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ON A CLASS OF PROJECTIVE MODULES OVER CENTRAL SEPARABLE ALGEBRAS II

BY
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Introduction. The purpose of this paper is to continue the work of [7]. Throughout the paper all notations shall have the same meanings as those in [7]; that is, the ring R is commutative with identity, $B(R)$ is the set of idempotents in R, Spec $B(R)$ is the set of prime ideals in $B(R)$, and U_e for e in $B(R)$ denotes the set $\{x/x \text{ in Spec } B(R) \text{ with } 1-e \text{ in } x\}$. Then from [6] we know that $\{U_e/e \text{ in } B(R)\}$ forms a basic open set for a topology imposed in Spec $B(R)$ and this topological space is totally disconnected, compact and Hausdorff. Furthermore, the Pierce sheaf is defined whose base space is Spec $B(R)$ and whose stalks are $R_x=R/xR$ for x in Spec $B(R)$. Then R is isomorphic with a global cross section of the sheaf. Some facts about this sheaf given in [8] will be used. In particular,

(1) For e in $B(R)$, the homomorphism $R\to Re$ establishes a homeomorphism $U_e\to\text{Spec } B(Re)$. Thus if M is a finitely generated R-module satisfying $M_x=0$ for all x in U_e, we may conclude from (2.11) in [8] that $eM=0$.

(2) Let $\{U_e\}$ be an open cover of Spec $B(R)$. Then by compactness of Spec $B(R)$, there is a finite subcover $\{U_{e_i}/i=1, 2, \ldots, n$ for some integer $n\}$ such that $\{U_{e_i}\}$ are disjoint. Thus $\{e_i\}$ are orthogonal and $\sum_i e_i=1$ (see [8], (2.10) and the end of §3). The author wishes to thank Professor D. Zelinsky for an example (see the remark on Theorem 1.3 below).

In [7], the finitely generated and projective modules over a central separable algebra are characterized in terms of the modules over a central separable R_x-algebra for x in Spec $B(R)$. Also, for any x in Spec $B(R)$ assume there is a finitely generated projective and indecomposable R-module M such that $M_x=R_x\otimes_R M\neq 0$. If R_x is semi-local for each x in Spec $B(R)$, then R is semi-local (see the proof of the theorem in [7]). As Zelinsky pointed out, the above assumption does not generally hold (see the remark in §1 below). The purpose of the present paper is to continue the study of the projective modules over the same kind of central separable algebras as in [7], but without the above assumption on R. First, we improve Lemma 1 in [7]. Next, it is proved that if M is a finitely generated projective and indecomposable module over a central separable algebra with $M_x\neq 0$ for x in Spec $B(R)$, then x is an isolated point. Then, using our results and a continuous rank function from Spec $B(R)$ to the set of integers with each point open we show a general Skolem-Noether theorem by a different method from the

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one given by DeMeyer and Magid in [2] and [4] respectively. Finally, Corollary 1.3 in [1] is extended.

§1. We begin with an improvement of Lemma 1 in [7] by eliminating the Noetherian condition on R.

LEMMA 1.1. *Let A be a central separable R-algebra and assume R_x is semi-local for each x in* Spec $B(R)$. *Let M and N be two finitely generated and projective A-modules. Then $M \simeq N$ as A-modules if and only if $M \simeq N$ as R-modules.*

Proof. The 'only if' part is clear. Conversely, since R_x is semi-local with no idempotents except 0 and 1 ([8], (2.13)) and since M_x and N_x are finitely generated and projective A_x-modules, M_x and N_x are isomorphic as A_x-modules [1, Theorem 1.1]. Hence $M_x \simeq N_x$ as A-modules. We then have a homomorphism $f: M \to N$. Noting that $N/f(M)$ is finitely generated with $(N/f(M))_x = 0$, we have a basic open set U_e of x such that $(N/f(M))_y = 0$ for each y in U_e. Then U_e is equal to Spec $B(Re)$, and so $e(N/f(M)) = 0$ by (1) in the introduction. Thus the sequence $0 \to \ker(f) \to eM \xrightarrow{f} eN \to 0$ is exact and splits. Furthermore, $\ker(f)$, a direct summand of eM, is finitely generated and is contained in xM, so $(\ker(f))_x = 0$. But then there is a basic open set U_{e_0} contained in U_e such that $(\ker(f))_y = 0$ for each y in U_{e_0}. Since $e_0 = ee_0$ by [8, (2.2)], $e_0 M \simeq e_0 N$ as $e_0 A$-modules. Finally let x vary over Spec $B(R)$ and cover Spec $B(R)$ by such U_{e_0}. Then by (2) in the introduction there are orthogonal idempotents $\{e_i, i = 1, 2, \ldots, n$ for some integer $n\}$ so that $R \simeq \oplus \sum_i Re_i$, $A \simeq \oplus \sum_i Ae_i$, $M \simeq \oplus \sum_i e_i M$ and $N \simeq \oplus \sum_i e_i N$ with $e_i M \simeq e_i N$ as Ae_i-modules for each i. Therefore $M \simeq N$ as A-modules.

THEOREM 1.2. *Let A be a central separable R-algebra and let M and N be finitely generated projective and indecomposable A-modules. Assume R_x is semi-local for each x in* Spec $B(R)$. *Then the following statements are equivalent:* (a) *$M \simeq N$ as A-modules,* (b) *$M \simeq N$ as R-modules,* (c) *$M_x \neq 0$ and $N_x \neq 0$ for some x in* Spec $B(R)$.

Proof. It is an immediate consequence of the lemma and the proof of Lemma 1 in [7]. For completeness, the proof is given here. (a)$\leftrightarrow$(b) by Lemma 1. (a)$\to$(c) is clear by (1) in the introduction. It suffices to show (c)$\to$(a). Since R_x is semi-local, all finitely generated projective and indecomposable A_x-modules are isomorphic. Hence we can assume that $M_x \simeq N_x \oplus N'_x$ for some finitely generated and projective A_x-module N'_x; and so there is a homomorphism from M_x to N_x. As in the proof of the lemma, there is an A-homomorphism f from M into N with $(N/f(M))_x = 0$. Using the argument in the proof of Lemma 1, we can show $e(N/f(M)) = 0$ for some idempotent e; and so $f(eM) = eN$. But M and N are indecomposable A-modules, then $eM \simeq M$ and $eN \simeq N$. Thus $M \simeq N$ as A-modules.

Let A be a central separable R-algebra. Then the projective A-modules are characterized in terms of the points in Spec $B(R)$.

THEOREM 1.3. *If M is a finitely generated projective and indecomposable A-module with $M_x \neq 0$ for x in* Spec $B(R)$, *then x is an isolated point.*

Proof. Since M is finitely generated and projective over the central separable R-algebra A, it is finitely generated and projective over R. First let f_M be the usual continuous rank function from Spec(R) to the set of integers Z with each point open; that is, $f_M(P) = \text{rank}_{R_P}(R_P \otimes_R M)$ where R_P is the local ring at the prime ideal P in Spec(R). Then a continuous rank function from Spec $B(R)$ to Z with each point open, g_M, is defined by $g_M = f_M \circ \phi^{-1}$ where ϕ is the homeomorphism from an identification space of Spec(R) to Spec $B(R)$ ([8], (2.4)). For each x in Spec $B(R)$, $g_M(x) = f_M \circ \phi^{-1}(x)$ which is $\text{rank}_{R_P}(R_P \otimes_R M)$ for x contained in P.

Now $M_x \neq 0$, so $g_M(x) \neq 0$. Let U_e be a basic open set containing x so that $g_M(y) = g_M(x)$ for y in U_e (for g_M is continuous and $g_M(x)$ is open in Z). Assume to the contrary that x is a limit point. Then there is y in U_e with $y \neq x$. On the other hand, Spec $B(R)$ is Hausdorff, so we have a basic open set containing x, $U_{e'}$, such that y is not in $U_{e'}$. Hence y is in $U_{1-e'}$. This gives a decomposition of R, $R \simeq Re' \oplus R(1-e')$ with x in $U_{e'} = \text{Spec } B(Re')$ and y in $U_{1-e'} = \text{Spec } B(R(1-e'))$ by (1) in the introduction. Thus $A \simeq Ae' \oplus A(1-e')$ and $M \simeq e'M \oplus (1-e')M$ with $(e'M) \neq 0$ and $(1-e')M \neq 0$. This leads to the contradiction that M is decomposable over A; and so the proof is completed.

A part of the converse of the above theorem is obtained.

THEOREM 1.4. *Let M be an indecomposable module over the central separable algebra A and assume M_x is a finitely generated and projective R_x-module with $M_x \neq 0$. If x is an isolated point in* Spec $B(R)$, *then M is a finitely generated and projective A-module.*

Proof. Since x is an isolated point, $R_x \simeq eR$ for some idempotent e [5, Lemma 2.10]. By hypothesis, M_x is a finitely generated and projective R_x-module, so M_x is a finitely generated and projective A_x-module. Hence $M_x = R_x \otimes_R M \simeq eR \otimes_R M \simeq eM$ as Ae-modules. Thus the indecomposable A-module $M \simeq eM$ is a finitely generated and projective A-module.

REMARK. The condition for Theorem 1.3 does not generally hold. The following example is due to Zelinsky.

Let R be all continuous functions from the following subspace B of the real line to a discrete field k: $B = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$. Then $B(R) = B$. Take x the point corresponding to 0, so $xR = \{f / f(0) = 0\}$. Each such f also vanishes on a neighborhood of 0 (since $\{0\}$ is open in k), so f is a k-linear combination of $e_1, e_2, \ldots, e_n, \ldots$ where e_i is the characteristic function of the one-point set $\{1/i\}$. Thus xR is generated by the e_i's.

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Now $R/xR \simeq k$ because $R \to k$ by $f \to f(0)$ with kernel xR; and R/xR is not projective because otherwise xR would be a direct summand in R, and the other summand would consist of continuous functions f with $xf=0$, that is $e_i f=0$, that is $e_i f=0$ for all i, that is $f(1/i)=0$ for all i, that is, $f=0$.

Finally, suppose M is an indecomposable module with $M_x=M/xM\neq 0$. We show that M is not projective. For every i, $M \simeq e_iM \oplus (1-e_i)M$. $xe_iM=e_iM$ because xe_i is generated by e_je_i for $j=1, 2, \ldots$ which are all 0 except for $e_ie_i=e_i$. Thus $M_x\neq 0$ implies $(1-e_i)M\neq 0$, and this for every i. M indecomposable then forces $e_iM=0$ for every i. This implies that $xM=0$, and so M is a module over the field R/xR. Therefore M is a direct sum of copies of R/xR. Since R/xR is not projective, neither is M.

§2. We are going to show a Skolem-Noether theorem by a different method from Theorem 6 in [2] and Theorem 2.3 in [4]. Also, Corollary 1.3 in [1] is extended.

THEOREM 2.1. *Assume R_x is semi-local for each x in* Spec $B(R)$. *Let A be a separable projective R-algebra with center K and let B be a separable R-subalgebra of A with center C containing K. Assume every idempotent in C belongs to R. Let f be an R-algebra monomorphism from B into A leaving K fixed, then f can be extended to an inner automorphism of A* (F. DeMeyer).

Proof. From the first part in the proof of Theorem 1.3, for a finitely generated and projective R-module M there is a basic open set U_e containing x for x in Spec $B(R)$ such that $\text{rank}_{Re}(eM)$ is defined. Hence, given finitely generated and projective R-modules, $M_1, M_2, \ldots, M_n$, for some integer n, there is a basic open set $U_{e'}$ such that $\text{rank}_{Re'}(eM_i)$ is defined for each i by taking the intersection of some finite basic open sets.

Now let D denote $B\otimes_R A^0$ where A^0 is the opposite ring of A. Then from the conditions D is a central C-algebra. Let A be an D-module defined by $(b\otimes_R a)(m)=bma$ for b in B, a in A and m in A. Then A is a finitely generated and projective D-module. Similarly, define the operation by $(b\otimes_R a)(m)=f(b)ma$, where f is the monomorphism from B into A leaving K fixed. Then A is a finitely generated and projective D-module denoted by A'. Since A and A' are finitely generated and projective C-module and $f(C)$-module respectively, $\text{rank}_{C_x}(A_x)$ and $\text{rank}_{C_x'}(A_x')$ are defined and equal for each x in Spec $B(R)$, where $C'=f(C)$. Noting that Spec $B(C)=$Spec $B(R)$ by hypothesis we have a decomposition of C, $C\simeq\oplus\sum_{i=1}^m Ce_i$ for some integer m such that $\text{rank}_{Ce_i}(Ae_i)$ and $\text{rank}_{C'e_i}(A'e_i)$ are defined ($C'=f(C)$) by the above remark on rank function and by the compact property of Spec $B(R)=$ Spec $B(C)$ as given in (2) in the introduction. Furthermore, $\text{rank}_{C_x}(A_x)=\text{rank}_{C_x'}(A_x')$ for x in Spec $B(C)$, so $\text{rank}_{Ce_i}(Ae_i)=\text{rank}_{C'e_i}(A'e_i)$ for each i. Since R_x is semi-local, so are C_x and C_x' [1, Lemma]; and so $(Ae_i)_x$ is a free $(Ce_i)_x$-module and $(A'e_i)_x$ is a free $(C'e_i)_x$-module. That is, $(Ae_i)_x \cong \oplus \sum_{j=1}^t (Ce_i)_x^j$, a t-copies of $(Ce_i)_x$, and similarly $(A'e_i)_x \cong \oplus \sum_{j=1}^t (C'e_i)_x^j$, where $t = \text{rank}_{C'e_i}(A'e_i)$ for each x in Spec $B(Ce_i)$. Hence $Ae_i \cong \oplus \sum_{j=1}^t (Ce_i)^j$ and $A'e_i \cong \oplus \sum_{j=1}^t (C'e_i)^j$ as Ce_i- and $C'e_i$-modules respectively; and so $Ae_i \cong A'e_i$ as Ce_i-modules (for $Ce_i \cong C'e_i$). Thus $Ae_i \cong A'e_i$ as De_i-modules by Lemma 1.1. So, $A \cong A'$ as D-modules. Denote this isomorphism by F. Then $F(1 \cdot a) = F(1)a$ for any a in A; and so $F(1)$ is a unit of A. For any b in B, this implies that $F(b) = F(1)b = f(b)F(1)$, so $f(b) = F(1)bF(1)^{-1}$. Therefore f is extended to an inner automorphism of A. This completes the proof.

THEOREM 2.2. *Assume R_x is semi-local for each x in* Spec $B(R)$. *Let A be a central separable R-algebra and let B be a separable subalgebra of A with center K such that $B(K) = B(R)$. If A^B denotes the commutator of B in A, then there is a decomposition of R, $R \cong \oplus \sum_{i=1}^n Re_i$, for some orthogonal idempotents e_i such that*

(a) $\text{rank}_{Re_i}(Ae_i)$ *and* $\text{rank}_{Re_i}(Be_i)$ *and* $\text{rank}_{Re_i}(Ae_i^{Be_i})$ *are defined,*

(b) $\text{rank}_{Re_i}(Ae_i) = \text{rank}_{Re_i}(Be_i) \cdot \text{rank}_{Re_i}(Ae_i^{Be_i})$, *for each i.*

Proof. It is not hard to see that K is finitely generated and projective over R, so B and A^B are finitely generated and projective over R. Since $(A^B)_x \cong (A_x)^B x$ [4, (1.5)] for each x in Spec $B(R)$, part (a) is proved by the first paragraph in the proof of Theorem 2.1. That is, there are basic open sets $\{U_{e_i}, i = 1, 2, \ldots, n\}$ with e_i orthogonal and summing to 1 so that $R \cong \oplus \sum_{i=1}^n Re_i$, $A \cong \oplus \sum_{i=1}^n Ae_i$, $B \cong \oplus \sum_{i=1}^n Be_i$ and part (b) holds. Now by Corollary 1.3 in [1], for x in U_{e_i},

$$\text{rank}_{(Re_i)_x}((Ae_i)_x) = \text{rank}_{(Re_i)_x}((Be_i)_x) \cdot \text{rank}_{(Re_i)_x}((Ae_i)_x^{(Be_i)}x).$$

But $((Ae_i)_x^{(Be_i)}x) \cong (Ae_i^{Be_i})_x$ by (1.5) in [4], then the above rank-value equality is part (b) for each i. This completes the proof.

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