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GREENBERG'S THEOREM FOR QUASICONVEX SUBGROUPS OF WORD HYPERBOLIC GROUPS

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ABSTRACT. Analogues of a theorem of Greenberg about finitely generated subgroups of free groups are proved for quasiconvex subgroups of word hyperbolic groups. It is shown that a quasiconvex subgroup of a word hyperbolic group is a finite index subgroup of only finitely many other subgroups.

1. Introduction.

Introductory remarks. There has been much interest recently in reintroducing geometric ideas into group theory. One of the most interesting examples is Gromov's work on the class of word hyperbolic groups [Gr]. The notion of a word hyperbolic group arose in the work of J. Cannon [C] and M. Gromov [Gr] as a generalization of certain group theoretic properties of discrete groups of isometries of classical hyperbolic spaces $\mathbb{H}^n$, the origins of which are to be found in Dehn's solution of the word and conjugacy problems for surface groups (see [De]). Word hyperbolic groups are finitely presented, and examples are provided by finite groups, finitely generated free groups, fundamental groups of compact surfaces (except for the torus and the Klein Bottle), and groups which act properly discontinuously and cocompactly on hyperbolic space of any dimension. Finite extensions of word hyperbolic groups, and free products of finitely many word hyperbolic groups are also word hyperbolic, though a direct product of two infinite word hyperbolic groups is not word hyperbolic. We will give some background information about word hyperbolic groups in Sections 2 and 3 and refer the reader to Gromov's original article [Gr], and the several commentaries that now exist (for instance [ABC], [Bo1], [CDP], [GH]) for further information, and proofs of the basic results.

The object of this note is to prove for word hyperbolic groups analogues of a theorem of Greenberg for finitely generated subgroups of Fuchsian groups (see Theorem 1, Part (2) below). In our case these results concern *quasiconvex subgroups* (for the full definition see Section 2), which are a special kind of finitely generated subgroup. Indeed, in a free group a subgroup is finitely generated if and only if it is quasiconvex. In a word hyperbolic group, all finite subgroups, all cyclic subgroups, and all subgroups of finite index are quasiconvex. Also, a quasiconvex subgroup of a word hyperbolic group

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is word hyperbolic. It is not true in general that a finitely generated subgroup of a word hyperbolic group is word hyperbolic as there exist finitely generated subgroups of word hyperbolic groups which are not finitely related (see for instance [R], [BMS]). Moreover, recently N. Brady [Br] constructed a remarkable example of a torsion-free hyperbolic group which contains a finitely presented subgroup which is not itself word hyperbolic.

An important part of Gromov's work involves the introduction of a "boundary" which compactifies the Cayley graph of a word hyperbolic group (in fact it compactifies a δ-hyperbolic space). This construction generalizes the usual compactification of n-dimensional hyperbolic space by a $n-1$ dimensional sphere. A similar construction for groups was also investigated by Floyd in his thesis (see [F]).

In Section 2 of this paper we define a very rudimentary form of boundary for the group, which is not in fact a compactification. We show that several results about word hyperbolic groups and their subgroups can be proved using exactly the same techniques which are used in classical hyperbolic geometry, provided certain preparatory work is done. We introduce the notions of a boundary of a subgroup of a group, and of the convex hull of a subset of the boundary. We then show in 2.4(i) that for a quasiconvex subgroup A, the quotient of the convex hull of the boundary of A by the action of A is finite. Using this, we obtain:

THEOREM 1. *Let G be a word hyperbolic group, and let A be an infinite quasiconvex subgroup.*

(1) If B is an infinite quasiconvex subgroup of G, and $A \cap B$ has finite index in A and in B, then $A \cap B$ has finite index in $A \vee B$, the subgroup generated by A and B in G.

(2) The subgroup A has finite index in only finitely many distinct subgroups of G.

(3) The subgroup A has finite index in its virtual normalizer

$$VN_G(A) = \{g \in G \mid [A : A \cap gAg^{-1}] < \infty \text{ and } [gAg^{-1} : A \cap gAg^{-1}] < \infty\}.$$

In Section 3 we study the boundary of a δ-hyperbolic space (for instance the Cayley graph of a word hyperbolic group) as defined by Gromov, as a compactification of the space. The subgroup structure of word hyperbolic groups is now discussed from the point of view of understanding their limit sets in the boundary.

We improve on the results of Section 2, and show (Proposition 3.4) that an infinite subgroup is quasiconvex if and only if the quotient of a convex hull (appropriately defined) of its limit set by the action of the subgroup has finite diameter.

It is not possible to use the standard notion of convexity and convex hulls when working with word hyperbolic groups. Indeed, as it was demonstrated in a remarkable example of M. Bridson and G. Swarup [BS], it can happen that a (classical) convex hull of a finite subset of a Cayley graph of a group coincides with the whole group. Thus certain adjustments have to be made and an appropriate substitution for the classical notion of convex hull should be used.

Our new characterization of quasiconvex subgroups of word hyperbolic groups stresses the similarity between quasiconvexity and geometrical finiteness for classical hyperbolic groups and immediately yields the following result of G.Swarup [Swa].

THEOREM 2. *Let G be a torsion-free geometrically finite group of isometries of $\mathbb{H}^n$ without parabolics. Then G is word hyperbolic and a subgroup A of G is quasiconvex in G if and only if it is geometrically finite.*

We also obtain a criterion of commensurability for quasiconvex subgroups of word hyperbolic groups in terms of limit sets (Lemma 3.8) and obtain another proof of Theorem 1 as well as

THEOREM 3. *Let A be an infinite quasiconvex subgroup of a word hyperbolic group G and let B be a subgroup of G containing A. Then A has finite index in B if and only if A contains an infinite subgroup C which is normal in B.*

In Section 4 we remark that it is possible to give a criterion of quasiconvexity similar to Proposition 3.4 in terms of the elementary boundary we use in Section 1.

Theorem 3 has also been obtained by Mihalik and Towle [MT] and results similar to those of Section 3 were independently obtained by E. Swenson [Swe] and R. Gitik, M. Mitra, E. Rips and M. Sageev [GMRS]. Parts (1) and (2) were originally proved for finitely generated subgroups of free groups by L. Greenberg [G], using the usual compactification of the hyperbolic plane by a circle. He showed that part of the boundary can be naturally associated to a subgroup A of a discrete subgroup G of PSL$(2,\mathbb{R})$. He then showed that two subgroups have the same associated boundary if and only if they are finite extensions of a common subgroup.

We would also like to bring to the reader's attention Stallings' beautiful paper "The topology of finite graphs" [St] where most of Theorem 1 is proved for finitely generated subgroups of free groups.

Some comments on Theorem 1. Before proceeding to the definitions and proofs of our main theorem, we give some examples to illustrate the necessity of some conditions concerning infiniteness and quasiconvexity. Notice that a finite subgroup is always quasiconvex.

Statement (3), may fail if H is finite or if H is infinite but not quasiconvex. Consider the group $Q = \langle a, b \mid [a,b]^2 = 1\rangle$ and $H = \{1\}$. Then Q is word hyperbolic (being a small cancellation group) and there are infinitely many distinct conjugates of the subgroup $M = gp([a,b]) \cong \mathbb{Z}_2$ of Q (by the usual small cancellation arguments). Thus H is a subgroup of finite index in infinitely many subgroups of G although H is quasiconvex in Q.

When G contains an infinite cyclic subgroup, the trivial subgroup $H = \{1\}$ is contained in infinitely many cyclic subgroups (each of which is quasiconvex when G is word hyperbolic). Concerning Theorem 3, there is a word hyperbolic group G which has a normal subgroup H isomorphic to the fundamental group of a closed hyperbolic surface with the quotient group Greenberg's theorem for quasiconvex subgroups of word hyperbolic groups G/H isomorphic to $\mathbb{Z}$. Again, in this case, the normalizer of H is all of G, and $|G:H| = \infty$.

One can also construct an example of an infinite, finitely generated (but not quasiconvex) subgroup H of a word hyperbolic group G which has finite index in infinitely many subgroups of G, as follows. Let Q be the one-relator group with torsion defined above. Now apply the construction of E. Rips [R] to obtain a short exact sequence

$$1 \to K \to G \to Q \to 1$$

where G is a torsion-free word hyperbolic group (as it satisfies a small cancellation condition), and K is an infinite, normal two-generator subgroup of G. Let q be a conjugate of $[a,b]$ in Q and g be an element in the preimage of q in G. Put $K_q = gp(K,g)$. Then $g^2 \in K$, $g \notin K$ and K is a subgroup of index 2 in K_q. Moreover, the image of K_q in Q is the cyclic group of order two $gp(q)$. Thus for different conjugates q of g we get different subgroups K_q of G containing K as a subgroup of finite index.

Hyperbolic groups with all finitely generated subgroups quasiconvex. We say that a word hyperbolic group G belongs to class (Q) when a subgroup A of G is quasiconvex in G if and only if A is finitely generated.

Among the groups known to belong to the class (Q) are finitely generated free groups and fundamental groups of closed hyperbolic surfaces (see [Sho] and [Pi]). It is also clear that finite extensions and subgroups of finite index (in fact all quasiconvex subgroups) of groups in (Q) belong to (Q). G. Swarup has shown [Swa] that if G is a geometrically finite torsion-free Kleinian group without parabolics with a nonempty discontinuity domain (*i.e.* the limit set of G is not the whole sphere S^2) then G is in (Q). The last observation is based on a result of W. Thurston [Mo].

Notice that any group G in the class (Q) has the Howson property, that is the intersection of any two finitely generated subgroups is finitely generated (indeed, the intersection of two quasiconvex subgroups is quasiconvex and so is finitely generated, finitely presentable and even word hyperbolic). This approach to Howson property is discussed in more detail in [Sho].

In the class (Q), Theorem 1 can be stated in terms of finitely generated subgroups. In [Ba] H. Bass proved Theorem 1(2) in these terms for the case when G is the fundamental group of a finite graph of groups $\mathbb{Y}$ with finite vertex groups. Such a group G lies in (Q) since it is virtually free. There are two natural proper hyperbolic spaces on which G admits a properly discontinuous cocompact isometric action. One is the Cayley graph of G with respect to some finite generating set. Another is the Bass-Serre universal covering tree for the graph of groups $\mathbb{Y}$. The result can be established for G by studying either of these actions. The original proof of H. Bass employed the second action but the ideas involved in it are quite similar to ours.

In the case when G is a free group of finite rank, the Bass-Serre tree coincides with the Cayley graph of G with respect to a free basis. This action was studied by J. Stallings in [Sta] to obtain the same statement for free groups. The original proof of this result by L. Greenberg (which also applies to hyperbolic surface groups) used the fact that a free group G admits a discrete properly discontinuous geometrically finite action in $\mathbb{H}^2$ and therefore the action of G on the (classical) convex hull of its limit set is properly discontinuous and cocompact.

From our point of view the proof given in the present note captures the similarity of all these approaches.

2. **Definitions and Proof of Theorem 1.**

Some definitions. If (X,d) is a metric space and I is a connected subset of a real line, a map $f: I \longrightarrow X$ is called *geodesic* if $d\big(f(t), f(s)\big) = |t-s|$ for any $s, t \in I$. Sometimes the image of this map is also termed a geodesic. A metric space is termed *geodesic* if there is at least one geodesic joining each pair of points.

If Δ is a geodesic triangle (that is a triangle with geodesic sides) in a metric space (X, d) with vertices x, y, z and geodesic sides $[x,y]$, $[x,y]$, $[y,z]$ then there are unique points s, q, r on the sides $[x,y]$, $[x,z]$ and $[y,z]$ such that $d(x,s) = d(x,q)$, $d(z,q) = d(z,r)$ and $d(y,s) = d(y,r)$. The points s, q, r are called the *vertices* of the inscribed triangle in the triangle Δ. The triangle Δ is said to be δ-*slim* if, for any points s' on $[x,y]$ and q' on $[x,z]$ chosen such that $d(x,s') = d(x,q') \leq d(x,s) = d(x,q)$, we have $d(s',q') \leq \delta$ (and the analogous condition holds interchanging x, y and z).

A geodesic metric space (X,d) is called δ-*hyperbolic* if all geodesic triangles are δ-slim.

Let G be a group generated by a finite set S. We define the *S-length* of an element $g \in G$, denoted by $l_S(g)$, as the minimal number t such that g can be expressed in the form $g = s_1 \cdots s_t$ where $s_i \in S^{\pm 1}$. A minimizing expression is termed an *S-geodesic* representative of g. We will often confuse formal words in the alphabet S with the elements they represent in G.

The empty word e by convention represents the identity element of G. Define $d_S(g_1, g_2) = l_S(g_1^{-1} g_2), g_1, g_2 \in G$. This defines a metric on G called the *word metric* associated to S. The word metric d_S can be extended to a metric (also denoted d_S) on the Cayley graph $\Gamma(G,S)$, by declaring each edge to be isometric to the unit interval. This metric is clearly invariant under the natural left action of G.

A finitely generated group G is called *word hyperbolic* if for some finite generating set S of G there is $\delta \geq 0$ such that all geodesic triangles in the Cayley graph $\big(\Gamma(G,S),\ d_S\big)$ of G are δ-slim. (It turns out that this property is independent of the finite generating set chosen—see [Gr], [ABC], etc.).

In general, a subset A of a geodesic metric space (X,d) is termed ϵ-quasiconvex in X if any geodesic $[a_1, a_2]$ in X joining points $a_1, a_2 \in A$ is contained in an ϵ-neighbourhood of A. If G is a finitely generated group and A is a subgroup of G, we say that A is *quasiconvex* in G if for some finite generating set S of G A is a quasiconvex subset of the Cayley graph $\Gamma(G,S)$.

The following properties of quasiconvex subgroups and word hyperbolic groups and their quasiconvex subgroups are of importance to us here:

PROPOSITION 2.0. *Let G be a finitely generated group, and S a finite set of generators for G.*

(1) [GS] If A is a quasiconvex subgroup of G then A is finitely generated.

(2) [GS] If A is a quasiconvex subgroup of G then for any finite generating set T of A and for any finite generating set S of G there is a constant $B \geq 0$ such that $d_T(a_1, a_2) \leq \mathrm{B}d_S(a_1, a_2), a_1, a_2 \in A$.

(3) [Sho] If $A_1, \ldots, A_k$ are quasiconvex subgroups of G, then $A = A_1 \cap \cdots \cap A_k$ is quasiconvex in G.

If in addition, G is a word hyperbolic group, then:

(4) [BGSS, Theorem 3.1] If A is a finitely generated subgroup of G and for some finite generating set T of A and some finite generating set S of G there is a constant $B \geq 0$ such that $d_T(a_1, a_2) \leq \mathrm{B}d_S(a_1, a_2), a_1, a_2 \in A$, then A is a quasiconvex subset of $\Gamma(G, S)$. (Together with (2) this implies that the definition of quasiconvexity for a subgroup of a word hyperbolic group G does not depend on the choice of a finite generating set of G).

(5) [CDP, Chapter 3, Proposition 4.1] If A is a quasiconvex subgroup of G then for any finite generating set T of A and for any finite generating set S of G A is a quasiconvex subset of $\Gamma(G, S)$.

(6) [ABC, Lemma 3.8] If A is a quasiconvex subgroup of G then A is also word hyperbolic.

(7) If A, B are subgroups of G such that A has finite index in B, then A is quasiconvex in G if and only if B is quasiconvex in G.

(8) [GH, Chapter 4, Theorem 13] There are only finitely many conjugacy classes of elements of finite order in G.

(9) [GH, Chapter 8, Theorem 37] An infinite subgroup of G contains an element of infinite order.

(10) [ABC, Corollary 3.4] If C is a virtually cyclic subgroup of G then C is quasiconvex in G.

(11) [N] If $\phi \in \mathrm{Aut}(G)$ and A is a subgroup of G, then A is quasiconvex in G if and only if $\phi(A)$ is quasiconvex in G.

Other interesting properties of word hyperbolic groups, for instance that G is finitely presented and has a linear isoperimetric function, will not be needed here. For proof of these results and further information about quasiconvexity and word hyperbolic groups, we refer the reader to [Sho], [Gr], [ABC], [GH] and the other hyperbolic references.

An elementary boundary for a finitely generated group. In this section G is a finitely generated group, and S is a finite set of generators. The group G acts by left multiplication on the Cayley graph $\Gamma_S(G)$, by isometries. We construct a 'boundary' for G, which is essentially the boundary defined by Gromov for word hyperbolic groups deprived of its topology.

Consider the set of subsemigroups $\mathfrak{S} = \{\{g^i \mid i > 0\}, g \in G\} =_{def} \{\{g^+\} \mid g \in G\}$.

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We now extend to a larger set (where there is an obvious left action induced by left multiplication), and define $\Sigma = G \times \mathfrak{S} = \big\{\{hg^i\}, h, g \in G\big\} = \{hg^+, h, g \in G\}$.

We define an equivalence relation on Σ by: $hg^+ \sim h'g'^+$ if $\{hg^i; i > 0\}$ lies in a bounded neighbourhood of $\{h'g'^j; j > 0\}$ and $\{h'g'^j; j > 0\}$ lies in a bounded neighbourhood of $\{hg^i; i > 0\}$.

This is clearly an equivalence relation, and we denote the equivalence class of hg^+ by $[hg^+]$ (and $[hg^-] = [h(g^{-1})^+]$). Notice that $[g^+] = [g'^+]$ if and only if g and g' have a common (positive) power: there is some word d and infinitely many pairs (i,j) of positive numbers such that $g^i = g'^j d$.

The elementary boundary of G is defined to be $\tilde{\partial} G = (\Sigma/\sim) - \{[1]\}$.

We remove the equivalence class of the identity element due to its special nature—it clearly contains all elements of finite order. One consequence of this is that a torsion group will have empty boundary.

It is clear from the construction that G acts on $\tilde{\partial} G$ on the left by $h \cdot [h'g^+] = [hh'g^+]$.

LEMMA 2.1. *(i) If* $[g^+] = [(g^-]$ *then* g *has finite order in* G, *and* $[g^+] = [1]$;

(ii) The left action of G *on* $\tilde{\partial} G$ *by left multiplication is the same as the action induced by conjugation* $h \cdot [h'g^+] = [(hh'h^{-1})(hgh^{-1})^+]$.

(iii) The definition of $\tilde{\partial} G$ *is independent of the choice of finite generating set.*

PROOF. (i) If the positive powers of G lie in a bounded neighbourhood of the negative powers of g, then there is some element d in G such that $g^i = g^{-j}d$, and $g^k = g^{-\ell}d$ for some $i, j, k, \ell > 0$. It follows that $d = g^{i+j} = g^{k+\ell}$ in the group G. As this happens for infinitely many such quadruples, g has finite order.

(ii) Notice that $d(hg^ih^{-1}, hg^i) = \ell_S(h)$, so that $\{(hh'h^{-1}) \cdot hg^i = hh'g^i \mid i > 0\}$ lies in a $\ell_S(h)$-neighbourhood of $\{(hh'h^{-1}) \cdot hg^ih^{-1} = hh'g^ih^{-1} \mid i > 0\}$ and vice versa.

(iii) Choosing a finite set of generators Y for G changes length in the group by at most a constant factor, so that equivalence is unchanged.

Notice that as a result of part (ii) of this lemma, in each equivalence class there is a representative of the form $\{g^+\}$, *i.e.* $\Sigma/\sim = \mathfrak{S}/\sim$. We now study subsets of this boundary which correspond to subgroups.

DEFINITION. Let A be a subset of G. We define the boundary of A in G, to be: $\tilde{\partial}_G A = \{[a^i] \mid a \in A\}$. Notice that, by (ii) above, $\tilde{\partial}_G G = \tilde{\partial} G$.

For a subset Y of the boundary $\tilde{\partial} G$, we define the *stabilizer* of Y to be:

$$\mathrm{Stab}_G(Y) = \{g \in G \mid gY = Y\}.$$

For a fixed finite set S of generators for G, define the *convex hull* of Y to be: $\mathcal{H}_S(Y) = \{v \in G$ such that there are $g, g', h, h' \in G$ such that $[hg^+], [h'g'^+] \in Y$, and there are two infinite, strictly increasing sequences n_i, m_i, such that v lies on a S-geodesic from hg^{n_i} to $h'g'^{m_i}\}$.

Notice that, while the definition of the stabilizer is independent of the choice of generating set, the convex hull *does* depend on the choice. We will however refer to Stab(H) and $\mathcal{H}(Y)$ when the ambient group and generating set are clear from the context.

EXAMPLES. 0. By Lemma 2.1(i) above, the boundary of a finite group, (or indeed of an infinite torsion group) is empty.

1. The boundary of the infinite cyclic group has two elements, (which we can think of as $\pm\infty$), and the stabilizer of each of these elements is the entire group. With the single generator 1, the convex hull of the boundary is clearly the whole group, as is the case for any finite set of generators.

2. The free abelian group of rank two: each maximal cyclic subgroup has a naturally associated "slope", and each rational number $\frac{m}{n}, n > 0$ (and $0, \infty$) defines a cyclic subgroup $\langle x^m y^n \rangle$. The boundary of this cyclic subgroup is two points, and the boundary of the group is a countable set (of rational slopes). All rank two subgroups have the same boundary, that of the whole group. Notice that the convex hull of the boundary of any nontrivial subgroup is the whole group.

3. The free group of rank two: each cyclically reduced element g which is not a proper power has its own equivalence class $[g^+]$. The Cayley graph with respect to a set of free generators is a tree, and the convex hull of the boundary of a cyclic subgroup generated by a cyclically reduced word g is the 'axis'—the line containing the elements of this subgroup (*i.e.* vertices corresponding to initial segments of powers of g).

For a noncyclically reduced word, the convex hull is the translated axis of the cyclically reduced part; if $w = vuv^{-1}$ with u cyclically reduced, then the elements of $\mathcal{H}(\tilde{\partial}\langle w \rangle)$ are the vertices of the Cayley graph corresponding to vu', where u' is an initial segment of the word u^m, for $m \in \mathbb{Z}$.

LEMMA 2.2. *Let G be a finitely generated group, S a finite set of generators and $Y \subset \tilde{\partial}G$.*

(i) Stab(Y) *acts freely on the convex hull $\mathcal{H}_S(Y)$.*

(ii) If A is a subgroup of G, $A \subset H_G(A) \subset$ Stab($\tilde{\partial}_G A$).

(iii) If A and B are subgroups of G, and A has finite index in B, then $\tilde{\partial}_G A = \tilde{\partial}_G B$.

PROOF. (i) G acts freely on itself on the left. It suffices to show that if $v \in \mathcal{H}_S(Y)$, and $b \in$ Stab(Y), then $bv \in \mathcal{H}_S(Y)$.

Suppose that v lies on geodesics from hg^i to $h'g'^j$, $[hg^+], [h'g'^+] \in Y$. For $b \in G$, bv lies on a geodesic from bhg^i to $bh'g'^j$, and for $b \in$ Stab(Y), it follows that $bv \in \mathcal{H}_S(Y)$.

(ii) As in Lemma 2.1 (ii) above, for $a \in A, d(ag^i, ag^i a^{-1}) \leq |a|$, so $\{ag^i\} \sim \{ag^i a^{-1}\}$. In fact, as long as a normalizes A, this is an element of $\tilde{\partial}_G A$.

(iii) If $b \in B$, then $b^k = a$ for some $a \in A, k > 0$, so $[b^+] = [a^+]$. ∎

(Notice that in the proof of (iii) we did not use the fact that k can be chosen to be bounded. Thus it was actually proved that if A is a subgroup of B and for each element of B some power of it belongs to A, then $\tilde{\partial}_G A = \tilde{\partial}_G B$).

We now concentrate on word hyperbolic groups and their quasiconvex subgroups, which have particularly nice properties. Recall that all cyclic subgroups in a word hyperbolic group, are quasiconvex.

LEMMA 2.3. *Let A be a quasiconvex infinite cyclic subgroup of the finitely generated group G. Then:*

(i) $\mathcal{H}(\tilde{\partial}_G A)$ is infinite;

(ii) if, in addition, G is a word hyperbolic group, then $\mathcal{H}(\tilde{\partial}_G A)$ is contained in a bounded neighbourhood of A, and is thus quasigeodesic too.

PROOF. Fix a finite set of generators S for G.

(i) Let $\langle c \rangle$ be a K-quasiconvex infinite cyclic subgroup. We show that *all* geodesics from c^m to c^{-m} pass through a ball of bounded radius about the identity vertex, for all m sufficiently large. Notice that these geodesics all lie in a K neighbourhood of the quasiconvex subgroup $\langle c \rangle$.

Consider a geodesic triangle with vertices $1, c^m, c^{-m}$. Let v_1, v_2 be adjacent vertices on the geodesic α joining c^m and c^{-m} such that there are positive numbers $p, q > 0$ with $d(v_1, c^{-p}) \leq K$ and $d(v_2, c^q) \leq K$. Then $d(c^{-p}, c^q) < 2K + 1$; this bounds $p + q$, and in particular bounds p and q. Hence the geodesic α passes through a bounded neighbourhood B of the identity element. It follows that there is an infinite sequence $\{m_i\}$ and geodesics γ_i joining c^{m_i} and c^{-m_i}, and a vertex $v \in B$, such that every γ_i passes through the vertex v. Thus $H_S(\tilde{\partial}_G A)$ is nonempty, and by 2.2(i) it is therefore infinite.

(ii) Suppose that $v \in \mathcal{H}_S(\tilde{\partial}_G A)$. Let $\{hg^i\}$ and $\{h'g'^j\}$ be sequences representing elements of $\tilde{\partial}_G A$ such that v lies on infinitely many geodesics joining the points hg^i and $h'g'^j$. There is a constant $C > 0$, and $a, b \in A$ such that $\{hg^i \mid i > 0\}$ and $\{h'g'^j \mid j > 0\}$ are contained in C neighbourhoods of the cyclic subgroups $\langle a \rangle$ and $\langle b \rangle$.

Choose i, j sufficiently large so that $\min\{d(v, g^i), d(v, g'^j)\} > 2\delta + C$. Let $a', b' \in A$ be nearest elements of A to hg^i and $h'g'^j$.

By studying the quadrilateral with geodesic sides and vertices $a', b', hg^i, h'g'^j$, we see that the vertex v is at distance at most 2δ from some point on a geodesic joining a' and b'. As A is quasigeodesic with constant K, this means that v is at distance at most $K + 2\delta$ from A. ∎

COROLLARY 2.4. *Let A be a quasiconvex subgroup of the word hyperbolic group G.*

(i) The quotient space $\mathcal{H}_X(A)/A$ is finite.

(ii) A has finite index in Stab$(\tilde{\partial}_G A)$.

PROOF. As $\mathcal{H}(A)$ lies in a bounded K-neighbourhood of A, and A acts freely on $\mathcal{H}(A)$, each element of the quotient space has a representative of length at most K. These represent the cosets of A in Stab$(\tilde{\partial} A)$. ∎

It is now easy to deduce Theorem 1:

PROOF OF THEOREM 1. (1) If $A \cap B$ has finite index in A, and A is quasiconvex in G, then $A \cap B$ is also quasiconvex in G. By Lemma 2.2 it follows that $\tilde{\partial}_G A = \tilde{\partial}_G(A \cap B) = \tilde{\partial}_G B$. As A and B both stabilize $\tilde{\partial}_G(A \cap B)$, so does $A \vee B$. Hence $A \cap B < A \vee B <$ Stab$\big(\tilde{\partial}_G(A \cap B)\big)$, and applying 2.4(ii) and the quasiconvexity of $A \cap B$ gives the result.

(2) If A has finite index in B, then $\tilde{\partial}_G A = \tilde{\partial}_G B$, and $A < B <$ Stab$(\tilde{\partial}_G A) = C$. If A is quasiconvex then B is a finite index subgroup of C, of index at most $[C : A]$.

There are only finitely many such subgroups. In fact, if $\{1, g_1, \dots, g_k\}$ is a set of coset representatives for A in C, then B is obtained by adjoining some subset of the $\{g_i\}$ to A.

(3) Suppose that $g \in VN_G(A)$. Put $U = gAg^{-1} \cap A$. Since U is of finite index in A and in gAg^{-1}, $K = \tilde{\partial}_G(U) = \tilde{\partial}_G(A) = \tilde{\partial}_X(gAg^{-1})$. However $\tilde{\partial}_X(gAg^{-1})$ is equal to the set $\{g[a^+] \mid a \in A\}$ that is to gK. Thus $gK = K$ and $g \in \mathrm{Stab}_G(K)$. Since $g \in VN_G(A)$ was chosen arbitrarily, we conclude that $A \leq VN_G(A) \leq \mathrm{Stab}_G(K)$. By Corollary 2.4(ii) this implies that A has finite index in $VN_G(A)$.

Notice that part (2) of the theorem gives another way to see that $\langle a, b \mid b^{-1}ab = a^2 \rangle$ is not word hyperbolic. The subgroup $\langle a \rangle$ has index 2 in $\langle bab^{-1} \rangle$, and index 2^n in $\langle b^n a b^{-n} \rangle$.

3. **Gromov boundary and quasiconvexity in hyperbolic groups.**

Definitions and notations. In order to obtain stronger results than those obtained in the Section 2, we require a boundary with a topological structure. Such a boundary has been defined by Gromov for δ-hyperbolic spaces (see also Floyd's work [F]). For the detailed discussion and proofs of the basic properties of this compactification (for instance that the elementary boundary of Section 2 can be viewed as a dense subset), the reader is referred to [Gr, Chapter 1,3,7], [CDP, Chapter 2], [GH, Chapter 5-7] and [ABC, Chapter 4]. We here give some of the basic definitions that we shall need.

A metric space (X, d) is said to be *proper* if all closed metric balls in X are compact. Any proper δ-hyperbolic metric space (X, d) can be compactified by adding points of the *ideal boundary* (or visual boundary) ∂X to X. Each point of the boundary ∂X is an equivalence class of geodesic rays $r\colon [0, \infty) \longrightarrow X$ where rays r_1 and r_2 are equivalent if $\sup\big\{d\big(r_1(t), r_2(t)\big)\big\} < \infty$. It is not hard to see that if r_1 and r_2 are equivalent then $\sup\big\{d\big(r_1(t), r_2([0,\infty))\big) \mid t \in [0, \infty)\big\} \leq \max\big\{d\big(r_1(0), r_2(0)\big), 2\delta\big\}$.

There is another way to think about points of the boundary. Fix a point $p \in X$ and put

$$(x, y)_p = (1/2)\big(d(x, p) + d(y, p) - d(x, y)\big)$$

for any $x, y \in X$. Notice that, considering a geodesic triangle with vertices p, x, y, we see that $(x, y)_p$ is equal to the distance from p to the two vertices of the inscribed triangle which lie on the sides $[p, x]$ and $[p, y]$ (see the introduction to Section 2).

We say that a sequence $(x_i)_{i \in \mathbb{N}}$ of points in X *converges to infinity* if

$$\lim_{n \to \infty} \{\inf_{i,j \geq n} (x_i, x_j)_p\} = \infty.$$

Two sequences $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ converging to infinity are said to be equivalent if

$$\lim_{n \to \infty} \{\inf_{i,j \geq n} (x_i, y_j)_p\} = \infty.$$

One can check that the last two definitions do not depend on the choice of a basepoint $p \in X$. It follows from the definition that if $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ are sequences of points in X such that for some finite K, $d(x_i, y_i) \leq K$ for each $i \in \mathbb{N}$ then $(x_i)_{i \in \mathbb{N}}$ converges to infinity if and only if $(y_i)_{i \in \mathbb{N}}$ converges to infinity, and the sequences are equivalent in this case.

The points of the ideal boundary ∂X can be identified with equivalence classes of sequences converging to infinity. If a is the equivalence class of $(x_i)_{i\in\mathbb{N}}$ we write $\lim_{i\to\infty} x_i = a$. It is obvious that if $r\colon[0,\infty)\longrightarrow X$ is a geodesic ray then the sequence $x_i = r(i)$ converges to infinity. It is also clear that if r_1, r_2 are equivalent geodesic rays then the sequences $\big(r_1(i)\big)_{i\in\mathbb{N}}$ and $\big(r_2(i)\big)_{i\in\mathbb{N}}$ are also equivalent.

Moreover, since (X,d) is proper, any sequence converging to infinity is equivalent to the sequence $\big(r(i)\big)_{i\in\mathbb{N}}$ for some geodesic ray r. This gives an identification between the points of ideal boundary and equivalence classes of sequences converging to infinity [Gr, 7.5].

It can be shown (see [CDP, Chapter 2, Theorem 2.1]) that for any two distinct points $a, b\in\partial X$ there exists a (not necessarily unique) bi-infinite geodesic $r\colon(-\infty,\infty)\longrightarrow X$ such that $\lim_{i\to\infty} r(-i) = a$ and $\lim_{i\to\infty} r(i) = b$. This geodesic r is sometimes denoted by (a,b) and referred to as a geodesic joining a and b.

Analogously, for any point $a\in\partial X$ and for any point $x\in X$ there is a geodesic ray $r\colon[0,\infty)\longrightarrow X$ such that $r(0) = x$ and $\lim_{i\to\infty} r(i) = a$. Such a ray r is referred to as a geodesic joining x and a and is sometimes denoted $[x,a)$.

Fix a basepoint p of X and let $r_1, r_2\colon[0,\infty)\longrightarrow X$ be geodesic rays joining p to points a and b in the boundary. We put

$$(r_1,r_2)_p = \sup_{t\in[0,\infty)} \big(r_1(t), r_2(t)\big)_p = \lim_{t\to\infty}\big(r_1(t), r_2(t)\big)_p.$$

We define $(a,b)_p$ to be the infimum of the numbers $(r_1,r_2)_p$ where r_1, r_2 vary over all geodesic rays joining p to a and b. In fact it can be shown that for any such r_1 and r_2, we have $|(r_1,r_2)_p - (a,b)_p| \leq 10\delta$.

The boundary ∂X of X is topologized by putting $\lim_{i\to\infty} a_i = a$ if $\lim_{i\to\infty}(a_i,a)_p = \infty$, where $a, a_i\in\partial X$.

It is easy to describe this topology in the language of sequences converging to infinity. Namely, let a_n, $n = 1,2,\ldots$, be the equivalence class of a sequence $(x_{n,i})_{i\in\mathbb{N}}$ and let a be the equivalence class of a sequence $(x_i)_{i\in\mathbb{N}}$. Then $\lim_{i\to\infty} a_n = a$ if

$$\lim_{m\to\infty}\{\inf_{i,j\geq n}(x_i, x_{m,j})_p\}\longrightarrow\infty.$$

All the definitions given above do not depend on the choice of a basepoint. Moreover ∂X and $X\cup\partial X$ are compact with the topologies described above [CDP, Chapter 2, Proposition 3.2].

We need one more piece of notation. If $x\in X$ and r is a geodesic ray joining p with a point $a\in\partial X$, we put $(x,r)_p = \lim\big(x, r(t)\big)_p = \sup\{\big(x,r(t)\big)_p \mid t\in[0,\infty)\}$. One observes that if a, b are distinct points in ∂X, and points $x\in r_1 = [p,a), y\in r_2 = [p,b)$ are chosen such that $M = (x,b)_p = (y,a)_p$, and $d(x,p) > 50(M+1)\delta$, $d(y,p) > 100(M+1)\delta$, then $(x,y)_p$, $(x,r_2)_p$, and $\big(r_1,y)_p\big)$ are within 10δ of $(r_1,r_2)_p$.

Convex hulls and limit sets in hyperbolic spaces. Let X be a proper δ-hyperbolic space for some $\delta > 1$.

We need the following obvious lemma.

LEMMA 3.1. *Any ideal geodesic quadrilateral, that is a quadrilateral with vertices on the boundary of X, is a 100δ-quasiconvex subset of X.*

PROOF. This follows immediately from the fact that ideal quadrilaterals in X are 10δ-thin, that is any side is contained in a 10δ-neighborhood of the union of other sides.

We can now define the convex hull of a subset of the boundary. We can now use infinite rays where in Section 2 we had to make explicit reference to infinite families of geodesics.

DEFINITION. (see [Gr, 7.5.A]) Let X be a proper δ-hyperbolic space with $\delta > 1$, and let K be a subset of ∂X containing at least two points. Then the set $CK = \bigcup_{a,b\in K}(a,b)$ is called the *convex hull* of K in X.

LEMMA 3.2. *Let X and K be as above. Then CK is 100δ-quasiconvex in X.*

PROOF. Let $x,y \in CK$. Then $x \in (a,b)$ and $y \in (c,d)$ for some $a,b,c,d \in K$. Assume that the points a,b,c,d are distinct (it will be clear from the argument that the case when some of them coincide is completely analogous).

Join points b and c by a geodesic (b,c) and points d and a by a geodesic (d,a) in X. By definition of CK the ideal quadrilateral

$$D = (a,b) \cup (b,c) \cup (c,d) \cup (d,a)$$

is contained in CK. By Lemma 3.1 a geodesic (x,y) is contained in 100δ-neighborhood of D and so in 100δ-neighborhood of CK. Thus CK is 100δ-quasiconvex.

Let (X,d) be as above and suppose a group G acts on X by isometries so that the action is *properly discontinuous* (*i.e.* for any compact Q in X the set $\{g \in G \mid Q \cap gQ \neq \emptyset\}$ is finite) and *cocompact* (that is X/G with the quotient topology is compact). The action of G on X induces an action of G on ∂X by homeomorphisms: namely an element g of G takes the equivalence class of a geodesic ray r to the equivalence class of the ray gr. Equivalently, if $(a_i)_{i\in\mathbb{N}}$ is a sequence converging to infinity, then an element g of G takes its equivalence class to the equivalence class of the sequence $(ga_i)_{i\in\mathbb{N}}$ (which obviously also converges to infinity).

Pick a basepoint $p \in X$. For any subgroup $A \leq G$ define the *X-limit set* of A, denoted by $\partial_X(A)$, as the collection of all those $b \in \partial X$ which are the limits of some sequences of points from Ap (in the topology of $X \cup \partial X$). It is easy to see that this definition does not depend on the choice of the basepoint $p \in X$ and that $\partial_X(A)$ is always closed.

Analogous to the results of Section 2, the limit sets defined above have the following properties:

LEMMA 3.3. *Let (X,d) be a proper, δ-hyperbolic space, and G a group acting properly discontinuously and cocompactly by isometries on X. Let A and B be subgroups of G. Then:*

(1) The group G is word hyperbolic.

(2) $\partial_X(A) = \emptyset$ if and only if A is finite.

(3) *If $\langle c\rangle$ is an infinite cyclic subgroup of G then $\partial_X(\langle c\rangle)$ consists of two distinct points $c^+ = \lim_{i\to\infty} c^i$ and $c^- = \lim_{i\to\infty} c^{-i}$.*
(4) *If A is contained in B then $\partial_X(A) \subset \partial_X(B)$.*
(5) *If A has finite index in B then $\partial_X(A) = \partial_X(B)$.*
(6) *If A is infinite then $\partial_X(A)$ contains at least two distinct points.*
(7) *If A is infinite then the sets $\partial_X(A)$ and $C\partial_X(A)$ are A-invariant.*

PROOF. Statement (1) follows from the fact that in these circumstances, X and G are quasi-isometric (see for instance [ECHLPT, Theorem 3.3.6]) and so G is word hyperbolic as a group quasi-isometric to a proper hyperbolic metric space [GH, Chapter 5, Theorem 12]. Statements (4), (5) and (7) are immediate from the definitions, while (2) follows immediately from (6). Statement (3) follows from the fact that an infinite cyclic subgroup is quasiconvex in G (though this is not used in 2.1) and from [CDP, Chapter 3, Theorem 2.2]. Statement (6) is a consequence of Proposition 2.0(9) together with (3) and (4).

We give a sketch of a proof of (6) independent of Proposition 2.0(9), using Proposition 2.0(8), which is easier to prove.

If A has an element of infinite order, the statement is trivial. Suppose now that every element in A has finite order. Since there are only finitely many conjugacy classes of elements of finite order in G, we can find an integer N such that the order of any element of A divides N.

Fix a basepoint $p \in X$. Since Ap is infinite and $X \cup \partial X$ is compact, the set $\partial_X(A)$ contains at least one point q. Assume $q = \partial_X(A)$. Then $Aq = q$. Fix a geodesic ray $r = [p,q)$ in X. Let $A = \{a_i | i = 1,2,\ldots\}$ be an enumeration of A such that $a_i \neq a_j$ when $i \neq j$. Since $a_i q = q$ the rays $a_i r$ and r are 2δ-close far away from p. Thus the action of a_i on r far away from p is 4δ-close to a translation by t_i. Since the action of G on the set of vertices in X (and so in r) is free and A is infinite, we conclude that $\lim_{i\to\infty} t_i = \infty$. Thus for big enough i we have $t_i > 1000N\delta$. But for such i the action of a_i^N on r far away from p is $16N\delta$-close to a translation by Nt_i. This contradicts the fact that $a_i^N = 1$ and (6) is proved. ∎

As the set $C\partial_X(A)$ is A-equivariant, the quotient space $C\partial_X(A)/A$ inherits a metric from the restriction of d to $C\partial_X(A)$: put $\bar{d}(Ax,Ay) = \inf\{d(a_1x,a_2y) \mid a_1,a_2 \in A\}$ for $x,y \in X$.

Clearly if $x,y \in X$ then $\inf\{d(a_1x,a_2y) \mid a_1,a_2 \in A\} = \inf\{d(x,ay) \mid a \in A\}$. Moreover, for any $x,y \in X$ $\inf\{d(x,ay) \mid a \in A\} = \min\{d(x,ay) \mid a \in A\}$. Indeed, let $M = \inf\{d(x,ay) \mid a \in A\}$. Then there is $a_0 \in A$ such that $d(x,a_0y) \leq M+1$. Since the closed metric ball $B = B(y,2M+2)$ is compact and the action of G on X is properly discontinuous, the set $\{a \in A \mid d(a_0y,ay) \leq 2M+2\}$ is finite. Therefore the set $\{a \in A \mid d(x,ay) \leq M+1\}$ is finite and so $\inf\{d(x,ay) \mid a \in A\} = \min\{d(x,ay) \mid a \in A\}$. Thus whenever $\bar{d}(Ax,Ay) = 0$ we have $Ax = Ay$. Besides for any $z \in X$, $a_1 \in A$ $\bar{d}(Ax,Ay) = \min\{d(x,ay) \mid a \in A\} \leq d(x,a_1z) + \min\{d(ay,a_1z) \mid a \in A\} = \bar{d}(x,a_1z) + \bar{d}(Ay,Az)$. Since $a_1 \in A$ was chosen arbitrarily, this implies that $\bar{d}(Ax,Ay) \leq \bar{d}(Ax,Az) + \bar{d}(Az,Ay)$. Thus $(C\partial_X(A)/A,\ \bar{d})$ is indeed a metric space.

We are now in a position to prove an improved, metric version of Proposition 2.4 (the necessary condition has become an equivalent condition).

PROPOSITION 3.4. *Let G be a word hyperbolic group and A be a subgroup of G. The following conditions are equivalent*

(i) A is quasiconvex in G;

(ii) for any proper δ-hyperbolic space X and for any properly discontinuous cocompact action of G on X by isometries the quotient space $C\partial_X(A)/A$ has finite diameter;

(iii) for some proper δ-hyperbolic space X and some cocompact properly discontinuous action of G on X by isometries the quotient space $C\partial_X(A)/A$ has finite diameter.

Proposition 3.4 provides an alternative definition of quasiconvexity which highlights the connection of the theory of word hyperbolic groups with the theory of classical hyperbolic groups (*i.e.* discrete groups of isometries of $\mathbb{H}^n$).

Before proving Proposition 3.4, we must do a little preliminary work.

Recall that if $(X, d), (Y, d_1)$ are metric spaces, a map $f: X \longrightarrow Y$ is called a *K-quasi-isometry* of X if there is $K > 0$ such that for each $x_1, x_2 \in X$ $(1/K)\, d(x_1, x_2) - K \leq d_1\big(f(x_1), f(x_2)\big) \leq K\, d(x_1, x_2) + K$.

The map f is called a *K-quasi-isometry* of X *onto* Y if in addition the image $f(X)$ is *K-cobounded* in Y, that is, for each $y \in Y$ there is $x \in X$ such that $d_1\big(y, f(x)\big) \leq K$. In this latter case we say that the two spaces are quasi-isometric.

A K-quasi-isometry of a connected subset of the real line to Y is called a K-quasigeodesic. In a δ-hyperbolic space X, there is a constant ϵ, depending on X and K, such that if γ is a K-quasigeodesic joining two points x, y, and α is a geodesic joining the same points, then α is contained in a ϵ-neighbourhood of γ, and γ is contained in a ϵ-neighbourhood of α. For a proof of this essential result ("quasigeodecis are close to geodesics"), and more about quasi-isometries, see [Gr, Chapter 7], [GH, Chapter 5], [CDP, Chapter 4], etc.

A proper metric space (X, d) is said to be λ-quasigeodesic ($\lambda > 1$), if for any pair of points $x, y \in X$ there is a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) \leq \lambda$ for $i = 0, \ldots, n-1$ and $|i - j| \leq \lambda\, d(x_i, x_j) + \lambda$ for any i, j.

The following lemma will be needed in the proof of 3.4 (it is analogous to Theorem 3.3.6 of [ECHLPT]).

LEMMA 3.5. *Suppose a group G acts by isometries on the λ-quasigeodesic space X properly discontinuously and cocompactly. Choose a basepoint $p \in X$ Then the following hold:*

(1) The group G is finitely generated.

(2) If S is a finite generating set of G then the map $f: (G, d_S) \longrightarrow (X, d)$ defined by $f: g \longmapsto gp$ is a quasi-isometry of the Cayley graph $\big(\Gamma_S(G), d_S\big)$ onto (X, d).

PROOF. We use $B(x, r)$ to denote the closed metric ball of radius r centered at x.

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To see that G is finitely generated we follow the chain argument as in [ECHLPT, Theorem 3.3.6]. Put $M = \mathrm{diam}(X/G)$. Put $S = \{g \in G \mid B(p, M+\lambda) \cap gB(p, M+\lambda) \neq \emptyset\}$.

The set S is finite since $B(p, M+\lambda)$ is compact and the action of G is properly discontinuous.

We claim that S generates G. Indeed, let $g \in G$. Consider a λ-quasigeodesic sequence $p = x_0, x_1, \ldots, x_n = gp$ connecting p and gp. Put $B = B(p, M)$. Clearly $GB = X$. Therefore for any $i = 1, \ldots, n-1$ there is $g_i \in G$ such that $x_i \in g_i B$. Put $g_0 = 1, g_n = g$. Then $(x_{i+1}, g_i p) \leq d(x_{i+1}, x_i) + d(x_i, g_i p) \leq M + \lambda$ and $d(x_{i+1}, g_{i+1}p) \leq M$ for $i = 0, 1, \ldots, n-1$. Thus $g_i^{-1} x_{i+1} \in B(p, M+\lambda) \cap B(g_i^{-1} g_{i+1} p, M)$ and so $h_i = g_i^{-1} g_{i+1} \in S$. But clearly $g = h_0 h_1 \cdots h_{n-1}$ and hence G is generated by S.

We will prove statement (2) of Lemma 3.5 for this particular choice of a finite generating set S of G. The general statement will then follow immediately from the fact that changing a finite generating set of a group induces a quasi-isometry of the word metrics.

Put $N = \max\{d(p, sp) \mid s \in S\}$. By definition the set S is symmetric that is $S = S^{-1}$. Let $g_1, g_2 \in G$. We need to estimate $d(g_1 p, g_2 p)$ in terms of $d_S(g_1, g_2)$. If $s_1 \cdots s_k$ is a d_S-geodesic representative for $g_1^{-1} g_2$ then put $y_0 = g_1 p, y_i = g_1 s_1 \cdots s_i p$ for $i = 1, \ldots, k$. Then $d(y_i, y_{i+1}) = d(p, s_{i+1} p)$ and so $d(g_1 p, g_2 p) = d(y_0, y_k) \leq Nk = N\, d_S(g_1, g_2)$.

Now we need to estimate $d_S(g_1, g_2)$ in terms of $d(g_1 p, g_2 p)$. Consider a λ-quasigeodesic sequence $z_0 = g_1 p, \ldots, z_n = g_2 p$ connecting $g_1 p$ with $g_2 p$ (instead of a geodesic path $[g_1 p, g_2 p]$ as it is done in [ECHLPT, 3.3.6]). Now for any $i = 1, \ldots, n-1$ there is $f_i \in G$ such that $z_i \in f_i B(p, M)$. Put $f_0 = g_1, f_n = g_2$ and $h_i = f_i^{-1} f_{i+1}$. As before, $d(f_i p, f_{i+1} p) \leq 2M + \lambda$ and hence $h_i \in S$ for each $i = 0, 1, \ldots, n-1$. Again $g_1^{-1} g_2 = h_0 h_1 \cdots h_{n-1}$ and so $d_S(g_1, g_2) \leq n \leq \lambda\, d(g_1 p, g_2 p) + \lambda$ since the sequence $z_0, \ldots, z_n$ is λ-quasigeodesic. Thus

$$(1/N)\, d(g_1 p, g_2 p) \leq d_S(g_1, g_2) \leq \lambda\, d(g_1 p, g_2 p) + \lambda.$$

It remains to note that the set $f(G) = Gp$ is cobounded in G. But we have already seen that $GB = GB(p, M) = X$ and so for any $x \in X$ there is $gp \in Gp$ such that $d(x, gp) \leq M$. This completes the proof of Lemma 3.5.

We need to introduce one extra piece of notation before proving Proposition 3.4. If Y is a set and d_1, d_2 are real-valued functions on $Y \times Y$ we term them *equivalent* and write $d_1 \sim d_2$ if for some $C > 0$ we have $(1/C) d_1 - C \leq d_2 \leq C d_1 + C$. Also if Z is a set, Y is a set equipped with a real-valued function d on $Y \times Y$ and $h: Z \longrightarrow Y$ is a map, we term the function d_1 on $Z \times Z$ given by $d_1(z_1, z_2) = d\big(h(z_1), h(z_2)\big)$ the *h-pullback* of d and denote it $h_*(d)$.

Thus, for instance, two finite sets of generators for a finitely generated group give rise to equivalent distance functions.

PROOF OF PROPOSITION 3.4. (i) implies (ii): Let A be a quasiconvex subgroup of G and suppose that G acts on X as in (ii). If A is finite the the statement is obvious since $\partial_X(A)$ is empty. Assume now that A is infinite so that it has at least two distinct limit points in ∂X. Pick a basepoint $p \in X$ so that $p \in C\partial_X(A)$ and a finite generating set S

for G. As in Lemma 3.5, the map $f\colon (G, d_S) \longrightarrow (X, d)$ given by $f(g) = gp$ is a K-quasi-isometry for some $K > 0$. Consider a point $x \in C\partial_X(A)$; by definition $x \in (t_1, q_1)$ for some $t_1, q_1 \in \partial_X(A)$. This implies that for some points $t \in [x, t_1)$ and $q \in [x, q_1)$ far from x, and for some $a, b \in A$, the values $(t, ap)_x, (q, bp)_x$ are big.

More precisely, for some $L > 1000\delta$, the points t, q and $a, b \in A$ are such that $(t, ap)_x \geq L, (q, bp)_x \geq L$. Consideration of the quadrilateral formed by ap, bp, t and q, shows that there is a point $y \in [ap, bp]$ with $d(x, y) \leq 6\delta$.

Consider now a d_S-geodesic sequence of points $g_0 = a, g_1, \ldots, g_n = b$ in G, that is $d_S(g_i, g_j) = |i - j|$. Since f is a K-quasi-isometry, the broken line $\alpha = [ap, g_1p] \cup [g_1p, g_2p] \cup \cdots \cup [g_{n-1}p, bp]$ is a $2KM$-quasigeodesic where $M = \max\{d(p, sp), s \in S\}$. As quasigeodesics are close to geodesics, there is a constant $\epsilon = \epsilon(X, K, M)$ such that α is ϵ-close to the geodesic $[ap, bp]$, and so there is $k, 0 \leq k \leq n$ such that $d(y, g_kp) \leq M + \epsilon$. Therefore $d(x, g_kp) \leq 6\delta + M + \epsilon$. Now A is e-quasiconvex in (G, d_S) and $g_0 = a$, $g_1, \ldots, g_n = b$ is a d_S-geodesic sequence. Therefore there is $a_1 \in A$ such that $d_S(g_k, a_1) \leq e$. This implies $d(g_kp, a_1p) \leq Ke + K$ and so $d(x, a_1p) \leq 6\delta + M + \epsilon + Ke + K = K_1$. This implies $\operatorname{diam}(C\partial_X(A)/A) \leq K_1$ since $x \in \partial_X(A)$ was chosen arbitrarily.

(ii) implies (iii): Let (X, d) be the Cayley graph of G with respect to some finite generating set. Then X is δ-hyperbolic for some $\delta > 1$, geodesic, has compact closed metric balls and the action of G on X by left multiplication is properly discontinuous and co-compact.

(iii) implies (i): Assume (X, d) and an action of G on X are as in (iii). Fix a finite generating set S for G and word metric d_S on G. If A is finite then it is obviously quasi-convex in G. Assume now that A is infinite. By Lemma 3.3, the boundary $\partial_X(A)$ contains at least two points and so $C = C\partial_X(A)$ is nonempty. Moreover C is 100δ-quasiconvex in X by Lemma 3.1. Therefore the closure (in X) $\bar{C}$ is 102δ-quasiconvex and A-invariant. It is easy to see that $(C, d|_C)$ is λ-quasigeodesic for $\lambda = 10000\delta$, as follows. Consider any two points $x, y \in \bar{C}$ and a geodesic segment $[x, y]$ in X. Choose a sequence $v_0 = x, v_1, \ldots, v_n = y$ of points on $[x, y]$ so that $d(v_{n-1}, v_n \leq 1$ and $d(v_i, v_{i+1}) = 1$ for $i = 0, \ldots, n-2$. Then for each $i = 1, \ldots, n-1$ there is a point $x_i \in \overline{C}$ such that $d(v_i, x_i) \leq 102\delta$. Put $x_0 = x$ and $x_n = y$. Then the sequence $x = x_0, x_1, \ldots, x_n = y$ is a required λ-quasigeodesic sequence connecting x and y inside $\bar{C}$ (in fact it is a $(204\delta + 1)$-quasigeodesic).

Now $\bar{C}$ is A-invariant and $\bar{C}/A$ is compact since C/A is bounded and X has compact closed metric balls. Thus Lemma 3.5 implies that A is finitely generated. Pick a finite generating set T for A. Take a point $p \in C$ and define $f\colon G \longrightarrow X$ by $f(g) = gp$. Notice that $f(A) \subset C$ since $p \in C$ and C is A-invariant.

The word metric d_T on A is equivalent to the f-pullback of the restriction of d to $\bar{C}$ by Lemma 3.5. On the other hand the word metric d_S on G is equivalent to the f-pullback of d by [ECHLPT, 3.3.6].

Therefore

$$d_T \sim f_*(d|_{\bar{C}}) = f_*(d)|_A \sim d_S|_A.$$

Thus $d_T \sim d_S|_A$ and so A is quasiconvex in G.

LEMMA 3.6. *Let X be a δ-hyperbolic space ($\delta > 1$) and K be a closed subset of ∂X with at least two elements. Let $C = CK$ and K_1 be the limit set of C in ∂X, that is K_1 is the union of all limits of sequences of points of C converging to infinity.*

Then $K = K_1$.

PROOF. Clearly $K \subset K_1$.

Pick a point $p \in C$ and put $C_1 = C \cup \left(\cup_{a \in K}[p,a)\right)$. Then C_1 is contained in the 10δ-neighbourhood of C since ideal triangles in X are 10δ-thin. Indeed, $p \in (x,y)$ for some $x,y \in K$. Let $a \in K$ be an arbitrary element different from x,y. Then the ideal triangle with sides $[p,x)$, $[p,a)$ and (x,a) is 10δ-thin (here $[p,x) \subset (x,y)$). Thus $[p,a)$ is contained in the 10δ-neighbourhood of $[p,x) \cup (x,a)$ and so in the 10δ-neighbourhood of C.

Suppose $b = \lim_{i \to \infty} x_i$ where $x_i \in C$ and $b \in K_1$. By the previous remark, we may assume that each x_i lies on a geodesic $r_i = [p,a_i)$ for some $a_i \in K$. Since ∂X is compact, after passing to a subsequence, we may assume that $\lim a_i = a$. Clearly $a \in K$ since K is closed. Fix a geodesic ray $r = [p,a)$. We are going to show that $a = b$, and thus that $K_1 = K$. Since $\lim a_i = a$, we conclude that $\lim(r_i,r)_p = \infty$. If $b \neq a$ and $x_i \longrightarrow b$, there is $M > 1, M < \infty$ such that $(x_i,r)_p \leq M$ for each i. Now $d(p,x_i) \longrightarrow \infty$. Therefore we may assume, after passing to a subsequence, that for each $i (x_i,r)_p \leq M$ and $d(p,x_i) > 1000M\delta$. But the last two inequalities imply that $(r_i,r)_p$ is 10δ-close to $(x_i,r)_p$ which contradicts $\lim(r_i,r)_p = \infty$. ∎

Recall (see [Bo2] for details) that a discrete group G of isometries of a n-dimensional hyperbolic space $\mathbb{H}^n$ without parabolics (that is any element of infinite order fixes exactly two points in the boundary S^{n-1}) is termed *geometrically finite* if $\mathrm{Conv}\big(\Lambda(G)\big)/G$ is compact, where $\Lambda(G)$ is a limit set of G in S^{n-1} and $\mathrm{Conv}\big(\Lambda(G)\big)$ is a (classical) convex hull of the union of all geodesics with endpoints in $\Lambda(G)$. The following theorem, which is due to G. Swarup [Swa], follows immediately from our new definition of quasiconvexity.

THEOREM 3.7 [SWA]. *Let G be a torsion free geometrically finite group of isometries of $\mathbb{H}^n$ without parabolics. Then G is word hyperbolic and a subgroup A of G is quasiconvex in G if and only if A is geometrically finite.*

PROOF. Take X to be the (classical) convex hull of the limit set of G. As a convex subset of $\mathbb{H}^n$, X is 10-hyperbolic with the metric induced from $\mathbb{H}^n$. The group G is geometrically finite and without parabolics. Thus the action of G on X is properly discontinuous and X/G is compact.

Therefore by Theorem 3.3.6 of [ECHLPT] G is finitely generated and has a Cayley graph which is quasi-isometric to a 10-hyperbolic metric space X, so that G is word hyperbolic. The stated properties of quasiconvex subgroups now follow directly from Proposition 3.4 and the definition of geometrical finiteness.

Indeed, if $A \leq G$ is geometrically finite then $\mathrm{Conv}\big(\Lambda(A)\big)/A$ is compact and so has bounded diameter $M < \infty$. On the other hand $\Lambda(A) = \partial_X(A) = K$ and $CK \subset \mathrm{Conv}\big(\Lambda(A)\big)$ is A-invariant. Thus $\mathrm{diam}(CK/A) \leq M$ and A is quasiconvex in G by Proposition 3.4.

Suppose now that $A \le G$ is quasiconvex in G and $\Lambda(A) = \partial_X(A) = K$ as above. Then $\operatorname{diam}(CK/A) = N < \infty$. It is easy to see that $CK \subset \operatorname{Conv}(\Lambda(A))$ and $\operatorname{Conv}(\Lambda(A))$ is contained in a $1000n$-neighborhood of CK (where $n = \dim \mathbb{H}^n$). This implies that $\operatorname{Conv}(\Lambda(A))/A$ has bounded diameter and so is compact. Thus A is geometrically finite.

We now obtain the analogue of Lemma 2.2 (ii).

LEMMA 3.8. *Suppose G is a word hyperbolic group acting by isometries on a proper δ-hyperbolic space X so that the action is properly discontinuous and cocompact. Let A be an infinite subgroup of G, $K = \partial_X(A)$ and $H = \mathrm{Stab}_G(K) = \{g \in G \mid gK = K\}$. Then*

(1) For any subgroup E of H, containing A, $\partial_X(E) = \partial_X(A)$. In particular $\partial_X(H) = \partial_X(A)$.

(2) If A is normal in a subgroup E of G then $\partial_X(E) = \partial_X(A)$.

PROOF. (1) Let $C = CK$. Notice that K has at least two points so that C is nonempty. Pick a point $p \in C$. It is clear that $K = \partial_X(A) \subset \partial_X(E)$. Since K is E invariant, $C = CK$ is also E-invariant. Thus $Ep \subset C$ and therefore $\partial_X(E)$ is contained in the limit set of C which by Lemma 3.6 is equal to K. Therefore $\partial_X(A) = K = \partial_X(E)$.

(2) If A is normal in E, for any $g \in E$ we have $gAp = Agp$. The limit set of Ap is equal to the limit set of Agp and is equal to K. Thus $gK = K$ and so $A \le E \le \mathrm{Stab}_G(K)$. By Part (1) this implies $\partial_X(A) = \partial_X(E) = K$.

Commensurability in hyperbolic groups. It is possible to obtain the following criterion of commensurability for quasiconvex subgroups of hyperbolic groups; this is the promised strengthened form of 2.4.

LEMMA 3.9. *Let G be a word hyperbolic group and let E, H be subgroups of G with $H \le E$. Let (X, d) be a proper δ-hyperbolic metric space ($\delta > 1$) equipped with properly discontinuous isometric cocompact action of G. Suppose that H is infinite and quasiconvex in G and let $K = \partial_X(H)$. Then the following conditions are equivalent.*

(a) H has finite index in E;

(b) the set $C = CK$ is E-invariant;

(c) E is quasiconvex in G and $\partial_X(E) = \partial_X(H)$;

(d) E is contained in $\mathrm{Stab}_G(K)$ where $\mathrm{Stab}_G(K) = \{g \in G \mid gK = K\}$.

PROOF. Notice that if $p \in X$, $h \in H$ then $Hp = hHp$ and so $K = hK$. Thus $H \le \mathrm{Stab}_G(K)$. Now the implications (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), (a) $\Rightarrow$ (d), (d) $\Rightarrow$ (b) and (c) $\Rightarrow$ (b) are clear.

Suppose now that (b) is satisfied. Then the closure (in X) $\bar{C}$ is E-invariant and quasiconvex in X. Thus $\operatorname{diam}(C/E) \le \operatorname{diam}(C/H) < \infty$ and therefore $\bar{C}/E$ is compact. This implies by Lemma 3.5 that E is finitely generated. Moreover, if Y is a finite generating set for G, Z is a finite generating set for E, $p \in CK$ and $f\colon G \longrightarrow X$ is given by $f(g) = gp$, then again by Lemma 3.5:

$$d_Y \sim f_*(d),\ d_Z \sim (f_*(d|_{\bar{C}})|_E) \text{ and so } d_Z \sim \big(f_*(d|_{\bar{C}})\big)|_E = [f_*(d)]_E \sim (d_Y)|_E.$$

Thus E is quasiconvex in G.

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Observe that K is a closed subset of ∂X as it is a limit set of a subgroup and so by Lemma 3.6 the limit set of C is K. We claim that $\partial_X(E) = K$. Clearly $K \subset \partial_X(E)$ since $H \leq E$. On the other hand C is E-invariant, $p \in C$ and $Ep \subset C$. The limit set of E which is, by definition, the limit set of Ep is contained in the limit set of the larger set C, and this limit is K. Thus $\partial_X(E) = K$.

We have shown that (b) implies (c). We will now show that (b) implies (a).

Indeed, suppose Y and Z are as above and S is a finite generating set of H. Let e be an arbitrary element of E. Since $\mathrm{diam}(C/H) \leq M < \infty$ and $f(e) = ep \in C$, there is a point $hp \in C$ for some $h \in H$ such that $d(ep, hp) \leq 2M$. Since f is a λ-quasi-isometry from (G, d_Y) onto (X, d), we conclude that $d_Y(e, h) \leq 2M\lambda + \lambda = M_1$. Thus for any $e \in E$ there is an element $g_e \in G$ with $d(g_e, 1) \leq M_1$ such that $eg_e \in H$. Therefore H has finite index in E. This completes the proof of Lemma 3.9.

COROLLARY 3.10. *Let G be a word hyperbolic acting by isometries properly discontinuously and cocompactly on a proper hyperbolic space X. Let H be an infinite quasiconvex subgroup of G and $K = \partial_X(H)$. Then* $\mathrm{Stab}_G(K) = VN_G(H)$.

PROOF. It is clear that $H \leq VN_G(H) \leq \mathrm{Stab}_G(K)$. On the other hand H has finite index in $\mathrm{Stab}_G(K)$ by Lemma 3.9(d). Thus for any $g \in \mathrm{Stab}_G(K)$ we have $|H : gHg^{-1} \cap H| \leq \infty$ and $|gHg^{-1} : gHg^{-1} \cap H| \leq \infty$. Therefore $g \in VN_G(H)$ and so $VN_G(H) = \mathrm{Stab}_G(K)$.

Theorem 1 now follows immediately from this proposition, as in Section 2.

PROOF OF THEOREM 1. Let X be the Cayley graph of G with respect to some finite generating set S. Put $K = \partial_X(A)$.

(1) It is obvious that $\partial_X(A) = \partial_X(B) = \partial_X(A \cap B)$. Denote this set by K. Thus $A, B \leq \mathrm{Stab}_G(K)$ and so $E = sgp(A \cup B) \leq \mathrm{Stab}_G(K)$. Thus by Lemma 3.9(d) $A \cap B$ has finite index in E.

(2) Suppose there are infinitely many subgroups of G containing A as a subgroup of finite index. Let E be the subgroup of G generated by their union. Then A has infinite index in E. On the other hand any subgroup of G having A as a subgroup of finite index has K as its limit set and so leaves K invariant. Thus K is E-invariant, that is $E \leq \mathrm{Stab}_G(K)$. Therefore A has finite index in E by Lemma 3.9(d). This contradicts our assumptions.

(3) It is clear that $A \leq VN_G(A)$. Suppose now that $g \in VN_G(A)$. Put $U = gAg^{-1} \cap H$. Then U has finite index in A and in gAg^{-1}. Since U is of finite index in A and in gAg^{-1}, $K = \partial_X(U) = \partial_X(A) = \partial_X(gAg^{-1})$. However $\partial_X(gAg^{-1})$ is equal to the limit set of gA that is to gK. Thus $gK = K$ and $g \in \mathrm{Stab}_G(K)$. Since $g \in VN_G(A)$ was chosen arbitrarily, we conclude that $A \leq VN_G(A) \leq \mathrm{Stab}_G(K)$. Thus by Lemma 3.9 A has finite index in $VN_G(A)$.

PROOF OF THEOREM 3. Let X be the Cayley graph of G with respect to some finite generating set S. Put $K = \partial_X(A)$. The "if" implication is clear. So suppose that A contains an infinite subgroup C which is normal in B. Then by Lemma 3.8 $\partial_X(C) = \partial_X(B)$. Since $C \leq A \leq B$, we conclude that $K = \partial_X(A) = \partial_X(B)$. Thus $B \leq \mathrm{Stab}_G(K)$ and by Lemma 3.9(d) A has finite index in B.

4. **Some additional remarks about the elementary boundary.** Let S be a finite generating set for the word hyperbolic group G and (X, d_S) be the Cayley graph of G with respect to S. Fix a number $\delta > 1$ such that (X, d_S) is δ-hyperbolic. The elementary boundary $\tilde{\partial}G$ (in the sense of Section 2) is a subset of ∂X (in the sense of Section 3) and moreover, a dense subset (see [Gr]). The map from the elementary boundary into the Gromov boundary is given by $[hg^+] \longmapsto \lim_{n\mapsto\infty} hg^n h^{-1} = \lim_{n\mapsto\infty} hg^n$.

It is also not hard to see, using the compactness argument, that if $g \in \mathcal{H}_G(Y)$ for a subset Y of $\tilde{\partial}G$ then there is a pair of points $a(g), a(h) \in Y$ such that g lies on a certain geodesic $\big(a(g), a(h)\big)$ (this is a bi-infinite geodesic the sense of Section 3). Moreover, if $a(g), a(h) \in Y$ are two distinct points, then there is a geodesic $\big(a(g), a(h)\big)$ such that any point (vertex) of $\big(a(g), a(h)\big)$ belongs to $\mathcal{H}_G(Y)$.

This immediately implies

LEMMA 4.1. *Let G, X, δ be as above, and let Y be a subset of $\tilde{\partial}G$ with at least two elements.*

Then $\mathcal{H}_G(Y)$ is 100δ-quasiconvex in X.

PROOF. The proof is exactly the same as that of Lemma 3.2 if one takes into account the remarks above.

The previous observation combined with Corollary 2.4 allows one to obtain a criterion of quasiconvexity similar to Proposition 3.4 in terms of elementary boundary, making the condition of 2.4 necessary and sufficient.

PROPOSITION 4.2. *Let G be a word hyperbolic group, S a finite generating set and X be the Cayley graph of G with respect to S. Let A be an infinite subgroup of G.*

The following conditions are equivalent:

(i) A is quasiconvex in G;

(ii) the quotient $\mathcal{H}(\tilde{\partial}_G A)/A$ is finite.

PROOF. The implication (i) $\Rightarrow$ (ii) was established in Corollary 2.4. Suppose now that (ii) is satisfied. As A is infinite, $\tilde{\partial}A$ contains at least two points and so $C = \mathcal{H}(\tilde{\partial}_G A)$ is nonempty and even infinite. Now C is 30δ-quasiconvex in X by Lemma 4.1 and so it is λ-quasigeodesic for $\lambda = 3000\delta$ (the argument is exactly the same as in the proof of Proposition 3.4). Now C is discrete (as $C \subset G$) and has compact metric balls. Fix a basepoint $p \in C$. We know that C/A is finite and therefore compact. Thus by Lemma 3.5, A is finitely generated and the orbit map $a \longmapsto ap, a \in A$ is a quasi-isometry of A onto C. The rest of the argument goes exactly as in Proposition 3.4.

REFERENCES

[ABC] J. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro and H. Short, *Notes on hyperbolic groups*, In: Group theory from a geometric viewpoint, Proc. ICTP. Trieste, World Scientific, Singapore, 1991, 3–63.

[Ba] H. Bass, *Covering theory for graphs of groups*, J. Pure Appl. Algebra **89**(1993), 3–47.

[Bo1] B. Bowditch, *Notes on Gromov's hyperbolicity criterion for path-metric spaces*, In: Group theory from a geometric viewpoint, (eds. E. Ghys, A. Haefliger and A. Verkovsky), 1991, 64–167.

[Bo2] B. Bowdich, *Geometrical finiteness for hyperbolic groups*, J. Funct. Anal. (2)**113**(1993), 245–317.

[Br] N. Brady, *Branched coverings of cubical complexes and subgroups of hyperbolic groups*, University of Utah, 1995, preprint.

[BMS] G. Baumslag, C. F. Miller III and H. Short, *Unsolvable problems about small cancellation and word hyperbolic groups*, Bull. London Math. Soc. (1)**26**(1994), 97–101.

[BS] M. Bridson and G. Swarup, *On Hausdorff-Gromov convergence and a theorem of Paulin*, Enseign. Math. (2)**40**(1994), no. 3–4, 267–289.

[CDP] M. Coornaert, T. Delzant and A. Papadopoulos, *Notes sur les groupes hyerboliques de Gromov*, no. 1441, Springer-Verlag, 1990.

[De] M. Dehn, *Papers on Group Theory and Topology*, translated and introduced by J. Stillwell, Springer-Verlag, 1987.

[ECHLPT] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson and W. P. Thurston, *Word Processing in Groups*, Jones and Bartlett, MA, 1992.

[F] W. Floyd, *Group completions and limit sets of Kleinian groups*, Invent. Math. **57**(1980), 205–218.

[G] L. Greenberg, *Discrete Groups of Motions*, Canad. J. Math. **12**(1960), 414–425.

[GH] E. Ghys and P. de la Harpe (editors), *Sur les groupes hyperboliques d'aprés Mikhael Gromov*, In: Papers from the Swiss seminar on hyperbolic groups held in Bern, 1988, (eds. E. Ghys and P. de la Harpe), Progress in Mathematics **83**, 1990.

[Gr] M. Gromov, *Hyperbolic Groups*, In: Essays in group theory, (ed. S. M. Gersten), MSRI Publ. **8**, Springer, 1987, 75–263.

[GMRS] R. Gitik, M. Mitra, E. Rips and M. Sageev, *Widths of subgroups*, Trans. Amer. Math. Soc., to appear.

[GS] S. M. Gersten and H. Short, *Rational subgroups of biautomatic groups*, Ann. of Math. (2)**134**(1991), no. 1, 125–158.

[LS] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin-Heidleberg-New York, 1977.

[Mo] J. Morgan, *On Thurston's uniformization theorem for three-dimensional manifolds*, In: The Smith Conjecture, (eds. H. Bass and J. Morgan), Academic Press, 1984, 37–126.

[MT] M. Mihalik and W. Towle, *Quasiconvex subgroups of negatively curved groups*, J. Pure Appl. Algebra **3**(1995), 297–301.

[N] W. Neumann, *The fixed group of an automorphism of a word hyperbolic group is rational*, Invent. Math. **110**(1992), 147–150.

[Pi] Ch. Pittet, *Surface groups and quasiconvexity*, In: Geometric Group Theory **1**, Sussex, 1991, London Math. Soc. Lecture Notes Series **181**, Cambridge University Press, Cambridge, 1993, 169–175.

[R] E. Rips, *Subgroups of small cancellation groups*, Bull. London Math Soc. **14**(1982), 45–47.

[Sho] H. Short, *Quasiconvexity and a Theorem of Howson's*, In: Group theory from a geometric viewpoint, Proc. ICTP. Trieste, World Scientific, Singapore, 1991.

[Sta] J. R. Stallings, *Topology of finite graphs*, Invent. Math. **71**(1983), 551–565.

[Swa] G. A. Swarup, *Geometric finiteness and rationality*, J. Pure Appl. Algebra **86**(1993), 327–333.

[Swe] E. Swensen, *Limit sets in the boundary of word hyperbolic groups*, 1994, preprint.

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