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# CONCEPTS IN VECTOR SPACES WITH CONVERGENCE STRUCTURES

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1. **Introduction.** Limit vector spaces ('limitierte Vektorräume') were defined by Fischer [1] and concepts such as continuity, compactness, etc. were introduced and studied by him and others, e.g. by Kent [3]. In this paper the concepts of precompactness and boundedness in limit vector spaces are studied. It is shown that most of their properties in topological vector spaces hold in limit vector spaces.

The notions of 'espaces à bornés' (*ab*-spaces) and 'espaces à bornés complets' (*abc*-spaces) were introduced by Waelbroeck [4]. An *ab*-space $(E, \mathscr{B})$ is a vector space $E$ over the field of complex numbers in which a class $\mathscr{B}$ of subsets of $E$ is specified such that each finite subset of $E$ is an element of $\mathscr{B}$, the union and sum of two elements of $\mathscr{B}$ are elements of $\mathscr{B}$, the convex envelope, any scalar multiple and subset of an element of $\mathscr{B}$ are elements of $\mathscr{B}$, and no non-zero vector subspace of $E$ is an element of $\mathscr{B}$. An *abc*-space is an *ab*-space having, together with some other properties, the property that the absolutely convex envelope of an element of $\mathscr{B}$ is an element of $\mathscr{B}$. It is shown that under certain conditions a limit vector space, with $\mathscr{B}$ the class of bounded subsets, is an *ab*-space and vice versa.

2. **Precompact and bounded sets.** In this section all vector spaces are over the field $C$ of complex numbers. The following notation is used here. Let $E$ be a vector space, $x \in E$, $A \subset E$, and let $\mathscr{F}$ and $\mathscr{F}'$ be filters in $E$ and $\mathscr{G}$ a filter in $C$. Then $\dot{x}$, $\mathscr{F}-\mathscr{F}'$ and $\mathscr{G}A$ are respectively the filters generated by $\{x\}$, $\{F-F' : F \in \mathscr{F}, F' \in \mathscr{F}'\}$ and $\{GA : G \in \mathscr{G}\}$. The filter in $C$ generated by $\{B_\varepsilon(0)=\{\alpha \in C : |\alpha| \leq \varepsilon\} : \varepsilon>0\}$ is denoted by $\mathscr{N}(0)$. For a filter $\mathscr{F}$ in $E$, $[\mathscr{F}]=\{\mathscr{F}' : \mathscr{F}'$ is a filter in $E$ such that $\mathscr{F} \subset \mathscr{F}'\}$. In a limit vector space $(E, \tau)$ the class of filters 'converging to $x$' is denoted by $\tau x$.

DEFINITION 2.1. Let $(E, \tau)$ be a limit vector space. Then

(a) a filter $\mathscr{F}$ in $E$ is a Cauchy filter in $E$ if $\mathscr{F}-\mathscr{F} \in \tau 0$;

(b) a subset $A$ of $E$ is $\tau$-precompact if $A=\varnothing$ or if $A \neq \varnothing$ and, for each filter $\mathscr{F}$ in $E$ with $A \in \mathscr{F}$, there is a Cauchy filter $\mathscr{G}$ in $E$ such that $\mathscr{F} \subset \mathscr{G}$ (or, equivalently, each ultrafilter $\mathscr{U}$ in $E$ with $A \in \mathscr{U}$ is a Cauchy filter in $E$).

In a limit vector space $(E, \tau)$, $\mathscr{F}$ is a Cauchy filter in $E$ if $\mathscr{F} \in \tau x$ for some $x \in E$.

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Hence each $\tau$-compact subset of $E$ is $\tau$-precompact. The union of two $\tau$-precompact subsets of $E$ is $\tau$-precompact and any non-zero scalar multiple and any subset of a $\tau$-precompact subset of $E$ is $\tau$-precompact.

PROPOSITION 2.1. *Let $(E, \tau)$ be a limit vector space and let $A$ be a $\tau$-precompact subset of $E$. Then*

(a) *there are, for each $U \in \cap \{\mathscr{F}:\mathscr{F} \in \tau 0\}$, elements $x_1, x_2, \ldots, x_n$ in $E$ such that $A \subset \cup \{x_i+U: i=1, 2, \ldots, n\}$;*

(b) *$f(A) \subset E'$ is $\tau'$-precompact if $f$ is a $\tau-\tau'$-continuous linear operator on $E$ into $E'$, where $(E', \tau')$ is a limit vector space.*

**Proof.** (a) Suppose that there is no finite subset $F$ of $E$ such that $A \subset F+U$ and let $\mathscr{B}=\{B\subset E:$there is a finite subset $F$ of $E$ such that $B\subset F+U\}$. Then $\mathscr{B}'=\{A \cap (E\sim B): B \in \mathscr{B}\}$ is a filter base in $E$. Let $\mathscr{F}$ be the filter in $E$ generated by $\mathscr{B}'$. Then $A \in \mathscr{F}$ and so there is a Cauchy filter $\mathscr{G}$ in $E$ such that $\mathscr{F} \subset \mathscr{G}$. Hence, since $U \in \mathscr{G}-\mathscr{G}$, there is a $G \in \mathscr{G}$ such that $G-G\subset U$, and so $G \in \mathscr{B}$. Thus $(E\sim G) \cap G \in \mathscr{G}$—which is false.

(b) Let $\mathscr{F}'$ be a filter in $E'$ with $f(A) \in \mathscr{F}'$ and let $\mathscr{F}$ be the filter in $E$ generated by $\{A \cap f^{-1}(f(A) \cap F'): F' \in \mathscr{F}'\}$. Then there is a filter $\mathscr{G}$ in $E$ such that $\mathscr{F} \subset \mathscr{G}$ and $\mathscr{G}-\mathscr{G} \in \tau 0$. Now, let $\mathscr{G}'$ be the filter in $E'$ generated by $\{f(G): G \in \mathscr{G}\}$. Then $\mathscr{F}' \subset \mathscr{G}'$ and $\mathscr{G}'-\mathscr{G}' \in \tau' 0$.

A subset $B$ of a topological vector space $(E, \mathscr{T})$ is said to be $\mathscr{T}$-bounded if, for each $\mathscr{T}$-neighborhood $U$ of 0 in $E$, there is a positive real number $\rho$ such that $B\subset \rho U$. It is easy to show that $B(\neq \varnothing)$ is $\mathscr{T}$-bounded if and only if $\mathscr{N}(0)B$ $\mathscr{T}$-converges to 0. This fact leads to the following definition.

DEFINITION 2.2. Let $(E, \tau)$ be a limit vector space and let $A\subset E$. Then $A$ is $\tau$-bounded if $A=\varnothing$ or if $A\neq\varnothing$ and $\mathscr{N}(0)A \in \tau 0$.

PROPOSITION 2.2. *Let $(E, \tau)$ and $(E', \tau')$ be limit vector spaces and let $f$ be a linear operator on $E$ into $E'$. Then,*

(a) *if there is an $N \in \cap \{\mathscr{F}:\mathscr{F} \in \tau 0\}$ such that $f(N)$ is $\tau'$-bounded, $f$ is $\tau-\tau'$-continuous;*

(b) *if $A$ is $\tau$-bounded and $f$ is $\tau-\tau'$-continuous, $f(A)$ is $\tau'$-bounded.*

**Proof.** (a) Let $\mathscr{G} \in \tau 0$ and let $f(\mathscr{G})$ be the filter in $E'$ generated by $\{f(G): G \in \mathscr{G}\}$. If $X \in \mathscr{N}(0)f(N)$, then there is a positive real number $\varepsilon$ such that $B_\varepsilon(0)f(N)\subset X$. Hence $f(\varepsilon N)\subset X$. But $\varepsilon N \in \cap \{\mathscr{F}:\mathscr{F} \in \tau 0\} \in \mathscr{G}$. Thus $X \in f(\mathscr{G})$, i.e. $\mathscr{N}(0)f(N)\subset f(\mathscr{G})$. Therefore $f(\mathscr{G}) \in \tau' 0$.

(b) This part is left to the reader.

PROPOSITION 2.3. *Let $(E, \tau)$ be a limit vector space. Then,*

(a) *if $A$ is $\tau$-bounded,*

(i) *$\alpha A$ is $\tau$-bounded for each $\alpha \in C$;*

(ii) *$B$ is $\tau$-bounded if $B\subset A$;*

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(b) *if both $A$ and $B$ are $\tau$-bounded,*

(i) *$A+B$ is $\tau$-bounded;*

(ii) *$A \cup B$ is $\tau$-bounded;*

(c) *for each $x \in E$, $\{x\}$ is $\tau$-bounded;*

(d) *if $(E, \tau)$ is separated, $\{0\}$ is the only $\tau$-bounded vector subspaces of $E$.*

**Proof.** (b) (i) Let $X \in \mathcal{N}(0)A+\mathcal{N}(0)B$. Then there are positive real numbers $\varepsilon_1$ and $\varepsilon_2$ such that $B_{\varepsilon_1}(0)A+B_{\varepsilon_2}(0)B \subset X$. Let $\varepsilon=\min\{\varepsilon_1, \varepsilon_2\}$. Then $B_\varepsilon(0)(A+B) \subset X$, i.e. $X \in \mathcal{N}(0)(A+B)$. Thus $\mathcal{N}(0)(A+B) \in \tau 0$.

(b) (ii) Let $\mathscr{F}=(\mathcal{N}(0)A) \cap (\mathcal{N}(0)B)$. Then $\mathscr{F} \in \tau 0$ and it remains to be shown that $\mathscr{F} \subset \mathcal{N}(0)(A \cup B)$. Let $X \in \mathscr{F}$. Then there are positive real numbers $\varepsilon_1$ and $\varepsilon_2$ such that $(B_{\varepsilon_1}(0)A) \cup (B_{\varepsilon_2}(0)B) \subset X$. Let $\varepsilon=\min\{\varepsilon_1, \varepsilon_2\}$. Then $B_\varepsilon(0)(A \cup B) \subset X$, and so $\mathscr{F} \subset \mathcal{N}(0)(A \cup B)$ (in fact, $\mathscr{F}=\mathcal{N}(0)(A \cup B)$).

(d) Let $x \in E$ and let $E_1=\operatorname{span}\{x\}$. Suppose now that $E_1$ is $\tau$-bounded, i.e. $\mathcal{N}(0)E_1=\mathscr{F} \in \tau 0$. Let $X \in \mathscr{F}$. Then there is a positive real number $\varepsilon$ such that $B_\varepsilon(0)E_1 \subset X$. Hence $\varepsilon E_1 \subset X$, which implies that $x \in X$, i.e. $X \in \dot{x}$. Thus $\dot{x} \in \tau 0 \cap \tau x$, and so $x=0$.

The other verifications are left to the reader.

Definition 2.3. A limit vector space $(E, \tau)$ is locally convex if, for each $\mathscr{F} \in \tau 0$, there is an $\mathscr{F}' \in \tau 0$, having a base of convex sets, such that $\mathscr{F}' \subset \mathscr{F}$.

Let $(E, \mathscr{T})$ be a locally convex topological vector space. Then $(E, \tau)$, with $\tau 0=\{\mathscr{F} : \mathscr{F}$ is a filter in $E$ which $\mathscr{T}$-converges to 0 in $E\}$, is a locally convex limit vector space. Other examples will be given in Proposition 2.5.

Proposition 2.4. *Let $(E, \tau)$ be a locally convex limit vector space and let $A \subset E$ be $\tau$-bounded. Then the convex envelope ${}^cA$ of $A$ is $\tau$-bounded.*

**Proof.** Let $\mathcal{N}(0)A=\mathscr{F} \in \tau 0$. Then there is an $\mathscr{F}' \in \tau 0$, having a base $\mathscr{B}$ of convex sets, such that $\mathscr{F}' \subset \mathscr{F}$. Let $X \in \mathscr{F}'$. Then there is a $B \in \mathscr{B}$ such that $B \subset X$ and $B \in \mathscr{F}$. Hence there is a positive real number $\varepsilon$ such that $B_\varepsilon(0)A \subset B$. It follows that $B_\varepsilon(0)({}^cA) \subset X$. Thus $\mathcal{N}(0)({}^cA) \in \tau 0$.

The following theorem is a combination of Propositions 2.3 and 2.4.

Theorem 2.1. *Let $(E, \tau)$ be a separated locally convex limit vector space. Then $(E, \mathscr{B})$, with $\mathscr{B}$ the class of $\tau$-bounded subsets of $E$, is an ab-space.*

Proposition 2.5. *Let $(E, \mathscr{B})$ be an abc-space. Then $(E, \tau)$, with $\tau$ defined by $\tau 0=\{[\mathcal{N}(0)B] : B \in \mathscr{B}\}$ (cf. [2, pp. 9, 10]), is a separated locally convex limit vector space in which each $B \in \mathscr{B}$ is $\tau$-bounded.*

**Proof.** Let $\mathscr{F}$, $\mathscr{G} \in \tau 0$. Then there are elements $B_1$, $B_2 \in \mathscr{B}$ such that $\mathscr{F} \supset \mathcal{N}(0)B_1$ and $\mathscr{G} \supset \mathcal{N}(0)B_2$. Let $B'$ and $B''$ be respectively the absolutely convex envelopes of $B_1$ and $B_2$. Then $B', B'' \in \mathscr{B}$ and $\mathcal{N}(0)B'+\mathcal{N}(0)B'' \subset \mathscr{F}+\mathscr{G}$. Let $X \in \mathcal{N}(0)$

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$(B'+B'')$. Then there is a positive real number $\varepsilon$ such that $B_\varepsilon(0)\ (B'+B'')\subset X$. Hence $B_{\varepsilon/2}(0)B'+B_{\varepsilon/2}(0)B''\subset X$, i.e. $X\in\mathcal{N}(0)B'+\mathcal{N}(0)B''$. Thus $\mathcal{F}+\mathcal{G}\in\tau 0$. It is easily verified that all the other conditions of [1, Satz 9] are satisfied. Hence $(E, \tau)$ is a limit vector space. Also, since $\{0\}$ is the only vector subspace of $E$ that is an element of $\mathcal{B}$, $(E, \tau)$ is separated. Furthermore, let $\mathcal{F}\in\tau 0$. Then there is a $B\in\mathcal{B}$ such that $\mathcal{F}\supset\mathcal{N}(0)B$. Let $\mathcal{F}'=\mathcal{N}(0)B'$, with $B'$ the absolutely convex envelope of $B$. Then $\mathcal{F}'\in\tau 0$, $\mathcal{F}'\subset\mathcal{F}$ and $\mathcal{F}'$ has a base $\{B_\varepsilon(0)B':\varepsilon>0\}$ of absolutely convex sets. Thus $(E, \tau)$ is a separated locally convex limit vector space. Finally, let $B\in\mathcal{B}$. Then $\mathcal{N}(0)B\in\tau 0$, i.e. $B$ is $\tau$-bounded.

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