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Wild non-abelian Hodge theory on curves

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Abstract

On a complex curve, we establish a correspondence between integrable connections with irregular singularities, and Higgs bundles such that the Higgs field is meromorphic with poles of any order. Moduli spaces of these objects are obtained with fixed generic polar parts at each singularity, which amounts to fixing a coadjoint orbit of the group $GL_r(\mathbb{C}[z]/z^n)$. We prove that they carry complete hyper-Kähler metrics.

Introduction

On a compact Kähler manifold, there is a well-known correspondence, named non-abelian Hodge theory, and established by Simpson and Corlette (see [Sim92]), between representations of the fundamental group (or integrable connections) and Higgs bundles. In the case of a curve, the correspondence is due to Hitchin [Hit87] and Donaldson [Don87]. This correspondence has been extended to the case when the objects on both sides have logarithmic singularities, at least in the case when the singular locus is a smooth divisor, see [Sim90] on curves and [Biq97] in higher dimensions.

In this article, we extend the correspondence to the irregular case, on a curve. This means that we now look, on one side, at integrable connections with irregular singularities like

$$d + A_n \frac{dz}{z^n} + \cdots + A_1 \frac{dz}{z} + \text{holomorphic terms} \tag{0.1}$$

with $n > 1$, and, on the other side, at Higgs bundles $(\mathscr{E}, \theta)$ with the Higgs field θ having polar parts of the form

$$T_n \frac{dz}{z^n} + \cdots + T_1 \frac{dz}{z}. \tag{0.2}$$

In this paper we need the following hypothesis near each singularity.

Main assumption. The connections and Higgs fields are holomorphically gauge equivalent to ones with diagonal polar parts.

This is a generic condition and holds for example if the leading coefficients A_n, T_n are regular semisimple; see Lemma 1.1 for a more detailed statement.

Sabbah [Sab99] has constructed a harmonic metric for integrable connections with irregular singularities: this is a part of the correspondence. We construct the whole correspondence as follows.

Theorem 0.1. *Under our main assumption, there is a one-to-one correspondence between stable (parabolic) integrable connections with irregular singularities, and stable parabolic Higgs bundles*

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with singularities like those in (0.2). The correspondence between the singularities (0.1) and (0.2) at the punctures is given, after diagonalization, by

$$T_i = \tfrac{1}{2}A_i, \quad i \geqslant 2.$$

For $i = 1$ there is the same permutation between eigenvalues of A_1, T_1, and the parabolic weights as in [Sim90]; more precisely, if on the connection side we have the eigenvalues μ_i of A_1 and the parabolic weights β_i, and on the Higgs bundle side we have the eigenvalues λ_i of T_1 and the parabolic weights α_i, then

$$\alpha_i = \operatorname{Re}\mu_i - [\operatorname{Re}\mu_i], \quad \lambda_i = \frac{\mu_i - \beta_i}{2},$$

where $[\cdot]$ denotes the integer part.

See § 6 for details about stability. When the weights of the local system vanish ($\operatorname{Re}\lambda_i = 0$), stability for integrable connections reduces to irreducibility, see Remark 8.2. The proof of Theorem 0.1 also gives precise information about the asymptotics of the harmonic metric.

In [Boa01b] the second author studied the symplectic geometry of the moduli spaces of integrable connections with irregular singularities in the case when A_n is regular semisimple: complex symplectic moduli spaces were obtained by fixing the gauge equivalence class of the polar parts of the connections (0.1) at each pole. (Said differently this amounts to fixing a coadjoint orbit of the group $GL_r(\mathbb{C}[z]/z^n)$, or to fixing the formal type of the connections, at each pole.)

Now we will show that these moduli spaces carry more structure, namely we have the following.

Theorem 0.2. *Under our main assumption, the moduli space of integrable connections with fixed equivalence classes of polar parts is hyper-Kähler. If the moduli space is smooth then the metric is complete.*

The same result remains true if we add some compatible parabolic structure at each singularity. We remark that a generic choice of parabolic structure leads to smooth moduli spaces (no semistable points).

Also, at least generically, one can explicitly describe the underlying complex symplectic manifold as follows.

Theorem 0.3. *For generic eigenvalues of the residues, all integrable connections are stable, and, if the leading coefficients are regular semisimple, the moduli space can be identified, from the complex symplectic viewpoint, with the finite-dimensional quasi-Hamiltonian quotient of [Boa].*

For example, over the projective line the moduli space contains a dense open subset, parameterizing connections on trivial holomorphic bundles, which may be described as a complex symplectic quotient of finite-dimensional coadjoint orbits. However, in general there are stable connections on non-trivial holomorphic bundles and the quasi-Hamiltonian quotient incorporates these points as well. An example will be given in § 8.

Note that Martinet and Ramis [MR91] have constructed a 'wild fundamental group', so that connections with irregular singularities can be interpreted as finite-dimensional representations of this group. From this point of view, Theorem 0.1 really generalizes the earlier correspondences between representations of the fundamental group of the curve (or the punctured curve) and Higgs bundles.

We remark that moduli spaces of meromorphic Higgs bundles have previously been studied algebraically by Bottacin [Bot95] and Markman [Mar94], who have shown that they are algebraic completely integrable systems.

Also the study of Higgs bundles with such irregular singularities has a physical interest, since some of them arise from the Nahm transform of periodic monopoles, see [CK01]. The corresponding hyper-Kähler metrics have been studied in [CK02].

The correspondence of Theorem 0.1 can be extended to the case in which the connections are locally equivalent to ones with $A_n, \dots, A_2$ diagonal but with residue A_1 having a nilpotent part. Then the nilpotent part of the residue T_1 of the corresponding Higgs field is the same as that of A_1. We will not prove this here in order to lighten the definitions of the function spaces required; our main interest was to understand the new phenomena introduced by the more singular terms. (The phenomenon induced by such a nilpotent part in the residue is complicated, but has been carefully analysed in [Sim90, Biq97] – the two behaviors, coming from the higher-order poles and from the nilpotent part of the residue can be basically superposed.) The hyper-Kähler metrics will be incomplete in this case. See also Remark 2.2.

On the other hand, it is not clear at all how to extend the correspondence to the case where the leading coefficient A_n has a nilpotent part. For example, in the case of an order-two pole, if we suppose that T_2 in (0.2) is nilpotent, then the eigenvalues of θ have only a simple pole, so the Higgs bundle satisfies the 'tameness' condition of Simpson [Sim90], and we are back in the simple pole situation. In particular, in that case we cannot construct new metrics; actually, by a meromorphic gauge transformation, the Higgs field can be transformed into a Higgs field with only a simple pole. The same kind of consideration holds on the integrable connection side.

One of the main features of connections with irregular singularities is that formal equivalence does not come from holomorphic equivalence, resulting in the well-known Stokes phenomenon. Sabbah [Sab99] studies carefully this Stokes phenomenon around the puncture in order to construct a sufficiently good initial metric to which he can apply Simpson's existence theorem for harmonic metrics [Sim90].

Our method is different: we use a weighted Sobolev space approach, which enables us to forget the difficult structure of irregular singularities, at the expense of developing some analysis to handle the operators with highly singular coefficients that we encounter. In particular, we strengthen the Fréchet symplectic quotient of [Boa01b] into a hyper-Kähler quotient.

In § 1, we develop the local models which are a guide for the behavior of the correspondence, and we then define the admissible deformations in suitable Sobolev spaces in § 2. Next we study the local analysis needed on a disk in §§ 3 and 4. This enables us to construct the C^∞-moduli space of solutions of Hitchin's self-duality equations in § 5, and prove that it is hyper-Kähler. It remains to identify this moduli space with the 'De Rham moduli space' of integrable connections and the 'Dolbeault moduli space' of Higgs bundles: these moduli spaces are studied in §§ 7 and 8, and the correspondence is stated in § 6 and then proven in § 9.

1. Local model

Look at a rank r holomorphic bundle $\mathscr{F}$ in the unit disk, trivialized in a basis (τ_i) of holomorphic sections, and consider the holomorphic connection

$$D = d + A_n \frac{dz}{z^n} + \cdots + A_1 \frac{dz}{z}, \tag{1.1}$$

where the A_i are constant matrices; then D is an integrable connection, that is, the curvature $F_D = D^2$ vanishes.

We now rephrase our main assumption in the following way.

Main assumption. The matrices $A_n, \dots, A_1$ are diagonal.

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In fact we could have taken D to be any connection which is locally equivalent to a connection having diagonal polar part. To understand this, and see that it is a generic condition, we note the following sufficient conditions.

Lemma 1.1. *Let $l \geqslant 0$ be an integer and consider a connection*

$$\widetilde{D} = d + B_n \frac{dz}{z^n} + \cdots + B_1 \frac{dz}{z} + \textit{holomorphic terms.}$$

If B_n is regular semisimple (generic case), or more generally if

1) *for some k with $n \geqslant k > 1$, the stabilizer of $(B_n, \dots, B_k)$ (under the diagonal adjoint action) is a maximal torus, and $B_{k-1}, \dots, B_1$ are arbitrary, or*
2) *$B_n, \dots, B_1$ are already diagonal,*

then locally $\widetilde{D}$ is holomorphically gauge equivalent to a connection which differs by $O(z^l)$ from D in (1.1) (for some diagonal $A_n, \dots, A_1$).

Proof. First one inductively splits $\widetilde{D}$ up to $O(z^l)$ along the spectrum of B_n using transformations of the form $1 + Xz^m$. One then splits along the spectrum of the resulting $B_{n-1}, \dots, B_1$ in turn. For details, see e.g. [BV83, Lemma 1, p. 42]. Up to $O(z^l)$, the resulting connection is block diagonal with diagonal polar part. The first l holomorphic terms of each block may then be removed since the connection is integrable and the polar part of the connection in each block is scalar. □

This lemma also holds (with the same proof) in the case of a meromorphic Higgs field rather than a connection, provided we take $l = 0$.

We will suppose that $\mathscr{F}$ comes with a parabolic structure with weights $\beta_i \in [0, 1[$, meaning basically that we have on the bundle $\mathscr{F}$ a hermitian metric

$$h = \begin{pmatrix} |z|^{2\beta_1} & & \\ & \ddots & \\ & & |z|^{2\beta_r} \end{pmatrix}; \tag{1.2}$$

the fiber $\mathscr{F}_0$ at the origin is filtered by $\mathscr{F}_\beta = \{s(0), |s(z)| = O(|z|^\beta)\}$. In the orthonormal basis $(\tau_i/|z|^{\beta_i})$, we get the formula

$$D = d + \sum_1^n A_i \frac{dz}{z^i} - \beta \frac{dr}{r}, \tag{1.3}$$

where β is the diagonal matrix with coefficients $\beta_1, \dots, \beta_r$.

Recall that, in general, we have a decomposition of D into a unitary part and a self-adjoint part,

$$D = D^+ + \phi,$$

and we can define new operators

$$D'' = (D^+)^{0,1} + \phi^{1,0},$$
$$D' = (D^+)^{1,0} + \phi^{0,1}.$$

The operator

$$D'' = \bar{\partial}^E + \theta \tag{1.4}$$

is a candidate to define a Higgs bundle structure, and this is the case if the *pseudo-curvature*

$$G_D = -2(D'')^2 \tag{1.5}$$

vanishes. In the case of a Riemann surface, the equation reduces to $\bar{\partial}^E \theta = 0$.

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In our case, we get, still in the orthonormal basis $(\tau_i/|z|^{\beta_i})$,

$$D^+ = d + \frac{1}{2}\sum_1^n A_i\frac{dz}{z^i} - A_i^*\frac{d\bar z}{\bar z^i}, \quad \phi = \frac{1}{2}\sum_1^n A_i\frac{dz}{z^i} + A_i^*\frac{d\bar z}{\bar z^i} - \beta\frac{dr}{r},$$

$$\bar\partial^E = \bar\partial - \frac{1}{2}\sum_1^n A_i^*\frac{d\bar z}{\bar z^i}, \quad \theta = \frac{1}{2}\sum_1^n A_i\frac{dz}{z^i} - \frac{\beta}{2}\frac{dz}{z}.$$

It is clear that $G_D = 0$, so that we have a solution of Hitchin's self-duality equations.

Formulas become simpler if we replace the orthonormal basis $(\tau_i/|z|^{\beta_i})$ by the other orthonormal basis (e_i) given on the punctured disk by

$$e_i = |z|^{-i\operatorname{Im}\mu_i}\exp\left(\sum_2^n -\frac{A_i^*}{2(i-1)\bar z^{i-1}} + \frac{A_i}{2(i-1)z^{i-1}}\right)\frac{\tau_i}{|z|^{\beta_i}},$$

where the μ_i are the eigenvalues of the residue of D (the diagonal coefficients of A_1). Indeed, in the basis (e_i), the previous formulas become

$$D^+ = d + \operatorname{Re}(A_1)i\,d\theta, \quad \phi = \frac{1}{2}\sum_1^n A_i\frac{dz}{z^i} + A_i^*\frac{d\bar z}{\bar z^i} - \beta\frac{dr}{r} \tag{1.6}$$

and

$$\bar\partial^E = \bar\partial - \frac{1}{2}\operatorname{Re}(A_1)\frac{d\bar z}{\bar z}, \quad \theta = \frac{1}{2}\sum_1^n A_i\frac{dz}{z^i} - \frac{\beta}{2}\frac{dz}{z}. \tag{1.7}$$

This orthonormal basis (e_i) defines a hermitian extension E of the bundle over the puncture.

Now look at the holomorphic bundle induced by $\bar\partial^E$ on the punctured disk. A possible choice of a basis of holomorphic sections is

$$\sigma_i = |z|^{\operatorname{Re}\mu_i - [\operatorname{Re}\mu_i]}e_i. \tag{1.8}$$

We see that $|\sigma_i| = |z|^{\alpha_i}$, with

$$\alpha_i = \operatorname{Re}\mu_i - [\operatorname{Re}\mu_i]. \tag{1.9}$$

This choice of σ_i is the only possible choice for which $0 \leqslant \alpha_i < 1$. The sections (σ_i) define an extension $\mathscr{E}$ of the holomorphic bundle over the puncture, and the behavior of the metric means that this extension carries a parabolic structure with weights α_i. Finally, in this basis the Higgs field is still given by

$$\theta = \frac{1}{2}\sum_1^n A_i\frac{dz}{z^i} - \frac{\beta}{2}\frac{dz}{z}. \tag{1.10}$$

In particular, the eigenvalues of the residue of the Higgs field are

$$\lambda_i = \frac{\mu_i - \beta_i}{2}. \tag{1.11}$$

The formulas (1.9) and (1.11) give the same relations between parabolic weights and eigenvalues of the residue on both sides as in the case of regular singularities. This basically means that the behavior described by Simpson in the case of regular singularities still occurs here in the background of the behavior of the solutions in the presence of irregular singularities.

2. Deformations

We continue to consider connections in a disk, using the same notation as in § 1.

We want to construct a space $\mathscr{A}$ of admissible connections on E, with the same kind of singularity as D at the puncture. In order to be able to do some analysis, we need to define Sobolev spaces.

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First define a weighted L^2 space (using the function $r = |z|$)

$$L^2_\delta = \left\{ f, \frac{f}{r^{\delta+1}} \in L^2 \right\}.$$

The convention for the weight is chosen so that the function $r^x \in L^2_\delta$ if and only if $x > \delta$.

Now we want to define Sobolev spaces for sections f of E or of associated bundles, mainly $\mathrm{End}(E)$. Let us restrict to this case: we have to decompose $\mathrm{End}(E)$ under the action of the A_i.

A simple case is the *regular case* in which the stabilizer of A_n is the same as the stabilizer of all the matrices $A_1, \dots, A_n$. Then we decompose $\mathrm{End}(E)$ as

$$\mathrm{End}(E) = \mathrm{End}(E)_n \oplus \mathrm{End}(E)_0, \quad \begin{cases} \mathrm{End}(E)_0 = \ker \mathrm{ad}(A_n), \\ \mathrm{End}(E)_n = (\ker \mathrm{ad}(A_n))^\perp. \end{cases} \tag{2.1}$$

For example, if A_n is regular, then $\mathrm{End}(E)_0$ consists of diagonal matrices.

In the non-regular case, we need a more subtle decomposition,

$$\mathrm{End}(E) = \sum_0^n \mathrm{End}(E)_i, \tag{2.2}$$

defined by induction by

$$\mathrm{End}(E)_0 = \bigcap \ker \mathrm{ad}(A_i), \quad \mathrm{End}(E)_i = \mathrm{End}(E)_{i-1}^\perp \cap \Bigg(\bigcap_{j>i} \ker \mathrm{ad}(A_j) \Bigg). \tag{2.3}$$

We will therefore decompose a section f of $\mathrm{End}(E)$ as $f = f_0 + \dots + f_n$, where the indices mean that the highest order pole term acting on f_i is $A_i\, dz/z^i$.

Now we can define Sobolev spaces with k derivatives in L^2,

$$L^{k,2}_\delta = \left\{ f, \frac{\nabla^j f_i}{r^{i(k-j)}} \in L^2_\delta \text{ for } 0 \leqslant j \leqslant k,\ 0 \leqslant i \leqslant n \right\}; \tag{2.4}$$

in the whole paper, $\nabla = \nabla^{D^+}$ is the covariant derivative associated to the *unitary* connection D^+.

In this problem it is natural to look at deformations of D such that the curvature remains $O(r^{-2+\delta})$, that is slightly better than L^1. This motivates the following definition of the space $\mathscr{A}$ of admissible deformations of D:

$$\mathscr{A} = \{ D + a,\ a \in L^{1,2}_{-2+\delta}(\Omega^1 \otimes \mathrm{End}\, E) \}, \tag{2.5}$$

and of the gauge group,

$$\mathscr{G} = \{ g \in U(E),\ Dgg^{-1} \in L^{1,2}_{-2+\delta} \}. \tag{2.6}$$

The following lemma says that we have defined good objects for gauge theory.

Lemma 2.1. *The connections in $\mathscr{A}$ have their curvature in $L^2_{-2+\delta}$. Moreover, $\mathscr{G}$ is a Lie group, with Lie algebra*

$$\mathrm{Lie}(\mathscr{G}) = L^{2,2}_{-2+\delta}(\mathfrak{u}(E)), \tag{2.7}$$

and it acts smoothly on $\mathscr{A}$.

The proof of the lemma involves some nonlinear analysis, which we will develop in § 3.

Remark 2.2. In the case when A_1 has a nilpotent part, the analysis has to be refined as in [Biq97] in logarithmic scales in order to allow the curvature to be $O(r^{-2}|\ln r|^{-2-\delta})$. The analysis which will be developed here shall continue to be valid in this case, for components in $\mathrm{End}(E)_n \oplus \dots \oplus \mathrm{End}(E)_2$, and the tools in [Biq97] handle $\mathrm{End}(E)_1 \oplus \mathrm{End}(E)_0$, where the action of the irregular part is not seen. This is the basic reason why the results in this paper continue to be true also in this case.

3. Gauge theory

In this section, we give the tools to handle the nonlinearity in the gauge equations for connections with irregular singularities. We first need to define weighted L^p-spaces, for sections f of $\operatorname{End} E$, by

$$L^p_\delta = \left\{ f, \frac{f}{r^{\delta+2/p}} \in L^p \right\}, \quad L^{k,p}_\delta = \left\{ f, \frac{\nabla^j f_i}{r^{i(k-j)}} \in L^p_\delta \right\}.$$

It is convenient to also introduce spaces with a weighted condition on f_0,

$$\hat{L}^{k,p}_\delta = \left\{ f \in L^{k,p}_\delta, \frac{\nabla^j f_0}{r^{k-j}} \in L^p_\delta \right\}.$$

Again, the weight is chosen so that $r^x \in L^p_\delta$ if and only if $x > \delta$. This is a nice convention for products, since we get

$$L^p_{\delta_1} \cdot L^q_{\delta_2} \subset L^r_{\delta_1+\delta_2}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

We remark that the Sobolev space above for functions has a simple interpretation on the conformal half-cylinder with metric

$$\frac{|dz|^2}{|z|^2} = \frac{dr^2}{r^2} + d\theta^2 = dt^2 + d\theta^2$$

with $t = -\ln r$. Indeed, for a function f, the condition $f \in \hat{L}^{k,p}_\delta$ is equivalent to

$$e^{\delta t} f \in L^{k,p}\left(\frac{|dz|^2}{|z|^2}\right),$$

which is the standard weighted Sobolev space on the cylinder.

From this interpretation one easily deduces the following facts for a function f on the disk, see e.g. [Biq91, § 1].

i) *Sobolev embedding:* for $1/2 \geqslant 1/p - 1/r$ (with strict inequality for $r = \infty$), one has

$$\hat{L}^{1,p}_{\delta-1} \hookrightarrow L^r_\delta; \tag{3.1}$$

in particular, for $p > 2$, one has $\hat{L}^{1,p}_{\delta-1} \hookrightarrow C^0_\delta$.

ii) *Control of the function by its radial derivative:* if $\delta < 0$ and f vanishes on the boundary, or if $\delta > 0$ and f vanishes near the origin, then

$$\left\| \frac{\partial f}{\partial r} \right\|_{L^p_{\delta-1}} \geqslant c \|f\|_{L^p_\delta}; \tag{3.2}$$

in particular, the estimate (3.2) means that

$$\hat{L}^{1,p}_{\delta-1} = L^{1,p}_{\delta-1} \quad \text{if } \delta < 0. \tag{3.3}$$

iii) For $p > 2$ and $\delta > 0$, the condition $df \in L^p_{\delta-1}$ implies $f \in C^0$ and then, by applying (3.2),

$$\|f - f(0)\|_{\hat{L}^{1,p}_{\delta-1}} \leqslant c \|df\|_{L^p_{\delta-1}}.$$

From (3.3), connection forms $a \in L^{1,2}_{-2+\delta}(\Omega^1 \otimes (\operatorname{End}(E)_0 \oplus \operatorname{End}(E)_1))$ actually belong to $\hat{L}^{1,2}_{-2+\delta} \subset L^p_{-1+\delta}$ for any p by (3.1). Now for a gauge transformation g, again restricting to the component $\operatorname{End}(E)_0 \oplus \operatorname{End}(E)_1$, the condition $Dgg^{-1} \in L^{1,2}_{-2+\delta} \subset L^p_{-1+\delta}$ implies by (iii) that g is continuous, with $g(0) \in \operatorname{End}(E)_0$, and $g - g(0) \in C^0_\delta \cap L^p_\delta$.

Similarly, an infinitesimal gauge transformation u in $L^{2,2}_{-2+\delta}(\operatorname{End}(E)_0 \oplus \operatorname{End}(E)_1)$ has a well-defined value $u(0) \in \operatorname{End}(E)_0$, and

$$u - u(0) \in \hat{L}^{2,2}_{-2+\delta}. \tag{3.4}$$

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We have to generalize this picture from the regular singularity case to the irregular singularity case. The new ingredient is that now for sections of $\mathrm{End}(E)_k$ the weight $1/r^k$ is no longer 'equivalent' to a derivative as in (3.2), and this implies that the behavior of the weights in the Sobolev embedding $L^{1,2} \hookrightarrow L^p$ is more involved than in (3.1), because we somehow have to separate what comes from the bound on the derivative from what comes from the bound on the tensor itself.

Lemma 3.1. *For $p > 2$, one has the Sobolev injections*

$$L^{1,2}_\delta(\mathrm{End}(E)_k) \hookrightarrow L^p_{\delta+2k/p+(1-2/p)}(\mathrm{End}(E)_k),$$
$$L^{1,p}_\delta(\mathrm{End}(E)_k) \hookrightarrow C^0_{\delta+k-2(k-1)/p}(\mathrm{End}(E)_k).$$

Proof. Take f in $L^{1,2}_\delta(\mathrm{End}(E)_k)$, then

$$\|f\|^2_{L^{1,2}_\delta} = \int \left(\left| \frac{f}{r^{1+\delta+k}} \right|^2 + \left| \frac{\nabla f}{r^{1+\delta}} \right|^2 \right) |dz|^2.$$

Because of Kato's inequality $|\nabla f| \geqslant |d|f||$, we can restrict to the case where f is a function, and replace ∇f by df. With respect to the metric $|dz|^2/|z|^{2k}$, the above norm transforms into

$$\|f\|^2_{L^{1,2}_\delta} = \int \left(\left| \frac{f}{r^{1+\delta}} \right|^2 + \left| \frac{df}{r^{1+\delta}} \right|^2 \right) \frac{|dz|^2}{|z|^{2k}},$$

which is equivalent to the $L^{1,2}$-norm

$$\int (|g|^2 + |dg|^2) \frac{|dz|^2}{|z|^{2k}}$$

of $g = r^{-1-\delta} f$. The metric $|dz|^2/|z|^{2k}$ is flat; actually

$$\frac{|dz|^2}{|z|^{2k}} = |du|^2, \quad u = \frac{-1}{(k-1)z^{k-1}}.$$

The problem here is that $z \to u$ is a $(k-1)$-covering $\Delta - \{0\} \to \mathbb{C} - \bar{\Delta}$: this means that f must be interpreted on $\mathbb{C}$ as a section of a rank $(k-1)$ flat unitary bundle. Nevertheless, still using Kato's inequality, we can apply the standard Sobolev embedding on $\mathbb{C}$ to deduce that

$$\left(\int (|g|^2 + |dg|^2) |du|^2 \right)^{1/2} \geqslant c \left(\int |g|^p |du|^2 \right)^{1/p},$$

that is

$$\|f\|_{L^{1,2}_\delta} \geqslant c \|r^{1+\delta-(2k/p)} f\|_{L^p_{|dz|^2}},$$

which is exactly the first statement of the lemma. The proof for the second statement is similar. □

Remark 3.2. Since on a compact manifold these Sobolev embeddings are compact, it is easy to deduce that for $\delta' < \delta$ the Sobolev embeddings

$$L^{1,2}_\delta(\mathrm{End}(E)_k) \hookrightarrow L^p_{\delta'+2k/p+(1-2/p)}(\mathrm{End}(E)_k),$$
$$L^{1,p}_\delta(\mathrm{End}(E)_k) \hookrightarrow C^0_{\delta'+k-2(k-1)/p}(\mathrm{End}(E)_k),$$

are compact.

Remark 3.3. The covering $z^{-(k-1)} = u$ can be used to extend to L^p-spaces the L^2-theory which will be done in § 4.

Corollary 3.4. *For $k > 0$ and $\delta + k + 1 > 0$, one has the injection*

$$L^{2,2}_\delta(\mathrm{End}(E)_k) \hookrightarrow C^0_{\delta'+k+1}$$

for any $\delta' < \delta$.

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Proof. By the previous lemma, if $f \in L^{2,2}_\delta(\mathrm{End}(E)_k)$, then

$$\nabla f \in L^{1,2}_\delta(\mathrm{End}(E)_k) \subset L^p_{\delta+k-(k-1)(1-2/p)}.$$

Take $p > 2$ close enough to 2 so that $\delta' = \delta-(k-1)(1-2/p)$, then we get $\nabla f \in L^p_{\delta'+k}$. If $k+1+\delta' > 0$, this implies

$$f = f(0) + f', \quad f' \in C^0_{\delta'+k+1}.$$

Because $f \in L^{2,2}_\delta(\mathrm{End}(E)_k)$, we see that $f(0) = 0$. □

We now have the tools to prove that the spaces defined in § 2 give us a nice gauge theory.

Lemma 3.5. *The spaces*

$$L^{2,2}_{-2+\delta}(\mathrm{End}\, E) \quad \textit{and} \quad L^{1,p}_{-1+\delta}(\mathrm{End}\, E) \quad (\textit{for } p > 2)$$

are algebras, and $L^{1,2}_{-2+\delta}(\mathrm{End}\, E)$ is a module over both algebras.

Proof. We first prove that $L^{1,p}_{-1+\delta}(\mathrm{End}\, E)$ is an algebra.

We use the fact that $F_k = \oplus_{i\leqslant k}\mathrm{End}(E)_i$ is a filtration of $\mathrm{End}\, E$ by algebras. Take $u \in L^{1,p}_{-1+\delta}(\mathrm{End}(E)_k)$ and $v \in L^{1,p}_{-1+\delta}(F_{k-1})$; we want to prove that $w = uv \in L^{1,p}_{-1+\delta}$. As F_k is an algebra, it is sufficient to prove that $w/r^k \in L^p_{\delta-1}$ and $Dw \in L^p_{\delta-1}$; the statement on w/r^k is clear, since $v \in C^0$ and $u/r^k \in L^p_{\delta-1}$. Now $Dw = (Du)v + u(Dv)$, but $u, v \in C^0$ and $Du, Dv \in L^p_{-1+\delta}$ implies $Dw \in L^p_{\delta-1}$.

The other statements are proven in a similar way. □

Proof of Lemma 2.1. The curvature of a connection $D + a \in \mathscr{A}$ is $F = Da + a \wedge a$. From the definition of $\mathscr{A}$, it is clear that $Da \in L^2_{-2+\delta}$. On the other hand, $a \in L^{1,2}_{-2+\delta} \subset L^4_{-1+\delta}$, therefore $a \wedge a \in L^2_{-2+2\delta} \subset L^2_{-2+\delta}$, so $F \in L^2_{-2+\delta}$.

We want to analyse the condition $Dgg^{-1} \in L^{1,2}_{-2+\delta}$ defining the gauge group. Recall that we have a decomposition $D = D^+ + \phi$, and the fact that g is unitary actually implies that both D^+gg^{-1} and $[\phi, g]g^{-1}$ are in $L^{1,2}_{-2+\delta} \subset L^p_{-1+\delta}$ for any $p > 2$. The condition on $[\phi, g]$ implies $g_k \in L^p_{k-1+\delta}$, and therefore the condition on Dg implies $dg \in L^p_{-1+\delta}$; finally, we get $g \in L^{1,p}_{-1+\delta}(\mathrm{End}\, E)$, and by Lemma 3.1,

$$g_k \in C^0_{\delta+k-1-2(k-1)/p}(\mathrm{End}(E)_k).$$

By Lemma 3.5, we finally deduce that D^+g and $[\phi, g]$ are in $L^{1,2}_{-2+\delta}(\Omega^1 \otimes \mathrm{End}\, E)$, which implies $g \in L^{2,2}_{-2+\delta}(\mathrm{End}\, E)$.

It is now clear that $\mathscr{G}$ is a Lie group with Lie algebra $L^{2,2}_{-2+\delta}(\mathfrak{u}(E))$. From Lemma 3.5, it is easy to prove the other statements in Lemma 2.1. □

4. Analysis on the disk

In this section, we give some tools to handle the analysis of our operators with strongly singular coefficients. In order to remain as elementary as possible, we restrict to L^2-spaces, which is sufficient for our purposes. Moreover, we have to be careful about the dependence of the constants with respect to homotheties of the disk, since this is crucial for the compactness result that we will need later.

Lemma 4.1. *For a p-form u with values in $\mathrm{End}\, E$, with compact support in $\Delta - \{0\}$, one has*

$$\int |Du|^2 + |D^*u|^2 = \int |\nabla u|^2 + |\phi \otimes u|^2.$$

Proof. Integrate by parts the formula [Biq97, Theorem 5.4]

$$D^*D + DD^* = \nabla^*\nabla + (\phi\otimes)^*\phi\otimes . \qquad \square$$

In particular, we apply this formula to get the following consequence.

Corollary 4.2. *If we have a 1-form $u \in L^{1,2}_{-2+\delta}$, vanishing on $\partial\Delta$, then*

$$\|Du\|_{L^2_{-2+\delta}} + \|D^*u\|_{L^2_{-2+\delta}} \geqslant c(\|\nabla u\|_{L^2_{-2+\delta}} + \|\phi\otimes u\|_{L^2_{-2+\delta}}).$$

On the $\mathrm{End}(E)_k$ *part for $k \geqslant 2$, the estimate is valid for all weights δ. For $k = 0$ or 1, the estimate holds only for $\delta > 0$ sufficiently small.*

Finally, the same estimate holds with the same *constant c if we replace D by $h^*_\varpi D$, where h_ϖ is the homothety $h_\varpi(x) = \varpi x$ in Δ, for $\varpi < 1$.*

Proof. First consider the u section of $\mathrm{End}(E)_k$ for $k \geqslant 2$. For a positive function $\rho(r)$ to be fixed later, one has, by Lemma 4.1,

$$\|(D + D^*)(\rho^{1-\delta}u)\|^2_{L^2} = \|\nabla(\rho^{1-\delta}u)\|^2_{L^2} + \|\rho^{1-\delta}\phi\otimes u\|^2_{L^2}.$$

On the other hand,

$$|[D + D^*, \rho^{1-\delta}]u| + |[\nabla, \rho^{1-\delta}]u| \leqslant c|\rho'\rho^{-\delta}u|.$$

From these two estimates, we deduce

$$\begin{aligned}\|\rho^{1-\delta}(D + D^*)u\|_{L^2} &\geqslant \|(D + D^*)(\rho^{1-\delta}u)\|_{L^2} - \|[D + D^*, \rho^{1-\delta}]u\|_{L^2}\\ &\geqslant c(\|\rho^{1-\delta}\nabla u\|_{L^2} + \|\rho^{1-\delta}\phi\otimes u\|_{L^2}) - c'\|\rho'\rho^{-\delta}u\|_{L^2}.\end{aligned}$$

Since $k \geqslant 2$, we have an (algebraic) estimate

$$|\phi\otimes u| \geqslant \lambda_k \frac{|u|}{r^k},$$

where λ_k is the smallest modulus of non-zero eigenvalues of A_k. Now choose ρ in the following way:

$$\rho(r) = \begin{cases} r & \text{for } r \leqslant \dfrac{\varepsilon}{2},\\ \dfrac{3}{4}\varepsilon & \text{for } r \geqslant \varepsilon,\end{cases}$$
$$0 \leqslant \rho' \leqslant 1.$$

Using the fact that $\rho/r^2 \geqslant 1/2\varepsilon$ for $r \leqslant \varepsilon$, we get the estimate

$$\|\rho'\rho^{-\delta}u\|^2_{L^2} \leqslant 2\varepsilon\left\|\rho^{1-\delta}\frac{u}{r^2}\right\|^2_{L^2} \leqslant \frac{2\varepsilon}{\lambda_k}\|\rho^{1-\delta}\phi\otimes u\|^2_{L^2},$$

hence

$$\|\rho^{1-\delta}(D + D^*)u\|_{L^2} \geqslant \left(c - \frac{2\varepsilon c'}{\lambda_k}\right)(\|\rho^{1-\delta}(D + D^*)u\|_{L^2} + \|\rho^{1-\delta}\phi\otimes u\|^2_{L^2}).$$

Choosing ε small enough, finally we get

$$\|\rho^{1-\delta}(D + D^*)u\|_{L^2} \geqslant \frac{c}{2}(\|\rho^{1-\delta}(D + D^*)u\|_{L^2} + \|\rho^{1-\delta}\phi\otimes u\|^2_{L^2}).$$

As ρ coincides with r near zero, the norm $\|\rho^{1-\delta}\cdot\|_{L^2}$ is equivalent to the $L^2_{-2+\delta}$ norm, and the corollary is proven. If D is transformed into $h^*_\varpi D$, then λ_k becomes $\varpi^{1-k}\lambda_k$ which is bigger, so the estimate still holds (c and c' do not depend on D).

In the cases $k = 0$ or 1, one can prove the estimate directly, but this is a bit more complicated. Another way to prove the lemma is to observe that in this case, by a conformal change, the operator $D + D^*$ is transformed into a constant coefficient operator on the cylinder with metric $|dz|^2/|z|^2$, and it is clear that it has no kernel. It then follows that if δ is not a critical weight (see Remark 4.5), the existence of the estimate follows from general elliptic theory for operators on the cylinder. Also, the homothety h_ϖ leaves invariant the operator for $k=0$ or 1, and this implies that the constant does not change. □

Corollary 4.3. *For $k \geqslant 2$, if we have a L^2_δ-solution u with values in $\operatorname{End}(E)_k$ of the equation $(D+D^*)u=0$, then $|u| = O(r^\gamma)$ for any real γ.*

Proof. Let $\chi_\epsilon(r)$ be a cut-off function, so that $\chi_\epsilon u = u$ for $\epsilon < r < 1/2$. We can choose χ_ϵ so that $|d\chi_\epsilon| \leqslant c/r$. Then

$$|(D+D^*)(\chi_\epsilon u)| \leqslant \frac{c}{r}|u|.$$

By Corollary 4.2 this gives

$$\begin{aligned}\|\nabla(\chi_\epsilon u)\|_{L^2_{\delta-1}} + \|\chi_\epsilon u\|_{L^2_{\delta+k-1}} &\leqslant c'\|(D+D^*)(\chi_\epsilon u)\|_{L^2_{\delta-1}} \\ &\leqslant cc'\|u\|_{L^2_\delta}.\end{aligned}$$

Taking $\varepsilon \to 0$, we get $u \in L^2_{\delta+k-1}$ and $\nabla u \in L^2_{\delta-1}$. Since $k>1$, we can iterate the argument and deduce that u and ∇u (hence $d|u|$) belong to L^2_γ for any γ. The corollary follows. □

Lemma 4.4. *On the disk, the Laplacian*

$$D^*D + DD^* : L^{2,2}_{-2+\delta}(\Omega^i \otimes \operatorname{End} E) \to L^2_{-2+\delta}(\Omega^i \otimes \operatorname{End} E),$$

with Dirichlet condition on the boundary, is an isomorphism for small weights $\delta > 0$.

If we restrict to the components $\operatorname{End}(E)_n \oplus \cdots \oplus \operatorname{End}(E)_2$, then the same holds for any weight δ.

Proof. Begin by considering the components in $\operatorname{End}(E)_k$ for $k \geqslant 2$. We claim that a solution of $(DD^*+D^*D)u = v$ is obtained by minimizing the functional

$$\int \frac{1}{2}(|Du|^2 + |D^*u|^2) - \langle u, v\rangle$$

among $u \in L^{1,2}_{-1}(\operatorname{End}(E)_k)$ vanishing on the boundary; indeed, by Lemma 4.1, we have that, for such u,

$$\int (|Du|^2 + |D^*u|^2) \geqslant c\int \frac{|u|^2}{r^{2k}}, \tag{4.1}$$

so the functional is coercive and a solution can be found; note that it is sufficient to have $r^k v \in L^2$. The solution satisfies an equation

$$(dd^* + d^*d)u = v + P_0\left(\frac{u}{r^{2k}}\right) + P_1\left(\frac{\nabla u}{r^k}\right),$$

where P_0 and P_1 are bounded algebraic operators, so by elliptic regularity $u \in L^{2,2}_{-k-1}$. In conclusion we get, for the Dirichlet boundary condition, an isomorphism

$$DD^* + D^*D : L^{2,2}_{-k-1} \to L^2_{-k-1}. \tag{4.2}$$

Now we want to prove that this L^2-isomorphism actually extends to all weights: we proceed by proving that the L^2-inverse is continuous in the other weighted spaces. For any weight γ, it is sufficient to prove an estimate

$$\|\rho^\gamma(DD^*+D^*D)u\|_{L^2_{-k-1}} \geqslant c\|\rho^\gamma u\|_{L^{2,2}_{-k-1}},$$

where ρ is some function which coincides with r near zero, as in the proof of Corollary 4.2. As in the proof of this corollary, the estimate is deduced from a control on the commutator $[DD^*+D^*D, \rho^\gamma]$, obtained after a careful choice of ρ. The details are left to the reader.

Now let us look at the component $\mathrm{End}(E)_1$. In fact, the estimate (4.1) still holds, so therefore the isomorphism (4.2) is true, and remains true for small perturbations $-2+\delta$ of the weight -2.

Finally, for the component $\mathrm{End}(E)_0$, we simply have the usual Laplacian dd^*+d^*d on the disk to study in weighted Sobolev spaces, so it is a classical picture: the inverse is the usual solution of the Dirichlet problem on the disk. The question is to check the regularity in the weighted Sobolev spaces. It is useful to proceed in the following way, using general theory for elliptic operators on cylinders. The operator

$$dd^*+d^*d : \hat{L}^{2,2}_{-2+\delta} \to L^2_{-2+\delta} \tag{4.3}$$

translates on the cylinder into the operator

$$-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \theta^2} : e^{\delta t} L^{2,2} \longrightarrow e^{\delta t} L^2. \tag{4.4}$$

Note that the operator for $k=1$ is just the same, with an additional term λ_1^2, where λ_1 is the eigenvalue of the action of A_1.

Here the weight $\delta=0$ is critical, because of the solutions $a+bt$ in the kernel of (4.4). Nevertheless, for small $\delta \neq 0$ the operator (4.4) becomes Fredholm, and, because the operator is self-adjoint, we get by formula [LM85, Theorem 7.4] the index $+1$ for small negative δ and -1 for small positive δ. Coming back on the disk, this means the operator (4.3) has index -1. As $\hat{L}^{2,2}_{-2+\delta} \subset L^{2,2}_{-2+\delta}$ has codimension one by (3.4), this means that $dd^*+d^*d : L^{2,2}_{-2+\delta} \to L^2_{-2+\delta}$ has index zero. Therefore it is an isomorphism. □

Remark 4.5. On the $\mathrm{End}(E)_0 \oplus \mathrm{End}(E)_1$ part, the theory of elliptic operators on cylinders also gives information for all weights δ. Namely, the problem is Fredholm if δ avoids a discrete set of critical weights (corresponding to the existence of solutions $(a+bt)e^{-\delta t}$). This easily follows from the following fact: if one considers the problem (4.4) on the whole cylinder (or, equivalently, DD^*+D^*D on $\mathbb{R}^2-\{0\}$), then it is an isomorphism outside these critical weights.

Finally, we deduce the decay of the solutions of the self-duality equations.

Lemma 4.6. *If we have on the disk a solution $a \in L^{1,2}_{-2+\delta}(\Omega^1 \otimes \mathrm{End}\, E)$ of an equation $(D+D^*)a = a \odot a$, then in the decomposition $a=\sum a_k$ we have the following decay for a:*

1) *if $k \geqslant 2$, then $|a_k| = O(r^\gamma)$ for any γ;*
2) *if $k=0$ or 1, then $|a_k| = O(r^{-1+\delta})$.*

Proof. We have $L^{1,2}_{-2+\delta} \subset L^p_{-1+\delta}$ for any p. On the other hand, for $q>2$ close enough to 2, one has the inclusion $L^{1,2}_{-2+\delta}(\mathrm{End}(E)_k) \subset L^q_{-2+k}$. Now take p big enough so that $1/p+1/q=1$, so we get $a \odot a_k \in L^2_{-3+k+\delta}$. In particular, we deduce that $(a\odot a)_k \in L^2_{-3+k+\delta}$ for all $k \geqslant 2$. It is now easy to adapt the proof of Corollary 4.3 to get $a_k \in L^2_{-3+2k+\delta}$ for all $k \geqslant 2$, and therefore $a_k \in L^{1,2}_{-3+\delta+k} \subset L^{1,2}_{-1+\delta}$.

Iterating this, we get that for $k \geqslant 2$ one has $a_k \in L^{1,2}_\gamma$ for any γ, and we deduce the bound on a_k.

For the $b = a_0 + a_1$ part, we can write the problem as

$$(D+D^*)b = b \odot b + \text{small perturbation},$$

with an initial bound $b \in L^{1,2}_{-2+\delta}$, and therefore $b \odot b \in L^p_{-2+2\delta}$ for any $p>2$. This is now a problem which translates into a constant coefficient elliptic problem on the conformal cylinder, so that elliptic regularity gives at once $b \in L^{1,p}_{-2+\delta} \subset C^0_{-1+\delta}$. □

5. Moduli spaces

Consider now a compact Riemann surface X with finitely many marked points p_i, and a complex vector bundle E over X, with a hermitian metric h. Choose an initial connection D_0 on E, such that, in some unitary trivialization of E around each p_i, the connection D_0 coincides with the local model (1.6). Of course on the interior of $X - \{p_i\}$, the connection D_0 is not flat in general.

Define r to be a positive function which coincides with $|z|$ around each puncture. We can then define global Sobolev spaces on X as in § 2, and therefore a space of connections $\mathscr{A} = D_0 + L^{1,2}_{-2+\delta}(\Omega^1 \otimes \operatorname{End} E)$, and a gauge group $\mathscr{G}$ as in (2.6) acting on $\mathscr{A}$.

The Lemma 4.4 on the disk can now be extended globally.

Lemma 5.1. *If $A \in \mathscr{A}$, then the operator*

$$D_A^* D_A : L^{2,2}_{-2+\delta}(\mathfrak{u}(E)) \to L^2_{-2+\delta}(\mathfrak{u}(E))$$

is Fredholm, of index zero.

Proof. First it is sufficient to prove that the Laplacian $D_0^* D_0$ is Fredholm, because the 1-form $a \in L^{1,2}_{-2+\delta}(\Omega^1 \otimes \operatorname{End} E)$ gives only a compact perturbation, see Remark 3.2. For $D_0^* D_0$, we can glue the inverse coming from Lemma 4.4 near the punctures with a parametrix in the interior. This gives a parametrix which is an exact inverse near infinity, implying that the operator is Fredholm.

Let i_δ be the index of the slightly different operator,

$$P_\delta = D_0^* D_0 : \hat{L}^{2,2}_{-2+\delta}(\mathfrak{u}(E)) \to L^2_{-2+\delta}(\mathfrak{u}(E));$$

this means that we now do not allow non-zero values at the origin for the $\operatorname{End}(E)_0$ part. We claim that

$$i_\delta = -i_{-\delta}, \tag{5.1}$$

$$i_\delta - i_{-\delta} = -2 \dim \mathfrak{u}(E)_0. \tag{5.2}$$

From these two assertions, it follows immediately that $i_\delta = -\dim \mathfrak{u}(E)_0$, and therefore the index of the initial operator is zero.

Now let us prove first (5.2). We have to calculate the difference between the indices of P_δ and $P_{-\delta}$. The operator does not change in the interior of X, so by the excision principle the difference comes only from what happens at the punctures, and it is sufficient to calculate it for the model Dirichlet problem: this has been done in Lemma 4.4 and its proof.

Now we prove (5.1). This comes from formal L^2-self-adjointness of P_δ: observe that the dual of $L^2_{-2+\delta}$ is identified to $L^2_{-\delta}$, therefore the cokernel of P_δ consists of solutions $u \in L^2_{-\delta}$ of the equation $D_0^* D_0 u = 0$. The behavior of such a u comes from Lemma 4.4: near the punctures, the components u_k for $k \geqslant 2$ decay quicker than any r^γ. This fact combined with elliptic regularity implies $u \in \hat{L}^{2,2}_{-2-\delta}(\mathfrak{u}(E))$, that is $u \in \ker P_{-\delta}$. Therefore, $\operatorname{coker} P_\delta = \ker P_{-\delta}$. □

We want to consider the quotient space $\mathscr{A}/\mathscr{G}$. If $A \in \mathscr{A}$ is irreducible, then D_A has no kernel and the cokernel of D_A is simply the kernel of D_A^*. From this and the lemma, it is classical to deduce that the irreducible part $\mathscr{A}^{\mathrm{irr}}/\mathscr{G}$ of the quotient is a manifold, with tangent space at a connection A given by

$$T_{[A]}(\mathscr{A}^{\mathrm{irr}}/\mathscr{G}) = \{a \in L^{1,2}_{-2+\delta}(\Omega^1 \otimes \operatorname{End} E),\ \operatorname{Im}(D_A^* a) = 0\}. \tag{5.3}$$

The moduli space $\mathscr{M} \subset \mathscr{A}/\mathscr{G}$ we consider is defined by the equations

$$F_A = 0, \quad G_A = 0. \tag{5.4}$$

These equations are not independent, since writing $A = A^+ + \phi_A$ we have the decomposition of F_A and G_A into self-adjoint and anti-self-adjoint parts given by

$$F_A = D_{A^+}\phi_A + (F_A + \phi_A \wedge \phi_A),$$
$$\Lambda G_A = -\Lambda D_{A^+}\phi_A + iD^*_{A^+}\phi_A.$$

Therefore, as is well-known, Equations (5.4) are equivalent to

$$F_A = 0, \quad D^*_{A^+}\phi_A = 0. \tag{5.5}$$

The space $\mathscr{A}$ is a flat hyper-Kähler space, for the standard L^2-metric, and with complex structures I, J, and $K = IJ$ acting on $a \in \Omega^1 \otimes \operatorname{End}(E)$ by

$$I(a) = ia, \quad J(a) = i(a^{0,1})^* - i(a^{1,0})^*. \tag{5.6}$$

The complex structure I is the natural complex structure on connections, and J is the natural complex structure on Higgs bundles.

Hitchin [Hit87] observed that Equations (5.5) are the zero set of the hyper-Kähler moment map of the action of $\mathscr{G}$ on $\mathscr{A}$. The linearization of Equations (5.5) is simply

$$D_A a = 0, \quad \operatorname{Re}(D^*_A a) = 0. \tag{5.7}$$

If $[A] \in \mathscr{M}$, there is an elliptic deformation complex governing the deformations of A:

$$L^{2,2}_{-2+\delta}(\mathfrak{u}(E)) \xrightarrow{D_A} L^{1,2}_{-2+\delta}(\Omega^1 \otimes \operatorname{End} E) \xrightarrow{D_A + D^*_A} L^2_{-2+\delta}((\Omega^2 \otimes \operatorname{End} E) \oplus i\mathfrak{u}(E))$$

As in the proof of Lemma 5.1, the Laplacians of the complex are Fredholm, with index zero, and we get the following result.

Lemma 5.2. *The cohomology groups of the deformation complex are finite dimensional.*

Lemma 5.3. *If $A \in \mathscr{M}$, then there is a gauge in which, near a puncture, $A = D_0 + a$, and a decays as in the conclusion of Lemma 4.6.*

Proof. We can write globally $A = D_0 + a$. Let χ_ϵ be a cut-off function, such that:

1) $\chi_\epsilon = 1$ in a disk of radius ϵ near each puncture;
2) $\chi_\epsilon = 0$ outside the disks of radius 2ϵ near each puncture;
3) $|d\chi_\epsilon| \leqslant c/r$ for some constant c.

Consider the connections $A_\epsilon = D_0 + \chi_\epsilon a$. This is a continuous path of connections in $\mathscr{A}$, converging to D_0.

Claim (Coulomb gauge). For ϵ sufficiently small, there exists a gauge transformation $g_\epsilon \in \mathscr{G}$, such that

$$\operatorname{Im}(D^*_0(g_\epsilon(A_\epsilon) - D_0)) = 0. \tag{5.8}$$

Suppose the claim is proven, fix some ϵ for which we have a Coulomb gauge; since $A = A_\epsilon$ in a disk of radius ϵ near the punctures, this means that, restricting to this disk,

$$g_\epsilon(A) = D_0 + a, \quad \operatorname{Im} D^*_0 a = 0.$$

Using this condition, Equations (5.5), with linearization (5.7), can be written as $(D_0 + D^*_0)a = a \odot a$ and the result follows from Lemma 4.6.

It remains to prove the claim. We try to find $g_\epsilon = e^u$ solving Equation (5.8), with $u \in L^{2,2}_{-2+\delta}(\mathfrak{u}(E))$. The linearization of the problem is $D^*_0 D_0 u = \operatorname{Im} D^*_0(\chi_\epsilon a)$. But the operator $D^*_0 D_0$ is Fredholm of index zero by Lemma 5.1. If it is invertible then, by the implicit function theorem, we get the solution g_ϵ we wanted. If not, we still get an isomorphism after restricting to the space $\ker(D_0^*D_0)^\perp$; fortunately, the operator $\operatorname{Im} D_0^*(e^u(A)-D_0)$ takes its values in the same space, so we can still apply the implicit function theorem after restricting to it. ☐

If A is irreducible, then the kernel of D_A on $\operatorname{End}(E)$ vanishes, and is equal to the kernel of D_A^* on $\Omega^2\otimes\operatorname{End}E$. From Lemma 5.2, it now follows that Equations (5.4) are transverse, and we therefore get the following result.

Theorem 5.4. *The moduli space $\mathscr{M}^{\mathrm{irr}}$ is a smooth hyper-Kähler manifold, with tangent space at A given by*

$$T_{[A]}\mathscr{M}=\{a\in L^2(\Omega^1\otimes\operatorname{End}E),\ D_Aa=0,\ D_A^*a=0\}=L^2H^1(\operatorname{End}E).$$

The metric is the natural L^2-metric. It is a complete metric if the moduli space $\mathscr{M}$ does not contain reducible points.

Proof. First we prove that L^2-cohomology calculates the H^1 of the elliptic complex. We have to prove that a L^2-harmonic 1-form actually belongs to the space $L^{1,2}_{-2+\delta}$. This is the infinitesimal version of Lemma 4.6 and is even simpler to prove (and because the equations are conformally invariant, one can do the local calculations on the disk with respect to the flat metric).

The difficult point is to prove that the metric is complete. Suppose we have a geodesic curve $([A_t])$ in $\mathscr{M}$ parameterized by arclength, of finite length ℓ. We want to extend it a bit. We can lift it to a horizontal curve ($A_t=A_0+a_t$), hence $\dot a_t$ satisfies

$$(D_{A_t}+D_{A_t}^*)\dot a_t=0, \tag{5.9}$$

$$\int_X|\dot a_t|^2=1. \tag{5.10}$$

These two equalities do not depend on the metric on X. If we choose a metric g, then, decomposing $\nabla_{A_t}=\nabla_t^++\phi_t$, we get the Weitzenböck formula [Biq97, Theorem 5.4]:

$$(\nabla_t^+)^*\nabla_t^+\dot a_t+\phi_t^*\phi_t\dot a_t+\frac{\mathrm{scal}^g}{2}\dot a_t=0.$$

We cannot integrate this equation against $\dot a_t$ for a smooth g, because the integral is divergent. Nevertheless, we use the freedom of the metric to choose g which coincides near each puncture with the flat metric $|dz|^2/|z|^{2(1-\delta')}$ for some local coordinate z and some positive $\delta'<\delta$. Because $\dot a_t\in L^{1,2}_{-2+\delta}$, now one can integrate by parts and get

$$\int_X\left(|\nabla_t^+\dot a_t|^2+|\phi_t\dot a_t|^2+\frac{\mathrm{scal}^g}{2}|\dot a_t|^2\right)\mathrm{vol}^g=0.$$

Here all norms are taken with respect to g. Because the scalar curvature of g is bounded, and using Kato's inequality, we deduce that

$$\int_X|d|\dot a_t||^2\,\mathrm{vol}^g\leqslant c\int_X|\dot a_t|^2\leqslant c. \tag{5.11}$$

Since the L^2-norm of 1-forms in conformally invariant, we can write on each disk near a puncture this equality with respect to the flat metric $|dz|^2$:

$$\int_{r<1}|d|\dot a_t|_g|^2|dz|^2=\int_{r<1}|d|r^{1-\delta'}\dot a_t||^2|dz|^2\leqslant c. \tag{5.12}$$

Claim. One has the estimate

$$\int_{r<1}|r^{-1+\varepsilon}f|^2|dz|^2\leqslant c\bigg(\int_{r<1}|df|^2|dz|^2+\int_{\frac12<r<1}|f|^2|dz|^2\bigg).$$

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Using the claim, we deduce from (5.10) and (5.12), for some positive $\delta'' < \delta'$, the estimate

$$\int_{r<1} (|r^{1-\delta''} d|\dot{a}_t||^2 + |r^{-\delta''}\dot{a}_t|^2)|dz|^2 \leqslant c.$$

In particular, the L^2-estimate is now slightly better than (5.10). Furthermore, the Sobolev embedding (3.1) implies $\|\dot{a}_t\|_{L^p_{-1+\delta''}} \leqslant c$. This estimate holds on every disk near the singularities, but also in the interior of X, applying the Sobolev embedding to (5.11); hence it is now a global estimate on X. Since $a_t = \int_0^t \dot{a}_t\, dt$, we get also the estimate on X: $\|a_t\|_{L^p_{-1+\delta''}} \leqslant c$. Now rewrite Equation (5.9) in the form $(D_{A_0} + D^*_{A_0})\dot{a}_t = a_t \odot \dot{a}_t$. Choose δ'' so close to δ that $2\delta'' \geqslant \delta$. Then, from the multiplication $L^4_{-1+\delta''} \cdot L^4_{-1+\delta''} \subset L^2_{-2+2\delta''} \subset L^2_{-2+\delta}$, we deduce that $\dot{a}_t$ remains bounded in $L^{1,2}_{-2+\delta}$, which means that a_t has a limit a_ℓ in $L^{1,2}_{-2+\delta}$ when t goes to ℓ. The limiting $A_0 + a_\ell$ is again a solution of Hitchin's equations, so represents a point of $\mathscr{M}$. Since there is no reducible solution, it is a smooth point and the geodesic can be extended.

There remains to prove the claim. It is a simple application of (3.2):

$$\int |df|^2|dz|^2 \geqslant \int |r^{\varepsilon} df|^2|dz|^2 \geqslant c\int |r^{-1+\varepsilon} f|^2|dz|^2$$

if f vanishes at the boundary. If not, then use a cut-off function χ so that $\chi = 1$ for $r \leqslant \frac{1}{2}$, and apply the above inequality to χf:

$$\int |df|^2|dz|^2 + \int_{\frac{1}{2}<r<1} |f|^2|dz|^2 \geqslant \int |d(\chi f)|^2|dz|^2 \geqslant \int |r^{-1+\varepsilon}\chi f|^2|dz|^2.$$

The claim follows. □

6. Complex moduli spaces and harmonic metrics

There are two complex moduli spaces that we would like to consider. We still have some reference connection $D_0 \in \mathscr{A}$, and recall that we can decompose $D_0 = D_0^+ + \phi_0$.

We have defined the unitary gauge group $\mathscr{G}$ by the condition $D_0 g g^{-1} \in L^{1,2}_{-2+\delta}$. This condition implies that both $D_0^+ g g^{-1}$ and $g\phi_0 g^{-1}$ are in $L^{1,2}_{-2+\delta}$.

For complex transformations, this is no longer true, and we have to define directly the complexified gauge group $\mathscr{G}_{\mathbb{C}}$ as the space of transformations $g \in GL(E)$ such that $D_0^+ g g^{-1}$ and $g\phi_0 g^{-1}$ belong to $L^{1,2}_{-2+\delta}$. As in Lemma 2.1, this definition makes $\mathscr{G}_{\mathbb{C}}$ into a Lie group with Lie algebra $\mathrm{Lie}(\mathscr{G}_{\mathbb{C}}) = L^{2,2}_{-2+\delta}(\mathrm{End}\, E)$. We now complexify the action of $\mathscr{G}$ on $\mathscr{A}$ for both the complex structures I and J.

For the first complex structure, the action is simply given by

$$D \longrightarrow g \circ D \circ g^{-1} = D - D g g^{-1}$$

and the associated moduli space is the moduli space of flat connections on E, defined by

$$\mathscr{M}_{\mathrm{DR,an}} = \{A \in \mathscr{A},\ F_A = 0\}/\mathscr{G}_{\mathbb{C}}. \tag{6.1}$$

The DR subscript means 'De Rham' moduli space, as in Simpson's terminology, and the 'an' is for 'analytic', by contrast with the moduli space of flat connections $\mathscr{M}_{\mathrm{DR,alg}}$ with some fixed behavior at the punctures that one can define algebraically.

For the second complex structure, we get a different action, namely the action of $\mathscr{G}_{\mathbb{C}}$ on $\mathscr{A}$ seen as a space of Higgs bundles: a connection $A = D_0 + a \in \mathscr{A}$ can be identified by (1.4) with the Higgs

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bundle $(\bar\partial_A, \theta_A)$, writing

$$\bar\partial_A = \bar\partial_0 + \frac{a^{0,1} - (a^{1,0})^*}{2}, \quad \theta_A = \theta_0 + \frac{a^{1,0} + (a^{0,1})^*}{2},$$

and the action of the complexified gauge group is simply

$$(\bar\partial_A, \theta_A) \longrightarrow (g \circ \bar\partial_A \circ g^{-1}, g\theta_A g^{-1}).$$

The associated moduli space is

$$\mathscr{M}_{\mathrm{Dol,an}} = \{A \in \mathscr{A},\ G_A = 0\}/\mathscr{G}_{\mathbb{C}}. \tag{6.2}$$

Again the subscript 'Dol' stands for Dolbeault moduli space, and the 'an' is used to distinguish the algebraic moduli space $\mathscr{M}_{\mathrm{Dol,alg}}$ of Higgs bundles with Higgs field of fixed polar part at each puncture.

In the two cases there is a notion of (analytic) stability which leads to spaces $\mathscr{M}^s_{\mathrm{DR,an}} \subset \mathscr{M}_{\mathrm{DR,an}}$ and $\mathscr{M}^s_{\mathrm{Dol,an}} \subset \mathscr{M}_{\mathrm{Dol,an}}$. In both cases, stability means that for some class of subbundles the slope of the subbundle (the analytic degree defined by the metric, divided by the rank) must be smaller than the slope of the bundle. Here we will only recall the definition and refer the reader to [Sim90, § 6] for details.

In the Dolbeault case, the class of subbundles to consider is the class of holomorphic $L^{1,2}$-subbundles, that is holomorphic subbundles $\mathscr{F}$ (outside the punctures), stable under the Higgs field, and defined by an orthogonal projection π such that $\bar\partial_A \pi \in L^2$. The corresponding analytic degree is obtained by integrating the curvature of the connection induced on the subbundle by the metric,

$$\begin{aligned}\deg^{\mathrm{an}} \mathscr{F} &= \frac{i}{2\pi} \int \operatorname{tr}(\pi F_A) - |D''_A \pi|^2 \\ &= \frac{i}{2\pi} \int \operatorname{tr}(\pi F_{A^+}) - |\bar\partial_A \pi|^2.\end{aligned}$$

In the De Rham case, there is a similar picture: one has to consider flat subbundles defined by an orthogonal projection π such that $D_A \pi \in L^2$. On a compact manifold, the degree of a flat subbundle is always zero, and stability reduces to semi-simplicity, that is there is no flat subbundle; in the non-compact case, the parabolic structure at the punctures may have a non-zero contribution to the degree of a flat subbundle.

Theorem 6.1. *Suppose* $\deg^{\mathrm{an}} E = 0$. *Then the natural restriction maps*

$$\mathscr{M}^s_{\mathrm{Dol,an}} \longleftarrow \mathscr{M}^{\mathrm{irr}} \longrightarrow \mathscr{M}^s_{\mathrm{DR,an}}$$

are isomorphisms.

We will prove the theorem in § 9.

7. The Dolbeault moduli space

In this section, we prove that elements of the analytic Dolbeault moduli space $\mathscr{M}_{\mathrm{Dol,an}}$ actually correspond to true meromorphic Higgs bundles, with fixed parabolic structure and fixed polar part of the Higgs field, on the Riemann surface X. This gives a correspondence between $\mathscr{M}^s_{\mathrm{Dol,an}}$ and $\mathscr{M}^s_{\mathrm{Dol,alg}}$.

In $X - \{p_i\}$, a $L^{1,2}_{\mathrm{loc}}$-$\bar\partial$-operator has holomorphic sections which define a structure of holomorphic bundle. The remaining question is local near the punctures. Fix the local model D_0 around a

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puncture as in § 1, with underlying Higgs bundle given by (1.7):

$$\bar\partial_0 = \bar\partial - \frac{1}{2}\operatorname{Re}(A_1)\frac{d\bar z}{\bar z}, \quad \theta_0 = \frac{1}{2}\sum_1^n A_i \frac{dz}{z^i} - \frac{\beta}{2}\frac{dz}{z}.$$

Now consider a Higgs bundle $(\bar\partial_0 + a, \theta_0 + \vartheta) \in \mathscr{A}$. The following lemma enables one to take the $\bar\partial$-operator to a standard form.

Lemma 7.1. *There is a complex gauge transformation g, defined in a neighborhood of the origin, such that:*

1) *g is continuous, and in the decomposition $g = g_0 + \cdots + g_n$ one has $g_i/r^{i-1+\delta'}$ continuous for some $\delta' < \delta$;*
2) *$g(\bar\partial_0 + a) = \bar\partial_0$.*

To prove the lemma, we first need the following statement.

Lemma 7.2. *Take $\delta \in \mathbb{R} - \mathbb{Z}$ and $p > 2$. Then the problem*

$$\frac{\partial f}{\partial \bar z} = g \tag{7.1}$$

in the unit disk has a solution $f = T_0 g$ such that

$$\|f\|_{C^0_{-1+\delta}} \leqslant c\|g\|_{L^p_{-2+\delta}}. \tag{7.2}$$

The same is true, if $\delta - \operatorname{Re}\lambda \notin \mathbb{Z}$, for the problem

$$\frac{\partial f}{\partial \bar z} + \frac{\lambda}{2}\frac{f}{\bar z} = g.$$

Proof. First one can restrict to the case $0 < \delta < 1$. Indeed, if $\delta = [\delta] + \delta_0$, then the problem (7.1) is equivalent to

$$\frac{\partial(z^{-[\delta]}f)}{\partial \bar z} = z^{-[\delta]}g,$$

and the wanted estimate (7.2) becomes equivalent to the estimate

$$\|z^{-[\delta]}f\|_{C^0_{-1+\delta_0}} \leqslant c\|z^{-[\delta]}g\|_{L^p_{-2+\delta_0}}.$$

Hence we may suppose $0 < \delta < 1$.

Now, for this range of δ, we claim that the Cauchy kernel

$$f(z) = \int \frac{g(u)}{z-u}|du|^2$$

gives the inverse we need. Indeed, by the Hölder inequality, we get

$$|f(z)| \leqslant \|g\|_{L^p_{-2+\delta}} \left(\int \frac{|du|^2}{|u|^{2-\delta p/(p-1)}|z-u|^{p/(p-1)}} \right)^{(p-1)/p}$$

and the result follows from the estimate

$$\int \frac{|du|^2}{|u|^{2-\delta p/(p-1)}|z-u|^{p/(p-1)}} \leqslant \frac{c}{|z|^{(1+\delta)p/(p-1)}}$$

which is left to the reader.

For the second problem, observe that

$$\frac{\partial}{\partial \bar z} + \frac{\lambda}{2\bar z} = r^{-\lambda} \circ \frac{\partial}{\partial \bar z} \circ r^{\lambda}.$$

If $\delta - \operatorname{Re}\lambda \notin \mathbb{Z}$, we can take the inverse $f = r^{-\lambda} T_0 r^{\lambda} g$. □

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Proof of Lemma 7.1. Define spaces of $u \in \operatorname{End} E$ and $a \in \Omega^1 \otimes \operatorname{End} E$ by

$$U_\delta = \{u, u_k \in C^0_{k-1+\delta},\ u_0 \in C^0_\delta\},$$
$$A_\delta = \{a, a_k \in L^p_{-2+\delta+k},\ a_0 \in L^p_{-1+\delta}\}.$$

Observe that, since $a \in L^{1,2}_{-2+\delta}$, we have $a_k \in L^p_{\delta+k-2-\epsilon}$ by Lemma 3.1, and therefore $a \in A_{\delta'}$ for some weight $\delta' < \delta$.

The problem $g(\bar\partial_0 + a) = \bar\partial_0$ can be written (with $g = 1 + u$)

$$\bar\partial_0 u - ua = a. \tag{7.3}$$

The operator $\bar\partial_0$ is of the type studied in Lemma 7.2; hence we get a continuous right inverse $T : A_{\delta'} \to U_{\delta'}$. We find a solution $u \in U_{\delta'}$ of (7.3) by a fixed point problem, looking at a solution $u \in U_{\delta'}$ of $u = T(ua + a)$. For this, we need $u \to T(ua + a)$ to be contractible; but

$$\|T(ua + a) - T(va + a)\|_U \leqslant c\|u - v\|_U \|a\|_A,$$

so this is true if $\|a\|_A$ is small enough.

Let h_ϖ be a homothety taking the disk of radius 1 to the disk of radius ϖ; then it is easy to see that $\|h^*_\varpi a\|_A \leqslant \varpi^\delta \|a\|_A$.

On the other hand, the operator $\bar\partial_0$ is unchanged by the homothety h_ϖ, so for ϖ small enough the operator $u \to T(ua + a)$ becomes contractible, and we can solve the problem. ☐

From the lemma, we deduce a basis of holomorphic sections for $\bar\partial_A$, as in (1.8),

$$\sigma_i = g^{-1} |z|^{\operatorname{Re}\mu_i - [\operatorname{Re}\mu_i]} e_i.$$

This basis defines a holomorphic extension of the bundle $(E, \bar\partial_A)$ over the puncture, which is characterized by the fact the sheaf of holomorphic sections of this bundle is the sheaf of bounded holomorphic sections outside the puncture. Moreover, the growth of the holomorphic sections is the same as the model (1.9), that is

$$|\sigma_i| \sim |z|^{\alpha_i} \tag{7.4}$$

with $\alpha_i = \operatorname{Re}\mu_i - [\operatorname{Re}\mu_i]$, and these different orders of growth define on the extension a parabolic structure, whose weights are $\alpha_1, \dots, \alpha_r$.

Finally, in the holomorphic basis (σ_i), the Higgs field becomes

$$\theta = g(\theta_0 + \vartheta)g^{-1} = \theta_0 + \vartheta', \quad \vartheta' = g[\theta_0, g^{-1}] + g\vartheta g^{-1},$$

and we now have simply $\bar\partial\vartheta' = 0$. From the bounds on g in the lemma, we deduce that actually ϑ' is holomorphic, therefore the polar part of the Higgs field is exactly, as in (1.10),

$$\theta_0 = \frac{1}{2}\sum_1^n A_i \frac{dz}{z^i} - \frac{\beta}{2}\frac{dz}{z}.$$

The above construction means that we have constructed a map

$$\mathscr{M}_{\mathrm{Dol,an}} \to \mathscr{M}_{\mathrm{Dol,alg}} \tag{7.5}$$

by defining a canonical extension. Actually, writing (7.5) is not completely correct, and it would be better to say that we have a functor between the two corresponding categories (indeed one can prove that morphisms in the space $L^{2,2}_{-2+\delta}$ extend to holomorphic morphisms of the extensions). This functor is actually an equivalence of categories, because the arrow (7.5) can be inverted through the following lemma.

LEMMA 7.3. *Let $(\mathscr{E}, \theta)$ be a meromorphic Higgs bundle on X, with parabolic structure at the punctures having weights $\alpha_1, \dots, \alpha_r$, and polar part of the Higgs field given at each puncture by*

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$\sum_1^n B_i(dz/z^i)$, *with the* B_i *diagonal matrices. Then there exists a hermitian metric on* $\mathscr{E}$ *such that the induced connection* $A = \bar{\partial}^{\mathscr{E}} + \partial^{\mathscr{E}} + \theta + \theta^*$ *belongs to a space* $\mathscr{A}$ *of connections with data (1.9) and (1.10) at the punctures. The bundle* $(\mathscr{E}, \theta)$ *can be recovered from* $(\bar{\partial}_A, \theta_A)$ *as its canonical extension.*

Proof. The problem consists in constructing an initial metric h on $\mathscr{E}$. In order to simplify the ideas, we will restrict to the case where B_n is regular semisimple, but the general case is similar. Take a basis (σ_i) of eigenvectors of $\mathscr{E}_p$ at the puncture p, and extend it holomorphically in a neighborhood. The action of a *holomorphic* gauge transformation $g = e^u$ of $\mathscr{E}$ on the Higgs field is by

$$\theta \longrightarrow g\theta g^{-1} = \exp(\mathrm{ad}\, u)\theta.$$

From this it is easy to see that, by a careful choice of u, one can kill the off-diagonal coefficients of θ to any finite order. Therefore, we can suppose that in the basis (σ_i) we have

$$\theta = \sum_1^n B_i \frac{dz}{z^i} + \vartheta,$$

where ϑ is holomorphic, and the off-diagonal coefficients of ϑ vanish up to any fixed order. Now we choose the flat metric

$$h = \begin{pmatrix} |z|^{2\alpha_1} & & \\ & \ddots & \\ & & |z|^{2\alpha_r} \end{pmatrix}.$$

It is clear that, in the orthonormal basis $e_i = s_i/|z|^{\alpha_i}$, we get exactly the flat model (1.7), that is

$$A = \bar{\partial}^{\mathscr{E}} + \partial^{\mathscr{E}} + \theta + \theta^* = d + \mathrm{Re}(A_1) i\, d\theta + \frac{1}{2}\sum_1^n \left(A_i \frac{dz}{z^i} + A_i^* \frac{d\bar{z}}{\bar{z}^i} \right) - \beta \frac{dr}{r} + a,$$

with notation as in § 1. In the perturbation a, the diagonal terms are C^∞, and the off-diagonal terms can be taken to vanish to any fixed high order. In particular, we get an element of $\mathscr{A}$. Actually we have obtained much more, because F_A vanishes up to any fixed high order (the diagonal terms do not contribute to the curvature). □

Remark 7.4. The extension property for the holomorphic bundle alone (no Higgs field) follows from earlier work: the connection A^+ has curvature in some L^p for $p > 1$ and this implies that a canonical extension as above exists [Biq92]. Actually we also need to make precise the singularity of the Higgs field in the extension: this requires the calculations above.

Finally, we prove that stability on both sides of (7.5) coincide, transforming the equivalence of categories (7.5) into an isomorphism of the moduli spaces, $\mathscr{M}^s_{\mathrm{Dol,an}} \xrightarrow{\sim} \mathscr{M}^s_{\mathrm{Dol,alg}}$. First we have to introduce an algebraic notion of stability. A meromorphic Higgs bundle $(\mathscr{E}, \theta)$ as above has a parabolic degree

$$\text{p-deg}^{\mathrm{alg}}\, \mathscr{E} = c_1(\mathscr{E})[X] + \sum \alpha_i;$$

if there are several marked points, one must add the contribution of the weights of each marked point. Subbundles of $\mathscr{E}$ also inherit a parabolic degree, and this enables one to define stability.

Lemma 7.5. *Let* $(\bar{\partial}_A, \theta_A) \in \mathscr{A}$ *be a Higgs bundle and* $(\mathscr{E}, \theta)$ *its canonical extension on* X*. Then* $(\bar{\partial}_A, \theta_A)$ *is analytically stable if and only if* $(\mathscr{E}, \theta)$ *is algebraically stable.*

Proof. The point is to prove that a holomorphic $L^{1,2}$-subbundle (stable under the Higgs field) extends into an algebraic subbundle of $\mathscr{E}$, and that the algebraic and analytic degrees coincide. Because of Remark 7.4, this is a consequence of the same statements in [Sim90]. □

8. The De Rham moduli space

We will not give any detail here, since this is completely parallel to the results of § 7. We only prove the following technical lemma, which is necessary for solving the $\bar\partial$-problem on components of $\mathrm{End}(E)_k$ with $k \geqslant 2$.

Lemma 8.1. *Take $p > 2$, $k > 1$ and $\delta \in \mathbb{R} - \mathbb{Z}$. Then the problem*

$$\frac{\partial f}{\partial \bar z} + \frac{\lambda}{\bar z^k} f = g$$

has a solution $f = Tg$ such that

$$\|f\|_{C^0_\delta} \leqslant c\|g\|_{L^p_{-1+\delta}}.$$

The constant c does not depend on λ.

Proof. The function

$$\varphi = \exp\left(\frac{\lambda}{(k-1)\bar z^{k-1}} - \frac{\bar\lambda}{(k-1)z^{k-1}}\right)$$

is a solution of $\partial f/\partial \bar z + (\lambda/\bar z^k) f = 0$, such that $|\varphi| = 1$. Let T_0 be the inverse defined by Lemma 7.2; then $Tg = \varphi T_0(\varphi^{-1} g)$ satisfies the requirements of the lemma. □

An element of the moduli space $\mathscr{M}_{\mathrm{DR,an}}$ is represented by a flat connection $A \in \mathscr{A}$. The holomorphic bundle underlying A has the $\bar\partial$-operator $D_A^{0,1}$. As in the previous section, relying on Lemmas 7.2 and 8.1, near a puncture it is possible to produce a complex gauge transformation g such that

$$g(D_A^{0,1}) = D_0^{0,1} = \bar\partial + \frac{1}{2}\sum_2^n A_i^* \frac{d\bar z}{\bar z^i} - \frac{\beta + i\,\mathrm{Im}\,A_1}{2}\frac{d\bar z}{\bar z}.$$

Therefore, we have $D_A^{0,1}$-holomorphic sections (τ_i) given by

$$\tau_i = |z|^{\beta_i + i\,\mathrm{Im}\,\mu_i} g^{-1} \exp\left(\sum_2^n \frac{A_i^*}{2(i-1)\bar z^{i-1}} - \frac{A_i}{2(i-1)z^{i-1}}\right) e_i,$$

and this basis of holomorphic sections defines a canonical holomorphic extension $\mathscr{F}$ over the puncture.

With respect to this extension, A becomes an integrable connection with irregular singularities, and the polar part of A remains equal to that of the model,

$$d + A_n \frac{dz}{z^n} + \cdots + A_1 \frac{dz}{z},$$

so that we get an element of the moduli space $\mathscr{M}_{\mathrm{DR,alg}}$ of such integrable connections.

Also the extension has a parabolic structure with weights β_i, and this enables us to define a parabolic degree $\text{p-deg}^{\mathrm{alg}}\,\mathscr{F}$ and therefore the algebraic stability of $(\mathscr{F}, A)$.

We finally have all the ingredients of the isomorphism $\mathscr{M}^s_{\mathrm{DR,an}} \longrightarrow \mathscr{M}^s_{\mathrm{DR,alg}}$.

Remark 8.2. The weights γ_i of the local system are the order of growth of parallel sections on rays going to the singularity. From the above formula, they are equal to

$$\gamma_i = \beta_i - \mathrm{Re}\,\mu_i = -2\,\mathrm{Re}\,\lambda_i.$$

By [Biq97, Proposition 11.1], the parabolic degree of $\mathscr{F}$ is

$$\text{p-deg}^{\mathrm{alg}}\,\mathscr{F} = \sum \gamma_i,$$

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where the sum has to be understood for all punctures. In particular, if all weights γ_i are taken to be zero, then the same is true for subbundles, so the degree for subbundles is always zero, so that stability reduces to irreducibility of the connection.

8.1 Sufficient stability conditions

We will describe some simple conditions on the parameters such that all points of $\mathscr{M}_{\rm DR}$ are stable. Suppose A is a meromorphic connection on a holomorphic vector bundle $\mathscr{E} \to X$ as constructed from the extension procedure above. Thus in some local trivialization near the ith singularity the polar part of A takes the form of the model

$$d + {}^iA_{n_i}\frac{dz_i}{z_i^{n_i}} + \cdots + {}^iA_1\frac{dz_i}{z_i},$$

where z_i is a local coordinate and ${}^iA_j$ are diagonal matrices. We wish to assume now that all of the leading coefficients ${}^iA_{n_i}$ are regular (have distinct eigenvalues). Note that the eigenvalues of the residues ${}^iA_1$ are uniquely determined by A up to order (independent of the coordinate choice). Now, if $\mathscr{F}$ is a subbundle of $\mathscr{E}$ preserved by A, we may choose a trivialization of $\mathscr{F}$ by putting the induced connection on $\mathscr{F}$ in model form (with residues ${}^iB_1$ say), and then extend this to a trivialization of $\mathscr{E}$ as above. In particular, it follows that the eigenvalues of ${}^iB_1$ are a subset of the eigenvalues of ${}^iA_1$. However (by considering the induced connection on $\det(\mathscr{E})$), we know that the (usual) degree of $\mathscr{E}$ is minus the sum of traces of the residues:

$$\deg(\mathscr{E}) = -\sum_i \operatorname{tr}({}^iA_1)$$

and similarly for $\mathscr{F}$. Thus we can ensure that A has no proper non-trivial subconnections by choosing the models for A such that none of the (finite number of) 'subsums'

$$\sum_i \sum_{j\in S_i} ({}^iA_1)_{jj} \tag{8.1}$$

of the residues is an integer, where $S_i \subset \{1,\ldots,\operatorname{rank}(\mathscr{E})\}$ are finite subsets of size k and k ranges from 1 to $\operatorname{rank}(\mathscr{E}) - 1$. Thus under such (generic) conditions any such connection A is stable.

8.2 An example

Let $\mathscr{M}$ be a moduli space of integrable connections with two poles on the projective line P^1 of order two at zero and order one at infinity. Consider the subspace $\mathscr{M}^* \subset \mathscr{M}$ of connections such that the underlying holomorphic bundle is trivial. Therefore, points of $\mathscr{M}^*$ are (globally) represented by connections of the form

$$d + A_0\frac{dz}{z^2} + B\frac{dz}{z}. \tag{8.2}$$

We will assume that A_0 is diagonal with distinct eigenvalues and that none of the eigenvalues of B differ by integers but allow arbitrary parabolic weights. It then follows that the model connections at zero and infinity are

$$d + A_0\frac{dz}{z^2} + \Lambda\frac{dz}{z}, \quad d + B_0\frac{dz}{z} \tag{8.3}$$

respectively, where Λ is the diagonal part of B and B_0 is a diagonalization of B. Once these models are fixed we see that B is restricted to the adjoint orbit $\mathscr{O}$ containing B_0 and has diagonal part fixed to equal Λ. Since A_0 is fixed and regular, the remaining gauge freedom in (8.2) is just conjugation by $T^{\mathbb{C}}$. Now if we use the trace to identify $\mathscr{O}$ with a coadjoint orbit and so give it a complex symplectic structure, then the map $\delta : \mathscr{O} \to \mathfrak{t}_{\mathbb{C}}$ taking B to its diagonal part is a moment map for this torus action, and so we have an isomorphism $\mathscr{M}^* \cong \mathscr{O}/\!\!/T^{\mathbb{C}}$ of the moduli space with the complex symplectic quotient at the value Λ of the moment map.

On the other hand, the same symplectic quotient underlies a complete hyper-Kähler metric obtained by taking the hyper-Kähler quotient of Kronheimer's hyper-Kähler metric [Kro90] on $\mathscr{O}$ by the maximal compact torus. Nevertheless, in general this quotient metric on $\mathscr{M}^*$ does not coincide with the metric of Theorem 5.4, because that is a complete metric on $\mathscr{M}$, which is larger, as we will show below. Therefore, varying A_0 in the regular part of the Cartan subalgebra leads to a family of hyper-Kähler metrics on $\mathscr{O}/\!\!/T^{\mathbb{C}}$ which become complete in a larger space. We remark that in this example the full space $\mathscr{M}$ may be analytically identified (cf. [Boa01a, Boa]) with the complex symplectic quotient $\mathscr{L}/\!\!/T^{\mathbb{C}}$ of a symplectic leaf $\mathscr{L} \subset G^*$ of the simply connected Poisson Lie group G^* dual to $GL_r(\mathbb{C})$.

LEMMA 8.3. *There are stable connections on non-trivial bundles with models of the type considered in the above example.*

Proof. We will do this in the rank-three case (this is the simplest case since then $\dim_{\mathbb{C}} \mathscr{M} = 2$; one may easily generalize to higher rank). Suppose we have $g \in GL_3(\mathbb{C})$ and diagonal matrices A_0, B_0', such that $A_0, e^{2\pi i B_0'}$ have distinct eigenvalues and:

1) the matrix entry $(g^{-1}A_0 g)_{31}$ is zero; and
2) the pair of diagonal matrices $-B_0', \Lambda := \delta(gB_0'g^{-1})$ have no integral subsums (in the sense of (8.1)).

Then consider the meromorphic connection on the bundle $\mathscr{O}(1) \oplus \mathscr{O} \oplus \mathscr{O}(-1) \to P^1$ defined by the clutching map $h = \operatorname{diag}(z, 1, z^{-1})$ and equal to

$$d + g^{-1}A_0 g \frac{dz}{z^2} + B_0' \frac{dz}{z}$$

on $\mathbb{C} \subset P^1$. Assumption (1) implies this is equivalent to the models (8.3) at 0 and ∞, with $B_0 = B_0' + \operatorname{diag}(1, 0, -1)$. Then assumption (2) implies it is stable. Finally one may easily construct such matrices by observing that companion matrices have zero (31) entry. For example,

$$A_0 = \lambda \begin{pmatrix} 1 & & \\ & 2 & \\ & & 4 \end{pmatrix}, \quad B_0 = \begin{pmatrix} p & & \\ & q & \\ & & r \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 12 & 48 \end{pmatrix}$$

have the desired properties if λ is non-zero and p, q, r are sufficiently generic (e.g. if $\{1, p, q, r\}$ are linearly independent over the rational numbers). □

9. Proof of Theorem 6.1

We will prove only the most difficult way, that is the isomorphism with the Dolbeault moduli space. We start with a stable Higgs bundle $(\bar{\partial}_{A_0}, \phi_{A_0}) \in \mathscr{A}$ satisfying the integrability condition $(D''_{A_0})^2 = 0$, that is $\bar{\partial}_{A_0}\theta_{A_0} = 0$, and we try to find a complex gauge transformation $g \in \mathscr{G}_{\mathbb{C}}$ taking it to a solution $A = g(A_0)$ of the self-duality equations, that is satisfying the additional equation $F_A = 0$. This problem is equivalent to finding the Hermitian–Einstein metric $h = g^*g$ on the Higgs bundle.

The idea, as in [Biq97, § 8] is to minimize the Donaldson functional $M(h)$ for $h = g^*g$ under the constraint

$$\|\Lambda F^h_{(\bar{\partial}_{A_0}, \phi_{A_0})}\|_{L^2_{-2+\delta}} \leqslant B. \tag{9.1}$$

We need the two following technical lemmas.

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Lemma 9.1. *If $f \in L^{2,2}_{-2+\delta}$ and $g \in L^2_{-2+\delta}$ are positive functions such that $\Delta f \leqslant g$ and $\|g\|_{L^2_{-2+\delta}} \leqslant B$, then*

$$\|f\|_{C^0} \leqslant c_0(B) + c_1(B)\|f\|_{L^1},$$

and $\|f\|_{C^0}$ goes to zero when $\|f\|_{L^1}$ goes to zero.

Proof. This follows easily (see [Sim88, Proposition 2.1]) from the fact that if $v \in L^2_{-2+\delta}$, then the problem $\Delta u = v$ on the unit disk, with Dirichlet boundary condition, can be solved with $u \in L^{2,2}_{-2+\delta} \subset C^0$. □

Lemma 9.2. *If we have a sequence of metrics $h_j = h_0 u_j$ with*

1) *h_j has a C^0-limit h_∞,*
2) *$u_j \in L^{2,2}_{-2+\delta}$ and h_j satisfies the constraint (9.1),*
3) *$\|D''_{A_0} u_j\|_{L^2}$ is bounded,*

then the limit is actually a $L^{2,2}_{-2+\delta}$-limit: $h_\infty = h_0 u_\infty$ and $u_\infty \in L^{2,2}_{-2+\delta}$.

Proof. This is a local statement. Outside the puncture, the statement is proven for example in [Sim88, Lemma 6.4]; the point here is to prove that C^0-convergence implies the convergence in our weighted Sobolev spaces.

Note that $D = D_{A_0}$, $D'' = D''_{A_0}$, etc. We use the freedom, from the proof of Lemma 7.3, to choose an initial metric h with ΛF^h bounded. Observe that, because of the formula

$$(D'')^* D'' - (D')^* D' = i\Lambda F^h,$$

the hypothesis on $\|D'' u_j\|_{L^2}$ also implies that $\|D' u_j\|_{L^2}$ is bounded.

We have the formula

$$(D')^* D' u_j = i u_j(\Lambda F^{h_j} - \Lambda F^h) + i\Lambda(D'' u_j) u_j^{-1}(D' u_j).$$

Let χ be a cut-off function, with compact support in the disk, such that $\chi|_{\Delta_{1/2}} = 1$. Then we obtain

$$(D')^* D' \chi u_j = i\chi u_j(\Lambda F^{h_j} - \Lambda F^h) + i\Lambda(D'' u_j) u_j^{-1}(D' \chi u_j) + d\chi \odot D u_j + \tfrac{1}{2}(\Delta\chi) u_j. \tag{9.2}$$

In particular, we get

$$\|(D')^* D' \chi u_j\|_{L^2_{-2+\delta}} \leqslant c(1 + \|D'' u_j\|_{L^4_{-1}} \|D' \chi u_j\|_{L^4_{-1+\delta}}). \tag{9.3}$$

Notice that, by Corollary 4.2, because $(D')^2 = 0$, we have

$$\|(D')^* D' \chi u_j\|_{L^2_{-2+\delta}} \geqslant c\|D' \chi u_j\|_{L^{1,2}_{-2+\delta}} \geqslant c\|D' \chi u_j\|_{L^4_{-1+\delta}}$$

and therefore from (9.3)

$$\|D' \chi u_j\|_{L^{1,2}_{-2+\delta}}(1 - c\|D'' u_j\|_{L^4_{-1}}) \leqslant c. \tag{9.4}$$

The important point here is that all constants are invariant by homothety. Indeed the L^2-norm of 1-forms is conformally invariant, so $\|(D'' \oplus D')u\|_{L^2}$ remains bounded; the same is true for the L^4_{-1}-norm of 1-forms. In Corollary 4.2, the constants do not depend on an homothety; finally, in the Sobolev embedding $\|d(\chi f)\|_{L^2_{-2+\delta}} \geqslant c\|\chi f\|_{L^4_{-1+\delta}}$, the norms on both sides are rescaled by the same factor under an homothety.

Now suppose that there exists some disk Δ_ρ of radius ρ such that for all j one has $\|D'' u_j\|_{L^4_{-1}(\Delta_\rho)} < 1/2c$. Then, because this norm is invariant under homothety, we can rescale the disk Δ_ρ by the homothety h_ρ to the unit disk Δ, and applying (9.4) we get that $\chi h_\rho^* u_j$ is bounded in $L^{2,2}_{-2+\delta}$, so we get the lemma.

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Now suppose on the contrary that there exist radii $\rho_j \to 0$ such that $\|D''u_j\|_{L^4_{-1}(\Delta_{\rho_j})} \geqslant 1/2c$; by taking some smaller ρ_j, one can arrange it so that actually

$$\|D''u_j\|_{L^4_{-1}(\Delta_{\rho_j})} + \|D'u_j\|_{L^4_{-1}(\Delta_{\rho_j})} = \frac{1}{2c}. \tag{9.5}$$

We will see that this hypothesis leads to a contradiction. First we prove that the 'energy' in (9.5) cannot concentrate near the origin. Again, using the homothety h_{ρ_j}, we deduce from (9.4) that $D'(\chi h^*_{\rho_j} u_j)$ is bounded in $L^{1,2}_{-2+\delta}$. Of course one must be a bit careful here, because these $L^{1,2}$-norms do depend on j, so we actually mean (write $D_j = h^*_{\rho_j} D$)

$$\|D'_j(\chi h^*_{\rho_j} u_j)\|_{L^2_{-1+\delta}} + \|(\nabla^{D_j^+} \oplus \phi_j) D'_j(\chi h^*_{\rho_j} u_j)\|_{L^2_{-2+\delta}} \leqslant C; \tag{9.6}$$

in particular, $\|d|D'_j(\chi h^*_{\rho_j} u_j)|\|_{L^2_{-2+\delta}}$ is bounded, and because of the compact inclusion $L^{1,2}_{-2+\delta} \subset L^4_{-1}$ (see Remark 3.2), we deduce that the functions $|D'_j(\chi h^*_{\rho_j} u_j)|$ converge strongly in L^4_{-1} to a limit (and the same is true for $|D''_j(\chi h^*_{\rho_j} u_j)|$).

Actually, we can deduce a bit more from (9.6): indeed, the operators $D_j = h^*_{\rho_j} D$ become very close to the model $h^*_{\rho_j} D_0$ when j goes to infinity, so it is enough to suppose that $D = D_0$ near the puncture. Now for components $u(k)$ for $k \geqslant 2$, we have by (9.6)

$$\frac{\lambda_k}{\rho_j^{k-1}} \|D'_j(\chi h^*_{\rho_j} u_j(k))\|_{L^2_{-1+\delta}} \leqslant \|\phi_j \otimes D'_j(\chi h^*_{\rho_j} u_j(k))\|_{L^2_{-2+\delta}} \leqslant C;$$

and therefore the limit in L^4_{-1} of $|D'_j(\chi h^*_{\rho_j} u_j(k))|$ must be zero.

Therefore, we are left with only the limit of the components with $k = 0$ or $k = 1$: observe now that on these components the operator D_0 is homothety invariant, so it makes sense to look at the limit of the operators D_j. Recall that u_j has a C^0-limit, and $\rho_j \to 0$, so that $h^*_{\rho_j} u_j$ has a constant limit. We deduce that actually, for components of u_j with $k = 0$ or 1, the limit in L^4_{-1} of $(D''_j \oplus D'_j)(h^*_{\rho_j} \chi u_j)$ must be zero, which implies

$$\|D''u_j\|_{L^4_{-1}(\Delta_{\frac{1}{2}\rho_j})} + \|D'u_j\|_{L^4_{-1}(\Delta_{\frac{1}{2}\rho_j})} \to 0.$$

Therefore, we have proven, as announced, that the 'energy' (9.5) cannot concentrate near the origin. We deduce that there exist points $x_j \in \Delta_{\rho_j} - \Delta_{\frac{1}{2}\rho_j}$, such that

$$\|D''u_j\|_{L^4_{-1}(\Delta_{\frac{1}{8}\rho_j}(x_j))} \geqslant \frac{1}{100c}. \tag{9.7}$$

Now we rescale the disk $\Delta_{\frac{1}{4}\rho_j}(x_j)$ centered at x_j into the unit disk Δ by a homothety h'_j; a similar argument gives, from (9.2), the estimate

$$\|\chi(h'_j)^* u_j\|_{L^{2,2}} \leqslant c(1 + \|\nabla(h'_j)^* u_j\|^2_{L^4}) \leqslant C.$$

Hence we can extract a strongly $L^{1,4}$ convergent subsequence $(h'_j)^* u_j$, but the limit must again be flat, and this contradicts the fact that by (9.7) the L^4-norm of $(D' \oplus D'')((h'_j)^* u_j)$ on $\Delta_{\frac{1}{2}}$ is bounded below. □

Using these two lemmas, the proof is now a standard adaptation of that in [Biq97]. Indeed, from Lemma 9.1 one deduces by Simpson's method that if the bundle is stable, then Donaldson's functional is bounded below, and a minimizing sequence $h_j = hu_j$ must converge in C^0 to some limit; moreover, $\|D''u_j\|_{L^2}$ remains bounded. Lemma 9.2 gives the stronger convergence in Sobolev spaces $L^{2,2}_{-2+\delta}$, and one can then deduce that the limit actually solves the equation. □

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