6

Statistical interpretation of the field

6.1 Fluctuations and virtual processes

Although it arises naturally in quantum field theory from unitarity, the Feynman Green function does not arise naturally in classical field theory. It contains explicitly acausal terms which defy our experience of mechanics. It has special symmetry properties: it depends only on $|x - x'|$, and thus distinguishes no special direction in space or time. It seems to characterize the uniformity of spacetime, or of a physical system in an unperturbed state.

The significance of the Feynman Green function lies in the *effective* understanding of complex systems, where Brownian fluctuations in bulk have the macroscopic effect of mixing or stirring. In field theory, its use as an intuitive model for fluctuations allows the analysis of population distributions and the simulation of field decay, by spreading an energy source evenly about the possible modes of the system.

6.1.1 Fluctuation generators: $G_{\rm F}(x, x')$ and $G_{\rm E}(x, x')$

The Feynman Green function is related to the Green function for Euclidean space. Beginning with the expression in eqn. (5.99), one performs an anti-clockwise rotation of the integration contour (see figure 5.3):

$$k_0^{\rm E} = {\rm i}k_0. \tag{6.1}$$

There are no obstacles (poles) which prevent this rotation, so the two expressions are completely equivalent. With this contour definition the integrand is positive definite and no poles are encountered in an integral over $\hat{\mathbf{k}}_{0\rm E}$:

$$\frac{1}{-k_0^2 + \mathbf{k}^2 + m^2 - {\rm i}\epsilon} \rightarrow \frac{1}{k_{0\rm E}^2 + \mathbf{k}^2 + m^2}. \tag{6.2}$$

There are several implications to this equivalence between the Feynman Green function and the Euclidean Green function. The first is that Wick rotation to

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Euclidean space is a useful technique for evaluating Green function integrals, without the interference of poles and singularities. Another is that the Euclidean propagator implies the same special causal relationship between the source and the field as does the Feynman Green function. In quantum field theory, one would say that these Green functions formed time-ordered products.

In the classical theory, the important point is the spacetime symmetry of the Green functions. Owing to the quadratic nature of the integral above, it is clear that both the Feynman and Euclidean Green functions depend only on the absolute value of $|x-x'|$. They single out no special direction in time. Physically they represent processes which do not develop in time, or whose average effect over an infinitesimal time interval is zero.

These Green functions are a differential representation of a cycle of emission and absorption (see below). They enable one to represent *fluctuations* or *virtual processes* in the field which do not change the overall state. These are processes in which an excitation is emitted from a source and is absorbed by a sink over a measurable interval of time.[1] This is a doorway to the study of statistical equilibria.

Statistical (many-particle) effects are usually considered the domain of quantum field theory. their full description, particularly away from equilibrium, certainly requires the theory of interacting fields, but the essence of statistical mechanics is contained within classical concepts of ensembles. The fact that a differential formulation is possible through the Green function has profound consequences for field theory. Fluctuations are introduced implicitly through the boundary conditions on the Green functions. The quantum theory creates a more elaborate framework to justify this choice of boundary conditions, and takes it further. However, when it comes down to it, the idea of random fluctuations in physical systems is postulated from experience. It does not follow from any deeper physical principle, nor can it be derived. Its relationship to Fock space methods of counting states is fascinating though. This differential formulation of statistical processes is explored in this chapter.[2]

6.1.2 *Correlation functions and generating functionals*

The Feynman (time-ordered) Green function may be obtained from a generating functional W which involves the action. From this generating functional it is possible to see that a time-translation-invariant field theory, expressed in terms of the Feynman Green function, is analytically related to a statistically

[1] Actually, almost all processes can be studied in this way by assuming that the field tends to a constant value (usually zero) at infinity.

[2] In his work on source theory, Schwinger [118, 119] constructs quantum transitions and statistical expectation values from the Feynman Green function Δ_+, using the principle of spacetime uniformity (the Euclidean hypothesis). The classical discussion here is essentially equivalent to his treatment of weak sources.

weighted ensemble of static systems. The action $S[\phi(x)]$ is already a generating functional for the mechanics of the system, as noted in chapter 4. The additional generating functional $W[J]$ may be introduced in order to study the statistical correlations in the field. This is a new concept, and it requires a new generating functional, the effective action. The effective action plays a central role in quantum field theory, where the extension to interacting fields makes internal dynamics, and thence the statistical interpretation, even more pressing.

We begin by defining averages and correlated products of the fields. This is the route to a statistical interpretation. Consider a field theory with fields $\phi^A, \phi^{\dagger B}$ and action $S^{(2)}$. The superscript here denotes the fact that the action is one for free fields and is therefore of purely quadratic order in the fields. In the following sections, we use the complex field $\phi(x)$ to represent an arbitrary field. The same argument applies, with only irrelevant modifications, for general fields. We may write

$$S^{(2)} = \int (\mathrm{d}x)\; \phi^{\dagger A} \hat{\mathcal{O}}_{AB} \phi^B, \tag{6.3}$$

where the Gaussian weighted average, for statistical weight $\rho = \exp(\mathrm{i}S/s)$ is then defined by

$$\begin{aligned} \langle F[\phi] \rangle &= \frac{\mathrm{Tr}(\rho F)}{\mathrm{Tr}\,\rho} \\ &= \frac{\int \mathrm{d}\mu[\phi] F[\phi] \mathrm{e}^{\frac{\mathrm{i}}{s} S^{(2)}}}{\int \mathrm{d}\mu[\phi] \mathrm{e}^{\frac{\mathrm{i}}{s} S^{(2)}}}. \end{aligned} \tag{6.4}$$

where s is an arbitrary scale with the dimensions of action. In quantum field theory, it is normal to use $s = \hbar$, but here we keep it general to emphasize that the value of this constant cancels out of relevant formulae at this classical level. Do not be tempted to think that we are now dealing with quantum field theory, simply because this is a language which grew up around the second quantization. The language is only a convenient mathematical construction, which is not tied to a physical model. In this section, we shall show that the Gaussian average over pairs of fields results in the classical Feynman Green function. Consider the generating functional

$$Z\left[J, J^\dagger\right] = \int \mathrm{d}\mu\left[\phi, \phi^\dagger\right] \mathrm{e}^{\frac{\mathrm{i}}{s}\int (\mathrm{d}x)\left[\phi^{\dagger A} \hat{\mathcal{O}}_{AB} \phi^B - \phi^{\dagger A} J_A - J_B^\dagger \phi^B\right]}, \tag{6.5}$$

which bears notable similarities to the classical thermodynamical partition function. From the definitions above, we may write

$$\frac{Z\left[J, J^\dagger\right]}{Z\left[0, 0\right]} = \left\langle \exp\left(-\frac{\mathrm{i}}{s} \int (\mathrm{d}x) \phi^{\dagger A} J_A - \frac{\mathrm{i}}{s} \int (\mathrm{d}x) J_A^\dagger \phi^A \right) \right\rangle, \tag{6.6}$$

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where the currents J^A and $J^{\dagger B}$ are of the same type as ϕ^A and $\phi^{\dagger B}$, respectively. The effective action, as a function of the sources $W[J, J^\dagger]$, is defined by

$$\exp\left(\frac{\mathrm{i}}{s} W[J, J^\dagger]\right) = Z\left[J, J^\dagger\right], \tag{6.7}$$

thus $W[J, J^\dagger]$ is like the average value of the action, where the average is defined by the Gaussian integral. Now consider a shift of the fields in the action, which diagonalizes the exponent in eqn. (6.6):

$$\begin{aligned}&(\phi^{\dagger A} + K^A)\hat{\mathcal{O}}_{AB}(\phi^B + L^B) - K^A\hat{\mathcal{O}}_{AB}L^B \\ &= \phi^{\dagger A}\hat{\mathcal{O}}_{AB}\phi^B + \phi^{\dagger A}\hat{\mathcal{O}}_{AB}L^B + K^A\hat{\mathcal{O}}_{AB}\phi^B.\end{aligned} \tag{6.8}$$

The right hand side of this expression is the original exponent in eqn. (6.5), provided we identify

$$\hat{\mathcal{O}}_{AB}L^B(x) = J^A(x) \tag{6.9}$$

$$\Rightarrow \quad L^A(x) = \int (\mathrm{d}x')(\hat{\mathcal{O}}^{-1})^{AB}(x, x')J_B(x') \tag{6.10}$$

and

$$K^A(x)\hat{\mathcal{O}}_{AB} = J^\dagger_B(x) \tag{6.11}$$

$$\Rightarrow \quad K^A(x) = \int (\mathrm{d}x')J^\dagger_B(x')(\hat{\mathcal{O}}^{-1})^{AB}(x, x'), \tag{6.12}$$

where $\int(\mathrm{d}x')\hat{\mathcal{O}}^{-1\,AB}\hat{\mathcal{O}}_{BC} = \delta^A_C$. With these definitions, it follows that

$$K^A\hat{\mathcal{O}}_{AB}L^B = \int (\mathrm{d}x)(\mathrm{d}x')\; J^\dagger_A(\hat{\mathcal{O}}^{-1})^{AB}J_B \tag{6.13}$$

and so

$$Z[J, J^\dagger] = \int \mathrm{d}\mu[\phi, \phi^\dagger]\; \mathrm{e}^{\frac{\mathrm{i}}{s}\int(\mathrm{d}x)\left[(\phi^{\dagger A}+K^A)\hat{\mathcal{O}}_{AB}(\phi^B+L^B) - J^\dagger_A(\hat{\mathcal{O}}^{-1})^{AB}J_B\right]}. \tag{6.14}$$

We may now translate away L^A and K^A, assuming that the functional measure is invariant. This leaves

$$Z\left[J, J^\dagger\right] = \exp\left(-\frac{\mathrm{i}}{s}\int (\mathrm{d}x)(\mathrm{d}x')\; J^\dagger_A(\hat{\mathcal{O}}^{-1})^{AB}J_B\right)\; Z\left[0, 0\right] \tag{6.15}$$

or

$$W[J, J^\dagger] = -\int (\mathrm{d}x)(\mathrm{d}x')\; J^\dagger_A(\hat{\mathcal{O}}^{-1})^{AB}J_B + \text{const.} \tag{6.16}$$

By differentiating $W[J, J^\dagger]$ with respect to the source, we obtain

$$\langle \phi^{\dagger A} \rangle = \frac{\delta W}{\delta J_A^\dagger(x)} = \mathrm{i}s \int (\mathrm{d}x') J_A^\dagger (\hat{\mathcal{O}}^{-1})^{AB} \tag{6.17}$$

$$\langle \phi^{B} \rangle = \frac{\delta W}{\delta J_B(x)} = \mathrm{i}s \int (\mathrm{d}x') (\hat{\mathcal{O}}^{-1})^{AB} J_B \tag{6.18}$$

$$\langle \phi^{\dagger A} \phi^{\dagger B} \rangle = \mathrm{i}s \frac{\delta^2 W}{\delta J^\dagger \delta J_B^\dagger} = 0 \tag{6.19}$$

$$\langle \phi^{A} \phi^{B} \rangle = \mathrm{i}s \frac{\delta^2 W}{\delta J_A \delta J_B} = 0 \tag{6.20}$$

$$\langle \phi^{A} \phi^{\dagger B} \rangle = \mathrm{i}s \frac{\delta^2 W}{\delta J_A^\dagger \delta J_B} = \mathrm{i}s (\hat{\mathcal{O}}^{-1})^{AB} \tag{6.21}$$

One may now identify $(\hat{\mathcal{O}}^{-1})^{AB}$ as the inverse of the operator in the quadratic part of the action, which is clearly a Green function, i.e.

$$\langle \phi^{A} \phi^{\dagger B} \rangle = \mathrm{i}s G^{AB}(x, x'). \tag{6.22}$$

Moreover, we have evaluated the generator for correlations in the field $W[J]$. Returning to real scalar fields, we have

$$W[J] = -\frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}x')\; J_A(x) G^{AB}(x, x') J_B(x'). \tag{6.23}$$

We shall use this below to elucidate the significance of the Green function for the fluctuations postulated in the system. Notice that although the generator, in this classical case, is independent of the scale s, the definition of the correlation function in eqn. (6.21) does depend on this scale. This tells us simply the magnitude of the fluctuations compared with the scale of $W[J]$ (the typical energy scale or rate of work in the system over a time interval). If one takes $s = \hbar$, we place the fluctuations at the quantum level. If we take $s \sim \beta^{-1}$, we place fluctuations at the scale of thermal activity kT.[3] Quantum fluctuations become unimportant in the classical limit $\hbar \to 0$; thermal fluctuations become unimportant in the low-temperature limit $\beta \to \infty$. At the level of the present discussion, the results we can derive from the correlators are independent of this scale, so a macroscopic perturbation would be indistinguishable from a microscopic perturbation. It would be a mistake to assume that this scale were unimportant however. Changes in this scaling factor can lead to changes in the correlation lengths of a system and phase transitions. This, however, is the domain of an interacting (quantum) theory.

[3] These remarks reach forward to quantum field theories; they cannot be understood from the simple mechanical considerations of the classical field. However they do appeal to one's intuition and make the statistical postulate more plausible.

We have related the generating functional $W[J]$ to weighted-average products over the fields. These have an automatic symmetry in their spacetime arguments, so it is clear that the object $G^{AB}(x, x')$ plays the role of a correlation function for the field. The symmetry of the generating functional alone implies that $(\hat{\mathcal{O}}^{-1})^{ij}$ must be the Feynman Green function. We shall nevertheless examine this point more closely below.

A note of caution to end this section: the spacetime symmetry of the Green function follows from the fact that the integrand in

$$G(x, x') = \int (\mathrm{d}k) \frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2 + m^2} \tag{6.24}$$

is a purely quadratic quantity. A correlator must depend only on the signless difference between spacetime events $|x - x'|$, if it is to satisfy the relations in the remainder of this section on dissipation and transport. If the spectrum of excitations were to pick up, say, an absorbative term, which singled out a special direction in time, this symmetry property would be spoiled, and, after an infinitesimal time duration, the Green functions would give the wrong answer for the correlation functions. In that case, it would be necessary to analyse the system more carefully using methods of non-equilibrium field theory. In practice, the simple formulae given in the rest of this chapter can only be applied to derive instantaneous tendencies of the field, never prolonged instabilities.

6.1.3 Symmetry and causal boundary conditions

There are two Green functions which we might have used in eqn. (6.21) as the inverse of the Maxwell operator; the retarded Green function and the Feynman Green function. Both satisfy eqn. (5.62). The symmetry of the expression

$$W = -\frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}x') J(x) G(x, x') J(x') \tag{6.25}$$

precludes the retarded function however. The integral is spacetime-symmetrical, thus, only the symmetrical part of the Green function contributes to the integral. This immediately excludes the retarded Green function, since

$$\begin{aligned} W_{\mathrm{r}} &= -\frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}x') J(x) G_{\mathrm{r}}(x, x') J(x') \\ &= -\frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}x') J(x) [G^{(+)}(x, x') + G^{(-)}(x, x')] J(x') \\ &= -\frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}x') J(x) [G^{(+)}(x, x') - G^{(+)}(x', x)] J(x') \\ &= 0, \end{aligned} \tag{6.26}$$

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where the last line follows on re-labelling x, x' in the second term. This relation tells us that there is no dissipation in one-particle quantum theory. As we shall see, however, it does not preclude dissipation by re-distribution of energy in 'many-particle' or statistical systems coupled to sources. See an example of this in section 7.4.1. Again, the link to statistical systems is the Feynman Green function or correlation function. The Feynman Green function is symmetrical in its spacetime arguments. It is straightforward to show that

$$
\begin{aligned}
W &= -\frac{1}{2}\int (\mathrm{d}x)(\mathrm{d}x')J(x)G_{\mathrm{F}}(x,x')J(x') \\
&= -\frac{1}{2}\int (\mathrm{d}x)(\mathrm{d}x')J(x)\overline{G}(x,x')J(x').
\end{aligned} \tag{6.27}
$$

The imaginary part of $\overline{G}(x,x')$ is

$$\mathrm{Im}\,\overline{G}(x,x') = 2\,\mathrm{Im}\,G^{(+)}(x,x'). \tag{6.28}$$

6.1.4 Work and dissipation at steady state

Related to the idea of transport is the idea of energy dissipation. In the presence of a source J, the field can decay due to work done on the source. Of course, energy is conserved within the field, but the presence of fluctuations (briefly active sources) allows energy to be transferred from one part of the field to another; i.e. it allows energy to be mixed randomly into the system in a form which cannot be used to do further work. This is an increase in entropy.

The instantaneous rate at which the field decays is proportional to the imaginary part of the Feynman Green function. In order to appreciate why, we require a knowledge of the energy–momentum tensor and Lorentz transformations, so we must return to this in section 11.8.2. Nonetheless, it is possible to gain a partial understanding of the problem by examining the Green functions, their symmetry properties and the roles they play in generating solutions to the field equations. This is an important issue, which is reminiscent of the classical theory of hydrodynamics [53].

The power dissipated by the field is the rate at which field does work on external sources,[4]

$$P = \frac{\mathrm{d}w}{\mathrm{d}t}. \tag{6.29}$$

Although we cannot justify this until chapter 11, let us claim that the energy of the system is determined by

$$\text{energy} = -\left.\frac{\delta S[\phi]}{\delta t}\right|_{\phi=\int GJ}. \tag{6.30}$$

[4] Note that we use a small w for work since the symbol W is generally reserved to mean the value of the action, evaluated at the equations of motion.

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So, the rate of change of energy in the system is equal to minus the rate at which work is done by the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\text{energy} = -\frac{\mathrm{d}w}{\mathrm{d}t}. \tag{6.31}$$

Let us define the action functional W by

$$\frac{\delta W}{\delta J} = \frac{\delta S[\phi, J]}{\delta J}\bigg|_{\phi=\int GJ}, \tag{6.32}$$

where the minus sign is introduced so that this represents the work done by the system rather than the energy it possesses. The object W clearly has the dimensions of action, but we shall use it to identify the rate at which work is done. Eqn. (6.32) is most easily understood with the help of an example. The action for a scalar field is

$$\delta_J S = -\int (\mathrm{d}x)\ \delta J \phi(x), \tag{6.33}$$

so, evaluating this at

$$\phi(x) = \int (\mathrm{d}x')\ G(x, x')J(x'), \tag{6.34}$$

one may write, up to source-independent terms,

$$W[J] = -\frac{1}{2}\int (\mathrm{d}x)(\mathrm{d}x')\ J(x)G_{\mathrm{F}}(x, x')J(x'). \tag{6.35}$$

This bi-linear form recurs repeatedly in field theory. Schwinger's source theory view of quantum fields is based on this construction, for its spacetime symmetry properties. Notice that it is based on the Feynman Green function. Eqn. (6.34) could have been solved by either the Feynman Green function or the retarded Green function. The explanation follows shortly. The work done over an infinitesimal time interval is given by

$$\Delta w = \mathrm{Im}\frac{\mathrm{d}W}{\mathrm{d}t}. \tag{6.36}$$

Expressed in more useful terms, the instantaneous decay rate of the field is

$$\int \mathrm{d}t\gamma(t) = -\frac{2}{\chi_h}\mathrm{Im}W. \tag{6.37}$$

The sign, again, indicates the difference between work done and energy lost. The factor of χ_h is included because we need a scale which relates energy and time (frequency). In quantum mechanics, the appropriate scale is $\chi_h = \hbar$. In

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fact, any constant with the dimensions of action will do here. There is nothing specifically quantum mechanical about this relation. The power is proportional to the rate of work done. The more useful quantity, the power spectrum $P(\omega)$ or power at frequency ω, is

$$\int \mathrm{d}\omega \frac{P(\omega, t)}{\hbar\omega} = -\gamma(t), \tag{6.38}$$

giving the total power

$$P = \int \mathrm{d}\omega P(\omega). \tag{6.39}$$

We speak of the instantaneous decay rate because, in a real analysis of dissipation, the act of work being done acts back on all time varying quantities. Taking the imaginary part of W to be the decay rate for the field assumes that the system changes only adiabatically, as we shall see below.

6.1.5 Fluctuations

The interpretation of the field as a statistical phenomenon is made plausible by considering the effect of infinitesimal perturbations to the field. This may be approached in two equivalent ways: (i) through the introduction of linear perturbations to the action, or sources

$$S \rightarrow S - \int (\mathrm{d}x)\ J\phi, \tag{6.40}$$

where J is assumed to be weak, or (ii) by writing the field in terms of a fluctuating 'average' part $\langle\phi\rangle$ and a remainder part φ,

$$\phi(x) = \langle\phi(x)\rangle + \varphi(x). \tag{6.41}$$

These two constructions are equivalent for all dynamical calculations. This can be confirmed by the use of the above generating functionals, and a change of variable.

It is worth spending a moment to consider the meaning of the function $W[J]$. Although originally introduced as part of the apparatus of quantum field theory [113], we find it here completely divorced from such origins, with no trace of quantum field theoretical states or operators (see chapter 15). The structure of this relation is a direct representation of our model of fluctuations or virtual processes. $W[J]$ is the generator of fluctuations in the field. The Feynman Green function, in eqn. (6.25), is sandwiched between two sources symmetrically. The Green function itself is symmetrical: for retarded times, it propagates a field radiating from a past source to the present, and for advanced times it propagates the field from the present to a future source, where it is absorbed.

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The symmetry of advanced and retarded boundary conditions makes $W[J]$ an explicit representation of a virtual process, at the purely classical level.[5]

The first derivative of the effective action with respect to the source is

$$\frac{\delta W}{\delta J(x)} = \langle \phi(x) \rangle, \tag{6.42}$$

which implies that, for the duration of an infinitesimal fluctuation $J \neq 0$, the field has an average value. If it has an average value, then it also deviates from this value, thus we may write

$$\phi(x) = \frac{\delta W}{\delta J(x)} + \varphi(x), \tag{6.43}$$

where $\varphi(x)$ is the remainder of the field due to J. The average value vanishes once the source is switched off, meaning that the fluctuation is the momentary appearance of a non-zero average in the local field. This is a smearing, stirring or mixing of the field by the infinitesimal generalized force J. The rate of change of this average is

$$(\mathrm{i}\hbar)\frac{\delta^2 W[J]}{\delta^2 J} = \langle \phi(x)\phi(x') \rangle - \langle \phi(x) \rangle \langle \phi(x') \rangle. \tag{6.44}$$

This is the correlation function $C_{AB}(x, x')$, which becomes the Feynman Green function as $J \to 0$. It signifies the response of the field to its own fluctuations nearby, i.e. the extent to which the field has become mixed. The correlation functions become large as the field becomes extremely uniform. This is called (off-diagonal[6]) long-range order.

The correlation function interpretation is almost trivial at the classical (free-field) level, but becomes enormously important in the interacting quantum theory.

Instantaneous thermal fluctuations Fluctuations have basically the same form regardless of their origin. If we treat all thermal fluctuations as instantaneous, then we may account for them by a Euclidean Green function; the fluctuations of the zero-temperature field are generated by the Feynman Green function. In an approximately free theory, these two are the same thing. Consider a thermal Boltzmann distribution

$$\mathrm{Tr}(\rho(x, x')\phi(x)\phi(x')) = \mathrm{Tr}(\mathrm{e}^{-\beta\hbar\omega}\phi(\omega)\phi(-\omega)). \tag{6.45}$$

[5] For detailed discussions of these points in the framework of quantum field theory, see the original papers of Feynman [46, 47, 48] and Dyson [41]. The generator $W[J]$ was introduced by Schwinger in ref. [113].

[6] 'Off-diagonal' refers to $x \neq x'$.

Since the average weight is $e^{iS/\hbar}$, and the Green function in momentum space involves a factor $\exp(-i\omega(t - t'))$, one can form a representation of the Boltzmann exponential factor $\exp(-\beta E)$ by analytically continuing

$$t \to t - i\hbar\beta \tag{6.46}$$

or

$$t' \to t + i\hbar\beta. \tag{6.47}$$

This introduces an imaginary time element such as that obtained by Wick rotating to Euclidean space. It also turns the complex exponential into a real, decaying exponential. If the real part of the time variable plays no significant role in the dynamics (a static system), then it can be disregarded altogether. That is why Euclidean spacetime is essentially equivalent to equilibrium thermodynamics. However, from the spacetime symmetry of the correlation functions, we should have the same result if we re-label t and t' so

$$G(t - t' + i\hbar\beta) = G(t' - t + i\hbar\beta) \tag{6.48}$$

or, in the Wick-rotated theory,

$$G(t_E - t'_E + \hbar\beta) = G(t'_E - t_E + \hbar\beta). \tag{6.49}$$

This is only possible if

$$e^{i\omega_E(t_E - \hbar\beta - t'_E)} = e^{i\omega_E(t'_E - \hbar\beta - t_E)} \tag{6.50}$$

or

$$e^{i\hbar\beta\omega_E} = 1. \tag{6.51}$$

From this; we deduce that the Euclidean Green function must be periodic in imaginary time and that the Euclidean frequency

$$\omega_E(n) = \frac{2n\pi}{\beta}, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{6.52}$$

where $\omega_E(n)$ are called the Matsubara frequencies.

Thermal fluctuations in time Using the fluctuation model, we may represent a system in thermal equilibrium by the same idealization as that used in thermodynamics. We may think of the source and sink for thermal fluctuations as being a large reservoir of heat, so large that its temperature remains constant at $T = 1/k\beta$, even when we couple it to our system. The coupling to the heat bath is by sources. Consider the fluctuation model as depicted in figure 6.1.

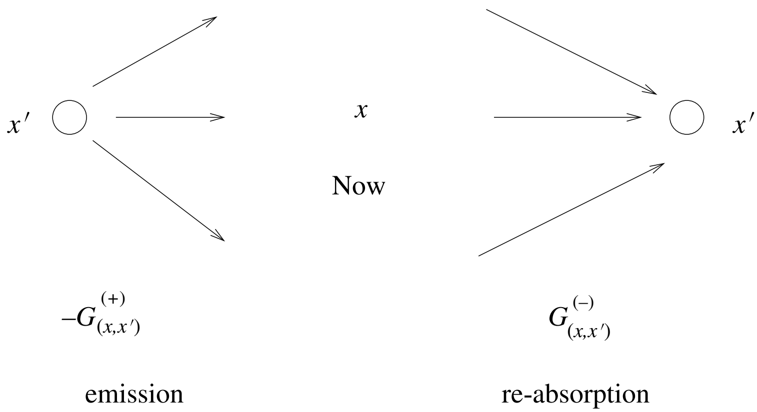

Fig. 6.1. Thermal fluctuations occur when the source is a heat reservoir at constant temperature.

Since the fluctuation generator is $W[J]$, which involves

$$W[J] = -\frac{1}{2}\int (\mathrm{d}x)(\mathrm{d}x')\ J(x)G_{\mathrm{F}}(x,x')J(x')$$
$$\sim J(x)\left[-G^{(+)}(\omega)\theta(\text{past}) + G^{(-)}(\omega)\theta(\text{future})\right]J(x'), \qquad (6.53)$$

then, during a fluctuation, the act of emission from the source is represented by $-G^{(+)}(\omega)$ and the act of re-absorption is represented by $G^{(-)}(\omega)$. In other words, these are the susceptibilities for thermal emission and absorption. In an isolated system in thermal equilibrium, we expect the number of fluctuations excited from the heat bath to be distributed according to a Boltzmann probability factor [107]:

$$\frac{\text{emission}}{\text{absorption}} = \frac{-G^{(+)}(\omega)}{G^{(-)}(\omega)} = \mathrm{e}^{\hbar\beta|\omega|}. \qquad (6.54)$$

We use $\hbar\omega$ for the energy of the mode with frequency ω by tradition, though $\hbar$ could be replaced by any more appropriate scale with the dimensions of action. This is a classical understanding of the well known Kubo–Martin–Schwinger relation [82, 93] from quantum field theory. In the usual derivation, one makes use of the quantum mechanical time-evolution operator and the cyclic property of the trace in eqn. (6.45) to derive this relation for a thermal equilibrium. What makes these two derivations equivalent is the principle of spacetime uniformity of fluctuations. The argument given here is identical to Einstein's argument for stimulated and spontaneous emission in a statistical two-state system, and the derivation of the well known A and B coefficients. It can be interpreted as the relative occupation numbers of particles with energy $\hbar\omega$. Here, the two states

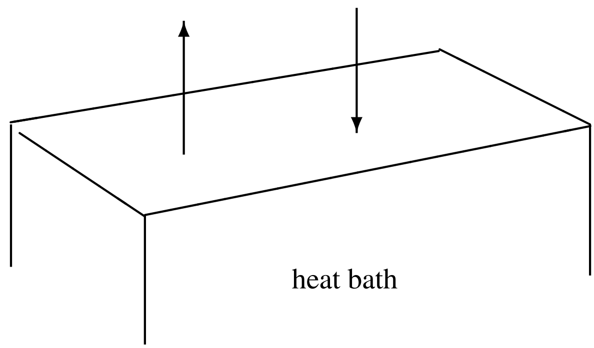

Fig. 6.2. Contact with a thermodynamic heat bath. Fluctuations represent emission and absorption from a large thermal field.

are the heat bath and the system (see figure 6.2). We can use eqn. (6.54) to find the thermal forms for the Wightman functions (and hence all the others). To do so we shall need the extra terms $X(k)$ mentioned in eqn. (5.41). Generalizing eqns. (5.66), we write,

$$\begin{aligned} G^{(+)}(k) &= -2\pi \mathrm{i}[\theta(k_0) + X]\delta(p^2c^2 + m^2c^2) \\ G^{(-)}(k) &= 2\pi \mathrm{i}[\theta(-k_0) + Y]\delta(p^2c^2 + m^2c^4) \end{aligned} \tag{6.55}$$

with X and Y to be determined. The commutator function $\tilde{G}(x, x')$ represents the difference between the emission and absorption processes, which cannot depend on the average state of the field since it represents the completeness of the dynamical system (see section 14.1.8 and eqn. (5.73)). It follows that $X = Y$. Then, using eqn. (6.54), we have

$$\theta(\omega) + X = \mathrm{e}^{\hbar\omega\beta}[\theta(-\omega) + X] \tag{6.56}$$

and hence

$$X(\mathrm{e}^{\beta\hbar|\omega|} - 1) = \theta(\omega), \tag{6.57}$$

since $\theta(-\omega)\mathrm{e}^{\beta\omega} = 0$. Thus, we have

$$X = \theta(\omega) f(\omega), \tag{6.58}$$

where

$$f(\omega) = \frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1}. \tag{6.59}$$

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This is the Bose–Einstein distribution. From this we deduce the following thermal Green functions by re-combining $G^{(\pm)}(k)$:

$$G^{(+)}(k) = -2\pi \mathrm{i}\theta(k_0)[1 + f(|k_0|)]\delta(p^2c^2 + m^2c^4) \tag{6.60}$$

$$G_{\mathrm{F}}(k) = \frac{1}{p^2c^2 + m^2c^4 - \mathrm{i}\epsilon} + 2\pi \mathrm{i} f(|k_0|)\delta(p^2c^2 + m^2c^4)\theta(k_0). \tag{6.61}$$

For a subtlety in the derivation and meaning of eqn. (6.59), see section 13.4.

The additional mixture of states which arise from the external (boundary) conditions on the system X thus plays the role of a macrostate close to steady state. Notice that the retarded and advanced Green functions are independent of X. This must be the case for unitarity to be preserved.

6.1.6 Divergent fluctuations: transport

The fluctuations model introduced above can be used to define instantaneous transport coefficients in a statistical system. Long-term, time-dependent expressions for these coefficients cannot be obtained because of the limiting assumptions of the fluctuation method. However, such a non-equilibrium situation could be described using the methods of non-equilibrium field theory.

Transport implies the propagation or flow of a physical property from one place to another over time. Examples include

- thermal conduction,
- electrical conduction (current),
- density conduction (diffusion).

The conduction of a property of the field from one place to another can only be accomplished by dynamical changes in the field. We can think of conduction as a persistent fluctuation, or a fluctuation with very long wavelength, which never dies. All forms of conduction are essentially equivalent to a diffusion process and can be analysed hydrodynamically, treating the field as though it were a fluid.

Suppose we wish to consider the transport of a quantity X: we are therefore interested in fluctuations in this quantity. In order to generate such fluctuations, we need a source term in the action which is generically conjugate to the fluctuation (see section 14.5). We add this as follows:

$$S \to S - \int (\mathrm{d}x) X \cdot F, \tag{6.62}$$

and consider the generating functional of fluctuations $W[F]$ as a function of the infinitesimal source $F(x)$; Taylor-expanding, one obtains

$$\delta W[F] = W[0] + \int (\mathrm{d}x) \frac{\delta W[0]}{\delta F(x)} F(x) + \int (\mathrm{d}x)(\mathrm{d}x') \frac{\delta^2 W[0]}{\delta F(x) \delta F(x')} F(x) \delta F(x') + \cdots . \quad (6.63)$$

Now, since

$$W[F] = \int (\mathrm{d}x)(\mathrm{d}x')\ F(x) \langle X(x) X(x') \rangle F(x'), \quad (6.64)$$

we have the first few terms of the expansion

$$\begin{aligned} W[0] &= 0 \\ \frac{\delta W[0]}{\delta F(x)} &= \langle X(x) \rangle \\ \frac{\delta^2 W[0]}{\delta F(x) \delta F(x')} &= -\frac{\mathrm{i}}{\hbar} \langle X(x) X(x') \rangle \end{aligned} \quad (6.65)$$

Thus, linear response theory gives us, generally,

$$\langle X(x) \rangle = \frac{\mathrm{i}}{\hbar} \int (\mathrm{d}x') \langle X(x) X(x') \rangle F(x'), \quad (6.66)$$

or

$$\frac{\delta \langle X(x) \rangle}{\delta F(x')} = \frac{\mathrm{i}}{\hbar} \langle X(x) X(x') \rangle. \quad (6.67)$$

Since the correlation functions have been generated by the fluctuation generator W, they satisfy Feynman boundary conditions; however, in eqn. (6.81) we shall derive a relation which may be used to relate this to the linear response of the field with retarded boundary conditions. It remains, of course, to express the correlation functions of the sources in terms of known quantities. Nevertheless, the above expression may be used to determine the transport coefficients for a number of physical properties. As an example, consider the electrical conductivity, as defined by Ohm's law,

$$J_i = \sigma_{ij} E_i. \quad (6.68)$$

If we write $E_i \sim \partial_t A_i$ in a suitable gauge, then we have

$$J_i(\omega) = \sigma_{ij} \omega A_j(\omega), \quad (6.69)$$

or

$$\frac{\delta J_i}{\delta A_j} = \sigma_{ij} \omega. \quad (6.70)$$

From eqn. (6.67), we may therefore identify the transport coefficient as the limit in which the microscopic fluctuations' wavelength tends to infinity and leads to a lasting effect across the whole system,

$$\sigma_{ij}(\omega) = \lim_{\mathbf{k}\to 0} \frac{\mathrm{i}}{\hbar\omega} \langle J_i(\omega) J_j(-\omega)\rangle. \tag{6.71}$$

The evaluation of the right hand side still needs to be performed. To do this, we need to know the dynamics of the sources J_i and the precise meaning of the averaging process, signified by $\langle \ldots \rangle$. Given this, the source method provides us with a recipe for calculating transport coefficients.

In most cases, one is interested in calculating the average transport coefficients in a finite temperature ensemble. Thermal fluctuations may be accounted for simply by noting the relationship between the Feynman boundary conditions used in the generating functional above and the retarded boundary conditions, which are easily computable from the mechanical response. We make use of eqn. (6.54) to write

$$\begin{aligned} G_{\mathrm{r}}(t,t') &= -\theta(t-t')\left[G^{(+)} + G^{(-)}\right] \\ &= -\theta(t-t')G^{(+)}\left[1 - \mathrm{e}^{-\hbar\beta\omega}\right]. \end{aligned} \tag{6.72}$$

The retarded part of the Feynman Green function is

$$G_{\mathrm{F}} = -\theta(t-t')G^{(+)}\theta(t-t'), \tag{6.73}$$

so, over the retarded region,

$$G_{\mathrm{r}}(x,x') = (1 - \mathrm{e}^{-\hbar\beta\omega})G_{\mathrm{F}}(x,x'), \tag{6.74}$$

giving

$$\sigma_{ij}(\omega) = \lim_{\mathbf{k}\to 0} \frac{(1 - \mathrm{e}^{-\hbar\beta\omega})}{\hbar\omega} \langle J_i(\omega) J_j(-\omega)\rangle, \tag{6.75}$$

for the conductivity tensor, assuming a causal response between source and field. The formula in eqn. (6.75) is one of a set of formulae which relate the fluctuations in the field to transport coefficients. The strategy for finding such relations is to identify the source which generates fluctuations in a particular quantity. We shall return to this problem in general in section 11.8.5.

6.1.7 Fluctuation dissipation theorem

In a quasi-static system, the time-averaged field may be defined by

$$\langle \phi \rangle = \frac{1}{T} \int_{\bar{t}-T/2}^{\bar{t}+T/2} \phi(x)\mathrm{d}t. \tag{6.76}$$

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From the generating functional in eqn. (6.5), we also have

$$\langle\phi(x)\rangle = \mathrm{i}\hbar\frac{\delta W[J]}{\delta J(x)}, \tag{6.77}$$

and further

$$\left.\frac{\delta\langle\phi\rangle}{\delta J}\right|_{J=0} = -\frac{\mathrm{i}}{\hbar}\langle\phi(x)\phi(x')\rangle = \mathrm{Im}\; G_{\mathrm{F}}(x,x'). \tag{6.78}$$

The field may always be causally expressed in terms of the source, using the retarded Green function in eqn. (6.76), provided the source is weak so that higher terms in the generating functional can be neglected; thus

$$\langle\phi\rangle = \frac{1}{T}\int_{\bar{t}-T/2}^{\bar{t}+T/2}\int(\mathrm{d}x')G_{\mathrm{r}}(x,x')J(x')\mathrm{d}t. \tag{6.79}$$

Now, using eqns. (6.78) and (6.79), we find that

$$\frac{\delta}{\delta J(x')}\frac{\delta}{\delta t}\phi(x) = -\mathrm{Im}\;\partial_t G_{\mathrm{F}}(x,x') = \frac{1}{T}G_{\mathrm{r}}(x,x'). \tag{6.80}$$

Thus, on analytically continuing to Euclidean space,

$$G_{\mathrm{r}}(\omega) = -\hbar\beta\omega\; G_{\mathrm{E}}(\omega). \tag{6.81}$$

This is the celebrated fluctuation dissipation theorem. It is as trivial as it is profound. It is clearly based on assumptions about the average behaviour of a statistical system over macroscopic times T, but also refers to the effects of microscopic fluctuations over times contained in $x - x'$. It relates the Feynman Green function to the retarded Green function for a time-averaged field; i.e. it relates the correlation function, which measures the spontaneous fluctuations

$$\varphi = \phi - \langle\phi\rangle \tag{6.82}$$

in the field, to the retarded Green function, which measures the purely mechanical response to an external source. The fluctuations might be either thermal or quantum in origin, it makes no difference. their existence is implicitly postulated through the use of the correlation function. Thus, the truth or falsity of this expression lies in the very assumption that microscopic fluctuations are present, even though the external source $J \to 0$ on average (over macroscopic time T). This requires further elaboration.

In deriving the above relation, we have introduced sources and then taken the limit in which they tend to zero. This implies that the result is only true for an infinitesimal but non-zero source J. The source appears and disappears, so that it is zero on average, but it is present long enough to change the distribution of modes in the system, little by little. An observer who could resolve only macroscopic behaviour would therefore be surprised to see the system changing, apparently without cause. This theorem is thus about the mixing of scales.

The fluctuation dissipation theorem tells us that an infinitesimal perturbation to the field, $J \to 0$, will lead to microscopic fluctuations, which can decay by mechanical response (mixing or diffusion). The decay rate may be related to the imaginary part of the correlation function, but this gives only an instantaneous rate of decay since the assumptions we use to derive the expression are valid only for the brief instant of the fluctuation.[7]

The Feynman Green function seems to have no place in a one-particle mechanical description, and yet here it is, at the classical level. But we have simply introduced it *ad hoc*, and the consequences are profound: we have introduced fluctuations into the system. This emphasizes the importance of boundary conditions and the generally complex nature of the field.

6.2 Spontaneous symmetry breaking

Another aspect of fluctuating statistical theories, which arises in connection with symmetry, is the extent to which the average state of the field, $\langle\phi\rangle$, displays the full symmetry afforded it by the action. In interacting theories, collective effects can lead to an average ordering of the field, known as long-range order. The classic example of this is the ferromagnetic state in which spin domains line up in an ordered fashion, even though the action allows them to point in any direction, and indeed the fluctuations in the system occur completely at random. However, it is energetically favourable for fluctuations to do this close to an average state in which all the spins are aligned, provided the fluctuations are small. Maximum stability is then achieved by an ordered state. As fluctuations grow, perhaps by increasing temperature, the stability is lost and a phase transition can occur. This problem is discussed in section 10.7, after the chapters on symmetry.

[7] The meaning of this 'theorem' for Schwinger's source theory viewpoint is now clear [119]. Spacetime uniformity in the quantum transformation function tells us that the Green function we should consider is the Feynman Green function. The symmetry of the arguments tells us that this is a correlation function and it generates fluctuations in the field. The infinitesimal source is a model for these fluctuations. Processes referred to as the decay of the vacuum in quantum field theory are therefore understood in a purely classical framework, by understanding the meaning of the Feynman boundary conditions.