.*

Proof. The assumptions that $\bar{r}$ is absolutely irreducible and $\bar{r} \not\cong \bar{r}(1)$ imply that $Z = \mathrm{Spec}\ R_{\bar{r}}[1/p]$ is smooth and irreducible.

Let $X = X(\bar{\rho}, W_\infty)^{\mathrm{aut}} \subset X(\bar{r})$ for a suitable choice of W_∞ as in Lemma 4.3. Our assumptions imply that X is non-empty and hence Theorem 4.14 implies that X has dense image in Z. Let $t \in R_{\bar{\rho}}$ be a function vanishing on all crystabelline points of weight $\mathbf{k}$. Then Corollary 2.7 implies that it vanishes on $X \cap \omega^{-1}(\mathcal{W}_{\mathbf{k},\mathrm{la}}^d)$ and hence by Theorem 4.18 it vanishes on X. The claim follows as X has dense image in Z. $\qquad\square$

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